Edge reduction for weakly non-negative quadratic forms

By

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Abstract. Let $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ be an integral quadratic form of the shape

$$q(x) = \sum_{i=1}^{n} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$$

with $q_i \leq 1$, for every $1 \leq i \leq n$. Several procedures have been introduced to study these forms. In this paper we consider the edge reduction procedure introduced in [7].

A form $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ is weakly non-negative if $q(x) \geq 0$ for every vector $x$ with non-negative coordinates. Let $q': \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}$ be obtained from $q$ by edge reduction, then $q$ is weakly non-negative if and only if so is $q'$. We propose an algorithm to decide if $q$ is weakly non-negative.

Let $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ be an integral quadratic form of the shape

$$q(x) = \sum_{i=1}^{n} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j.$$

We say that $q$ is a unit (resp. semi-unit) form if $q_i = 1$ (resp. $q_i \leq 1$) for every $1 \leq i \leq n$.

Unit forms appear associated to different algebraic structures. In the Representation Theory of Algebras they appear as the Tits and the homological forms of finite dimensional algebras, partially ordered sets and certain classes of bimodules. For those forms, the positive roots ($x \in \mathbb{N}^n$ with $q(x) = 1$) and the positive isotropic vectors are associated to important families of modules. In several situations, the representation type of the algebra is characterized by properties of the associated quadratic forms. For instance: let $A$ be a finite dimensional algebra over an algebraically closed field, if $A$ is representation-finite (resp. tame) algebra, then the Tits form $q_A$ is weakly positive (resp. weakly non-negative). We recall that a quadratic form $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ is weakly positive (resp. weakly non-negative) if $q(x) > 0$ (resp. $q(x) \geq 0$) for every vector $0 \neq x \in \mathbb{N}^n$. See [1, 2, 10, 12] for the consideration of unit forms in Representation Theory.

For the combinatorial study of semi-unit forms several procedures have been introduced. Gabrielov transformations of semi-unit forms have been studied in [3, 4, 5, 8]. In [7] the edge reduction for unit forms was introduced as an adaptation of the algorithm originally formulated for DGC’s and bocses by Roiter and Kleiner. It was shown there that edge reduction provides an algorithm to decide if a given unit form is weakly positive.
The purpose of this work is to study edge reduction of weakly non-negative forms. We show that weak non-negativity is preserved under edge reduction and in that case, the positive corank remains invariant (see Section 1 for definitions). In Section 2 we describe an algorithm deciding whether or not a given semi-unit form is weakly non-negative. A computer implementation of this algorithm is already working.

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1. Edge reductions.

1.1. We consider integral quadratic forms \( q : \mathbb{Z}^n \to \mathbb{Z} \) of the shape

\[
q(x) = \sum_{i=1}^{n} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j ;
\]

where \( q_i \equiv 1 \) for all \( i \), and call such quadratic forms semi-unit forms. If \( q_i \equiv 1 \) for all \( i \), then \( q \) is called unit form.

Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a semi-unit form and \( a \neq b \) be indices such that \( q_{ab} < 0 \). Following [7] we define a new semi-unit form \( q' : \mathbb{Z}^{n+1} \to \mathbb{Z} \) by the formula

\[
q'(y) = q \rho(y) + y_i y_j
\]

where \( \rho : \mathbb{Z}^{n+1} \to \mathbb{Z}^n \) denotes the linear map which maps the canonical basis vector \( e_i \in \mathbb{Z}^{n+1} \) onto \( e_i \in \mathbb{Z}^n \) if \( 1 \leq i \leq n \) and onto \( e_a + e_b \) if \( i = n + 1 \). We say that \( q' \) is obtained from \( q \) by reduction with respect to \( a \) and \( b \).

The quadratic form \( q \) can be recovered from \( q' \) using the non-linear map \( \sigma : \mathbb{Z}^n \to \mathbb{Z}^{n+1} \) defined as follows: \( \sigma(x)_k = x_k \) for \( k \in \{a, b, n+1\} \) and

\[
(\sigma(x)_a, \sigma(x)_b, \sigma(x)_n+1) = \begin{cases} 
(0, x_b - x_a, x_a) & \text{if } x_a \leq x_b , \\
(x_a - x_b, 0, x_b) & \text{if } x_a > x_b .
\end{cases}
\]

Indeed we have \( \rho \sigma = \text{id} \) and \( q(x) = q \rho \sigma(x) = q'(\sigma(x) - \sigma(x)_a \sigma(x)_b) = q' \sigma(x) \) for all \( x \in \mathbb{Z}^n \).

1.2. Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a semi-unit form. We recall that \( q \) is said to be weakly positive (resp. weakly non-negative) if \( 0 < q(x) \) (resp. \( 0 \leq q(x) \)) for all vectors \( 0 \neq x \in \mathbb{N}^n \).

Remark. (a) Let \( q \) be a weakly positive semi-unit form. Then \( q \) is a unit form. Moreover for every pair \( a, b \) such that \( q_{ab} < 0 \), then \( q_{ab} = -1 \).

(b) Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a weakly non-negative semi-unit form. Then \( q_i \in \{0, 1\} \) for every \( 1 \leq i \leq n \). Moreover for every pair \( a, b \) such that \( q_{ab} < 0 \), then \( -2 \leq q_{ab} \).

Let \( q' \) be obtained from \( q \) by an edge reduction as described above. The main result in [7] is the following:

**Theorem.** The unit form \( q \) is weakly positive if and only if \( q' \) is weakly positive. In this case the map \( \rho \) induces a bijection between the set of positive roots \( \Sigma^1(q) = q^{-1}(1) \cap \mathbb{N}^n \) and \( \Sigma^1(q') \).
1.3.

**Proposition.** Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a semi-unit form and \( q' \) be obtained from \( q \) by reduction with respect to \( a \) and \( b \).

Then \( q \) is weakly non-negative if and only if \( q' \) is weakly non-negative form. If this is the case, then \( \sigma \) and \( \rho \) induce inverse bijections between the sets of positive isotropic vectors \( \Sigma^0(q) = q^{-1}(0) \cap \mathbb{N}^n \) of \( q \) and \( \Sigma^0(q') \).

**Proof.** The equivalence of the weak non-negativity of \( q \) and \( q' \) follows from \( \rho(\mathbb{N}^{n+1}) \subseteq \mathbb{N}^n \), \( \sigma(\mathbb{N}^n) \subseteq \mathbb{N}^{n+1} \) and \( q = q' \sigma \). Also, \( \sigma(\Sigma^0(q)) \subseteq \Sigma^0(q') \) is clear. It remains to show that \( \rho(\Sigma^0(q')) \subseteq \Sigma^0(q) \) and \( \sigma \rho(S_{y}(q')) = id_{S_{y}(q)} \) if \( q \) is weakly non-negative. Indeed, the inequality \( 0 = q'(y) = q\rho(y) + y_a y_b \geq y_a y_b \geq 0 \) for \( y \in \Sigma^0(q') \) implies that \( y_a y_b = 0 \) and therefore \( \rho(y) \in \Sigma^0(q) \) and \( \sigma \rho(y) = y \). \( \square \)

Denote by \( q(x,y) = q(x+y) - q(x) - q(y) \in \mathbb{Z} \) the value of the symmetric bilinear form associated with \( q \). Then the associated bilinear form of \( q' \) looks as follows:

\[
q'(x, y) = q(\rho(x), \rho(y)) + x_a y_b + x_b y_a.
\]

We shall consider orthogonality with respect to these bilinear forms.

**Corollary.** If \( q \) is weakly non-negative and \( x, y \in \Sigma^0(q') \) are orthogonal, then we have \( x_a = y_a = 0 \) or \( x_b = y_b = 0 \). In particular, \( \rho(x), \rho(y) \in \Sigma^0(q) \) are orthogonal.

**Proof.** The orthogonality of \( x, y \in \Sigma^0(q') \) yields \( x + y \in \Sigma^0(q) \), and \( \rho(\Sigma^0(q')) \subseteq \Sigma^0(q) \) implies that \( x_a x_b = y_a y_b = 0 \) and \( 0 = (x + y)_a (x + y)_b = x_a y_b + x_b y_a \). Thus, if for instance \( x_a \neq 0 \) then necessarily \( x_b = 0 \) and \( y_b = 0 \). \( \square \)

1.4. Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a weakly non-negative semi-unit form. Following [11] we define the **positive corank** of \( q \) to be the maximal rank of an isotropic sublattice of \( \mathbb{Z}^n \) generated by elements of \( \mathbb{N}^n \).

Given two indices \( a \neq b \), we say that a subset \( \Sigma \subseteq \mathbb{N}^n \) is **conformal** with respect to \( a \) and \( b \) if \( (x_a - x_b)(y_a - y_b) \geq 0 \) for all \( x, y \in \Sigma \).

**Remarks.** (a) If \( \Sigma \) is conformal and \( x, y \in \Sigma \), then \( \sigma(x + y) = \sigma(x) + \sigma(y) \).

(b) If \( \Sigma \) is a linear independent subset of \( \mathbb{N}^n \) and \( x \in \Sigma \) is such that \( |x_a - x_b| \) is maximal then clearly the set \( x + \Sigma \) is linear independent and conformal.

(c) It follows from (b) that the positive corank of \( q \) is the maximal number of pairwise orthogonal vectors in \( \Sigma^0(q) \) forming a linear independent and conformal set.

(d) Let \( q \) be a weakly non-negative semi-unit form which accepts a sincere vector \( v \in \Sigma^0(q) \) (that is, \( v(i) = 0 \) for all \( 1 \leq i \leq n \)). Then \( q \) is non-negative.

**Proposition.** The positive corank of a weakly non-negative semi-unit form is preserved under reduction.

**Proof.** Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be weakly non-negative and \( q' \) be obtained from \( q \) by reduction with respect to \( a \) and \( b \).

If \( \Sigma \subseteq \Sigma^0(q) \) is a linear independent and conformal set of pairwise orthogonal, positive isotropic vectors, then \( \sigma(\Sigma) \) is linear independent since \( \rho \) is a linear retraction of \( \sigma \), and the
elements of $\sigma(\Sigma)$ are pairwise orthogonal since $\sigma$ is additive on $\Sigma$ and $q'\sigma = q$. It follows that $\text{corank}^+(q) \leq \text{corank}^+(q')$.

Conversely, consider a linear independent set $\Sigma' \subseteq \Sigma^0(q')$ of pairwise orthogonal, positive isotropic vectors of $q'$. By Corollary 1.3, the elements of $\rho(\Sigma')$ are pairwise orthogonal and, moreover, we have either $x_a = 0$ for all $x \in \Sigma'$ or $x_b = 0$ for all $x \in \Sigma'$. This shows that $\rho(\Sigma')$ is also linear independent since $\text{Ker} \rho = \mathbb{Z}(e_{n+1} - e_a - e_b)$. Thus, $\text{corank}^+(q') \leq \text{corank}^+(q)$. □

1.5. Let $q: \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form. A vector $y \in \mathbb{N}^n$ is primitive if it is not the sum $y = y' + y''$ of vectors $y', y'' \in \mathbb{N}^n$. We call a primitive non-zero vector $y \in \Sigma^0(q)$ critical if the support $\text{supp} y := \{ i \mid y_i > 0 \}$ is minimal with respect to the inclusion order and if $q_i \equiv 0$ and $q_{ij} \equiv -2$ for all $i, j \in \text{supp} y$; the set of all critical vectors of $q$ will be denoted by $\mathcal{C}_q$. By [6], for any $y \in \mathcal{C}_q$, the restriction $q|_y$ of $q$ to $\text{supp} y$ is non-negative and we have $\{ x \in \Sigma^0(q) \mid \text{supp} x = \text{supp} y \} = \mathbb{N}y$. The critical vectors $y$ of length $\sum_{i=1}^n y_i \leq 2$ are the vectors $e_i$ where $q_i = 0$, and $e_i + e_j$ where $q_i = q_j = 1$ and $q_{ij} = -2$; the critical vectors of length $> 2$ are the positive radical generators of the critical unit forms classified in [5]. The following lemma carries easily over from [11, 1.5] (see also [1]) and shows the relevance of the critical vectors.

**Lemma.** Let $q$ be a weakly non-negative semi-unit form. Then:

(a) For every $x \in \Sigma^0(q)$, we have $x = \sum_{i=1}^r \lambda_i v_i$, where $v_1, \ldots, v_r$ are orthogonal vectors in $\mathcal{C}_q$ and $\lambda_1, \ldots, \lambda_r$ are positive rational numbers.

(b) Let $v_1, \ldots, v_s$ be a maximal linearly independent set of orthogonal vectors in $\mathcal{C}_q$. Then $\text{corank}^+(q) = s$.

1.6. Now consider a semi-unit form $q'$ which is obtained from $q$ by reduction with respect to a pair $\{a, b\}$. We want to describe the set of critical vectors $\mathcal{C}_{q'}$ of $q'$ in terms of $\mathcal{C}_q$. For this purpose, denote by $\mathcal{N} = \mathcal{N}(q, a, b)$ the set of primitive vectors $z \in \mathbb{N}^n$ of the form $z = \lambda x + \mu y$ where $x, y \in \mathcal{C}_q$ are orthogonal and non-conformal (w.r.t $a$ and $b$) and $\lambda, \mu$ are positive rational numbers such that $z_a = z_b$.

**Proposition.** With the above notations we have:

a) $\sigma(\mathcal{C}_q) \subseteq \mathcal{C}_q$.

Under the assumption that $q$ is weakly non-negative we have additionally:

b) $\mathcal{C}_{q'} \subseteq \sigma(\mathcal{C}_q) \cup \sigma(\mathcal{N})$.

c) If $\text{corank}^+(q) = 1$ then $\mathcal{C}_{q'} = \sigma(\mathcal{C}_q)$.

d) If $\text{corank}^+(q) = 2$ then $\mathcal{C}_{q'} = \sigma(\mathcal{C}_q) \cup \sigma(\mathcal{N})$.

**Proof.** a) Let $y \in \mathcal{C}_q$. If $\{a, b\} \subseteq \text{supp} y$ then $\sigma(y) \in \mathcal{C}_{q'}$ since $\sigma(y) = y$ and $q'|_y = q|_y$. Now assume that $a, b \in \text{supp} y$. We then have $\{ z \in \Sigma^0(q') \mid \text{supp} z \subseteq \text{supp} \sigma(y) \cup \{a, b\} \} = \sigma(\{ x \in \Sigma^0(q) \mid \text{supp} x \subseteq \text{supp} y \}) = \sigma(\mathbb{N}y) = \mathbb{N}\sigma(y)$; the first equality holds by (1.3) since $q|_y$ is (weakly) non-negative, and the second since $y$ is critical. It follows that $\sigma(y)$ is critical.
b) Let \( y \in \Sigma^0(q') \) be a critical vector. By (1.3) we have \( \rho(y) \in \Sigma^0(q) \), and by the above lemma there are pairwise orthogonal and different vectors \( x^1, \ldots, x^r \in C_q \) and positive rational numbers \( \lambda_1, \ldots, \lambda_r \) such that \( \rho(y) = \sum_{i=1}^{r} \lambda_i x^i \). If \( x^1, \ldots, x^r \) are conformal (w.r.t. \( a \) and \( b \)), then \( y = \sigma\rho(y) = \sigma\left( \sum_{i=1}^{r} \lambda_i x^i \right) = \sum_{i=1}^{r} \lambda_i \sigma(x^i) \) is a positive linear combination of primitive vectors in \( \Sigma^0(q') \) and therefore \( r = 1 \) and \( \lambda_1 = 1 \), thus \( y \in \sigma(C_q) \). So we can assume that \( x^1, \ldots, x^r \) are not conformal, say \( x^1_a > x^1_b \) and \( x^2_a < x^2_b \).

Choose positive rational numbers \( \mu_1, \mu_2 \) such that the vector \( z = \mu_1 x^1 + \mu_2 x^2 \) belongs to \( \mathbb{N}^n \), is primitive and satisfies \( z_a = z_b \). Then \( \sigma(x) \in \Sigma^0(q') \) is primitive, and \( \text{supp} \sigma(z) \subseteq \text{supp} y \). Since \( y \) is critical it follows that \( y = \sigma(z) \), that is, \( y \in \sigma(C) \).

c) Follows from a) and b), since \( C = \emptyset \) if corank \( ^+ q = 1 \).

d) We have to show that \( \sigma(C) \subseteq C_{q'} \) if corank \( ^+ q = 1 \). Let \( z \) be an element of \( C \), that is, \( z \) is a primitive vector in \( \mathbb{N}^n \) of the form \( z = \lambda x + \mu y \) where \( x, y \in C_q \) are orthogonal and conformal (w.r.t. \( a \) and \( b \)), and \( \lambda_1, \mu \) are positive rational numbers such that \( z_a = z_b \). In particular, we have \( \sigma(z)_a = \sigma(z)_b = 0 \). It follows from (1.3) that \( \sigma(x), \sigma(y) \in \Sigma^0(q') \) are orthogonal and that \( \text{supp} \sigma(z) \subseteq I := \text{supp} \sigma(y) \cup \text{supp} \sigma(x) \). Thus the restriction \( q' \) is non-negative and corank \( ^{+} q' \mid_{\sigma(z)} < \text{corank} \) \( ^{+} q' \subseteq \text{corank} \) \( ^{+} q' = \text{corank} \) \( ^{+} q = 2 \). We infer that \( q' \mid_{\sigma(z)} \) is non-negative of positive corank 1 and therefore \( \sigma(z) \in C_{q'} \).

**Corollary.** Under the assumption of the above proposition. If \( q \) is weakly non-negative and \( C = \emptyset \), then \( C_{q'} = \sigma(C_q) \).

**Example.** Given a semi-unit form \( q: \mathbb{Z}^n \to \mathbb{Z} \) we draw a bigraph \( G_q \) with \( n \) vertices, and for two vertices \( i \neq j \) we joint \( i \) and \( j \) by \( -q_{ij} \) full edges if \( q_{ij} < 0 \) and by \( q_{ij} \) dotted edges in case \( q_{ij} > 0 \); there are \( 1 - q_i \) full loops at the vertex \( i \).

Consider the bigraphs of the unit forms \( q \) and \( q' \) as follows:

![Diagram](image)

Then we get:

\[
C_q = \left\{ \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \right\} \quad C_{q'} = \left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \right\}
\]

\[N(q, a, b) = \left\{ \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix} \right\}.\]
2. Exhaustive reductions.

2.1. Recall from [7] that a unit form is weakly positive if and only if an arbitrary sequence of iterated reductions stops after finitely many steps at some unit form which admits no further reduction. An analogous characterization of weak non-negativity for semi-unit forms cannot be expected since there can occur self-reproductions. For instance the reduction of the unit form \( q : \mathbb{Z}^3 \to \mathbb{Z} \) (shown below) with respect to \( \{2, 3\} \) yields the unit form \( q' \), which contains \( q \) as a restriction (up to permutation of the variables).

\[
\begin{align*}
q : & \quad 1 \to 2 \\
& \quad 3 \\
q' : & \quad 1 \to 4 \\
& \quad 3
\end{align*}
\]

We say that an iterated reduction of a semi-unit form \( q : \mathbb{Z}^n \to \mathbb{Z} \) is exhaustive if every reduction step only involves reductions with respect to indices \( \leq n \) and the resulting semi-unit form \( q' \) satisfies \( q'_{ij} \geq 0 \) for all \( 1 \leq i < j \leq n \). Clearly, such an exhaustive (iterated) reduction of \( q \) involves \( \sum_{q_{ij} < 0} q_{ij} \) (ordinary) reduction steps, but the resulting semi-unit form is far away from being unique.

For instance, the unit form \( q \) in the above example admits the following two exhaustive reductions, which are obtained by reductions with respect to the sequences \( \{1, 2\}, \{1, 2\}, \{2, 3\} \) and \( \{2, 3\}, \{1, 2\}, \{1, 2\} \), respectively.

Let \( q' : \mathbb{Z}^m \to \mathbb{Z} \) be obtained from \( q : \mathbb{Z}^n \to \mathbb{Z} \) by \( m - n \) consecutive reductions, then we denote by \( \sigma^{m-n} : \mathbb{Z}^n \to \mathbb{Z}^m \) the composition of the functions \( \sigma \) corresponding to the successive reductions.

**Lemma.** Let \( q : \mathbb{Z}^n \to \mathbb{Z} \) be a semi-unit form and let \( q' : \mathbb{Z}^m \to \mathbb{Z} \) be obtained from \( q \) by an exhaustive reduction.

a) If \( x \in \Sigma^0(q) \) satisfies \( q_i \geq 0 \) for all \( i \in \text{supp} x \), then we have either \( q_i = q_{ij} = 0 \) for all \( i < j \in \text{supp} x \) and \( \sigma^{m-n}(x) \) is identified with \( x \), or \( |\sigma^{m-n}(x)| < |x| = \sum x_i \).

b) If \( x \in \Sigma^0(q) \) is critical and \( |x| \geq 2 \), then \( |\sigma^{m-n}(x)| < |x| \).

**Proof.** Assertion b) is an immediate consequence of a). To prove a) let \( x \in \Sigma^0(q) \) be such that \( q_i \geq 0 \) for all \( i \in \text{supp} x \). We then have \( \sum_{i < j} q_{ij} x_i x_j \leq 0 \), and if \( q_{ij} = 0 \) for all \( i < j \in \text{supp} x \) then \( q_i = 0 \) for all \( i \in \text{supp} x \). On the other hand, if \( q_{ij} < 0 \) for some \( i < j \in \text{supp} x \) and \( \{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{m-n}, j_{m-n}\} \) denotes the sequence of pairs with respect to which \( q' \) is
obtained from $q$ by reductions, then there is a smallest index $\equiv m - n$ such that $i, j \in \text{supp} \chi$, and then $|\sigma^{m-n}(x)| \leq |\sigma'(x)| < |x|$. □

2.2. Now consider a semi-unit form $q: \mathbb{Z}^n \to \mathbb{Z}$ and a sequence $q^{(0)}, q^{(1)}, q^{(2)}, \ldots$ of semi-unit forms obtained from $q$ by iterated exhaustive reductions, that is, $q^{(0)} = q$ and, for $k > 0$, $q^{(k)}$ is obtained from $q^{(k-1)}$ by an exhaustive reduction. If $n_k$ denotes the number or variables of $q^{(k)}$ we set $\sigma^{(k)} = \sigma^{m-n}: \mathbb{Z}^n \to \mathbb{Z}^{n_k}$.

**Corollary 1.** If $q$ is weakly non-negative, then the following holds.

a) For every $x \in \Sigma^0(q)$ there is an $r < |x|$ such that $q^{(r)}_{ij} = 0$ for all $i < j \in \text{supp} \sigma^{(r)}(x)$.

b) For every $x \in \mathcal{C}_q$ there is an $r < \max \{2n - 4, 30\}$ such that $\sigma^{(r)}(x) = e_i$ for some $1 \leq i \leq n_r$.

**Proof.** Follows from the above lemma, $\sigma^{(k)}(\mathcal{C}_q) \subseteq \mathcal{C}_{q^{(k)}}$ (1.6), and $|x| \leq \max \{2n - 4, 30\}$ for all $x \in \mathcal{C}_q$. □

**Corollary 2.** If $q$ is weakly non-negative of positive corank 1, then there is an $r$ such that $q^{(r)}_{ij} \geq 0$ for all $1 \leq i < j \leq n_r$. In particular, $\Sigma^0(q) = \bigcup_{i \in I} \mathbb{N} \rho^{n_i-n}(e_i)$ where $I = \{i \leq n_r \mid q^{(r)}_{ij} = 0\}$.

**Proof.** By (1.6.c) we have that $\mathcal{C}_{q^{(i)}} = \sigma^{(s)}(\mathcal{C}_q)$ for all $s$, and by Corollary 1 there is an $s$ such that $\mathcal{C}_{q^{(s)}} = \{e_i \mid i \in I\}$ where $I = \{i \leq n_s \mid q^{(s)}_{ij} = 0\}$. Since $q^{(s)}$ is weakly non-negative, we have $q^{(s)}_{ij} \geq 0$ whenever $i \in I$ or $j \in I$, and the restriction of $q^{(s)}$ to $\{1, \ldots, n_s\} \setminus I$ is weakly positive. It follows that there is an $r \geq s$ such that $q^{(r)}_{ij} \geq 0$ for all $i < j \leq n_r$. Finally, we have $\Sigma^0(q) = \bigcup_{x \in \mathcal{C}_q} \mathbb{N} x$ (1.6) and $\mathcal{C}_q = \rho^{n_i-n}(\mathcal{C}_{q^{(s)}}) = \rho^{n_i-n}(\mathcal{C}_{q^{(r)}})$.

2.3. It would be interesting to know whether or not an arbitrary semi-unit form $q$ is weakly non-negative if and only if any sequence of iterated exhaustive reductions of $q$ stops after finitely many steps at some semi-unit form having only non-negative coefficients. We do not know if this is the case. However, we have the following criterion.

**Theorem.** Let $q: \mathbb{Z}^n \to \mathbb{Z}$ be a semi-unit form and $q^{(k)}: \mathbb{Z}^{n_k} \to \mathbb{Z}$, $k = 0, 1, 2, \ldots$, be a sequence of semi-unit forms obtained by iterated exhaustive reductions of $q$. Then $q$ is weakly non-negative if and only if $q^{(k)}_{ij} \geq 0$ for all $k \leq 31$ and $i \leq n_k$.

**Proof.** The necessity is clear by (1.3). To prove the converse assume that $q$ is not weakly non-negative. We can assume that $q_i \equiv 0$ for all $i \leq n$. If $q_{ij} < -2$ for some $i < j$ then $|\sigma^{m-n}(e_i + e_j)| = 1$ and $q^{(1)}_{ij} = q(e_i + e_j) = q_i + q_j + q_{ij} < 0$; so we can assume further on that $q_{ij} \equiv -2$ for all $i < j$.

It follows that there is a critical vector $y \in \mathcal{C}_q$ of length $\leq 30$ and an index $i \notin \text{supp} y$ such that $q(2y + e_i) < 0$, see [5, 10].

Now consider a semi-unit form $q': \mathbb{Z}^{n+1} \to \mathbb{Z}$ obtained from $q$ by reduction, say with respect to $a$ and $b$. We claim that $q'(2\sigma(y) + e_j) < 0$ for some $j \notin \text{supp} \sigma(y)$. Indeed, if $a = i$ and $b \in \text{supp} y$, then $q'(2\sigma(y) + e_{n+1}) = q\rho(2\sigma(y) + e_{n+1}) = q(2y + e_i + e_b) \leq q(2y + e_i)$,
since $q(y, e_b) = 0$, $q_b \leq 1$ and $q(e_i, e_b) = q_{ib} < 0$. On the other hand, if $i \notin \{a, b\} \cap \text{supp } y = \emptyset$, one easily checks that $q'(2\sigma(y) + e_i) = q(2y + e_i)$. Applying this iteratively we obtain $q^{(1)}(y) < 0$ for some $j_1 \notin \text{supp } q^{(1)}(y)$, where $q^{(1)}(y) \in \mathcal{C}_{q^{(1)}}$ and $|q^{(1)}(y)| < |y|$ (1.6 and 2.2).

Proceeding in this way iteratively, it follows that there is an $r \leq 29$ such that $q^{(r)}(y) = e_k \in \mathcal{C}_{q^{(r)}}$ and $q^{(r)}(2e_k + e_{j_r}) < 0$ for some $j_r \neq k$. If $q^{(r)}_{j_r} < 0$ we are done. Otherwise, if $q^{(r)}_{j_r} \geq 0$ we have $q^{(r)}_{kr} < 0 = q^{(r)}_{k}$, and one easily verifies that there is an $\ell \leq 2$ such that $q^{(r)}_{\ell} < 0$ for some $\ell \leq n_{r+1}$.

Remarks. (1) The bound 31 in the above theorem could be replaced by the number $b + 2$ where $b \in \mathbb{N}$ denotes the minimal number such that, for every critical unit form $q$ of type $\bar{A}_m$, $1 \leq m \leq 7$, $\bar{D}_m$, $4 \leq m \leq 8$, or $\bar{E}_m$, $6 \leq m \leq 8$, and for every sequence $q^{(0)}, q^{(1)}, q^{(2)}, \ldots$ obtained by iterated exhaustive reductions of $q$, the image $q^{(k)}(y)$ of the positive radical generator $y$ of $q$ has length 1 for some $k \leq b$.

(2) The above algorithm has been implemented as part of the CREP system.

References


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