SYMMETRIC SEMICLASSICAL STATES TO A MAGNETIC NONLINEAR SCHRÖDINGER EQUATION VIA EQUIVARIANT MORSE THEORY

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ABSTRACT. We consider the magnetic NLS equation

$$\left(-\varepsilon i\nabla + A(x)\right)^2 u + V(x)u = K(x)\left|u\right|^{p-2} u, \qquad x \in \mathbb{R}^N,$$

where $N \geq 3$, $2 < p < 2^* := 2N/(N-2)$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential and $V : \mathbb{R}^N \to \mathbb{R}$, $K : \mathbb{R}^N \to \mathbb{R}$ are bounded positive potentials. We consider a group G of orthogonal transformations of $\mathbb{R}^N$ and we assume that A is G-equivariant and V, K are G-invariant. Given a group homomorphism $\tau : G \to \mathbb{S}^1$ into the unit complex numbers we look for semiclassical solutions $u_\varepsilon : \mathbb{R}^N \to \mathbb{C}$ to the above equation which satisfy

$$u_\varepsilon(gx) = \tau(g)u_\varepsilon(x)$$

for all $g \in G$, $x \in \mathbb{R}^N$. Using equivariant Morse theory we obtain a lower bound for the number of solutions of this type.

1. Introduction

Let G be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$. We assume $A : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1-function and $V, K : \mathbb{R}^N \to \mathbb{R}$ are bounded C^2-functions with $\inf_{\mathbb{R}^N} V > 0$ and $\inf_{\mathbb{R}^N} K > 0$ which satisfy

$$A(gx) = gA(x) \quad \forall g \in G,\ x \in \mathbb{R}^N,$$

$$V(gx) = V(x), \quad K(gx) = K(x) \quad \forall g \in G,\ x \in \mathbb{R}^N. \tag{1.1}$$

Given a group homomorphism $\tau : G \to \mathbb{S}^1$ into the unit complex numbers we look for solutions $u : \mathbb{R}^N \to \mathbb{C}$ to the problem

$$(\wp_\varepsilon) \qquad \begin{cases} \left(-\varepsilon i\nabla + A\right)^2 u + Vu = K\left|u\right|^{p-2} u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \varepsilon\nabla u + iAu \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

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which satisfy

$$u(gx) = \tau(g)u(x) \quad \text{for all } g \in G, \ x \in \mathbb{R}^N. \tag{1.2}$$

That is, the absolute value $|u|$ of u is G-invariant and the phase of $u(gx)$ is that of $u(x)$ multiplied by the phase factor $\tau(g)$. Functions satisfying (1.2) are called *τ-intertwining.*

This paper is concerned with semiclassical τ-intertwining solutions to problem $(\wp_\varepsilon)$, i.e. solutions to $(\wp_\varepsilon)$ satisfying (1.2) for $\varepsilon > 0$ small enough.

The study of semiclassical phenomena for equation $(\wp_\varepsilon)$ in the non-equivariant case has been extensively pursued in recent years. We refer e.g. to [20, 27, 17, 1, 23, 3] for the nonmagnetic case $A = 0$, and to [22, 9, 12, 13, 4, 11] for the magnetic case $A \neq 0$. See also [19, 2, 26, 7, 14, 16] for existence results for fixed ε. Here we are interested in investigating how the competing potentials V and K interact among themselves and with the homomorphism τ to produce τ-intertwining semiclassical states of $(\wp_\varepsilon)$. In a recent paper [10] we dealt with the case $K \equiv 1$ and showed that there is a combined effect of the symmetries and the electric potential V on the number of semiclassical τ-intertwining solutions to $(\wp_\varepsilon)$. More precisely, we showed that the Lusternik-Schnirelmann category of the G-orbit space of a suitable set M_τ, depending on V and τ, furnishes a lower bound on the number of solutions of this type. See also [15] for a multiplicity result of G-invariant solutions in the nonmagnetic case. In this work we shall apply equivariant Morse theory. Equivariant Morse theory provides information on the local behavior of a functional around a critical orbit. It also provides, in many cases, better multiplicity results than those given by Lusternik-Schnirelmann category, see Example 1 below.

In order to state our main result we briefly recall some definitions and introduce some notation. A detailed discussion will be given in the following sections. Set $\nabla_{\varepsilon,A}u := \varepsilon\nabla u + iAu$ and consider the magnetic Sobolev space $H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau$ consisting of all functions $u \in L^2(\mathbb{R}^N, \mathbb{C})$ such that $\nabla_{\varepsilon,A}u \in L^2(\mathbb{R}^N, \mathbb{C}^N)$ and u is τ-intertwining. The nontrivial solutions of $(\wp_\varepsilon)$ which satisfy (1.2) are the critical points of the C^2-functional $J_\varepsilon : \mathcal{N}^\tau_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{2}\int \left(|\nabla_{\varepsilon,A}u|^2 + V(x)\,|u|^2\right) - \frac{1}{p}\int K(x)\,|u|^p\,,$$

where

$$\mathcal{N}^\tau_\varepsilon := \{u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau \smallsetminus \{0\} : \ \int \left(|\nabla_{\varepsilon,A}u|^2 + V(x)\,|u|^2\right) = \int K(x)\,|u|^p\}$$

is the Nehari manifold. This is a complete C^2-Hilbert manifold. The group $\mathbb{S}^1$ of unit complex numbers acts on it by multiplication and J_ε is invariant under this action, i.e. $\gamma u \in \mathcal{N}_\varepsilon^\tau$ and $J_\varepsilon(\gamma u) = J_\varepsilon(u)$ for all $u \in \mathcal{N}_\varepsilon^\tau$ and $\gamma \in \mathbb{S}^1$. Therefore, if u is a critical point of J_ε on $\mathcal{N}_\varepsilon^\tau$ then so is γu for every $\gamma \in \mathbb{S}^1$. The set $\mathbb{S}^1 u := \{\gamma u : \gamma \in \mathbb{S}^1\}$ is then called a *τ-intertwining critical $\mathbb{S}^1$-orbit of* J_ε. Two solutions of $(\wp_\varepsilon)$ are said to be *geometrically different* if their $\mathbb{S}^1$-orbits are different.

Let $\mathcal{H}^*$ be Alexander-Spanier cohomology with coefficients in $\mathbb{K}$. If $A \subset X$ are $\mathbb{S}^1$-invariant subsets of $\mathcal{N}_\varepsilon^\tau$ we denote by $X/\mathbb{S}^1$ and $A/\mathbb{S}^1$ their $\mathbb{S}^1$-orbit spaces and write

$$\mathcal{H}^*_{\mathbb{S}^1}(X, A) := \mathcal{H}^*(X/\mathbb{S}^1, A/\mathbb{S}^1).$$

If $\mathbb{S}^1 u$ is an isolated critical $\mathbb{S}^1$-orbit of J_ε its *k-th critical group* is defined as

$$C^k_{\mathbb{S}^1}(J_\varepsilon, \mathbb{S}^1 u) := \mathcal{H}^k_{\mathbb{S}^1}(J_\varepsilon^c \cap U, (J_\varepsilon^c \smallsetminus \mathbb{S}^1 u) \cap U),$$

where U is an $\mathbb{S}^1$-invariant neighborhood of $\mathbb{S}^1 u$ in $\mathcal{N}_\varepsilon^\tau$, $c := J_\varepsilon(u)$ and $J_\varepsilon^c := \{u \in \mathcal{N}_\varepsilon^\tau : J_\varepsilon(u) \leq c\}$. Its total dimension

$$\mu(J_\varepsilon, \mathbb{S}^1 u) := \sum_{k=0}^{\infty} \dim C^k_\Gamma(J_\varepsilon, \mathbb{S}^1 u)$$

is called the *multiplicity of* $\mathbb{S}^1 u$. If $\mathbb{S}^1 u$ is nondegenerate and J_ε satisfies the Palais-Smale condition in some neighborhood of c, then $\dim C^k_\Gamma(J_\varepsilon, \mathbb{S}^1 u) = 1$ if k is the Morse index of J_ε at the critical submanifold $\mathbb{S}^1 u$ of $\mathcal{N}_\varepsilon^\tau$ and it is 0 otherwise.

Let $q := \frac{p}{p-2} - \frac{N}{2}$ and

$$\ell_G := \min_{x \in \mathbb{R}^N} (\#Gx) \frac{V^q(x)}{K^{2/(p-2)}(x)},$$

and consider the set

$$M_\tau := \{x \in \mathbb{R}^N : (\#Gx) \frac{V^q(x)}{K^{2/(p-2)}(x)} = \ell_G,\ G_x \subset \ker \tau\}.$$

Here $Gx := \{gx : g \in G\}$ is the G-orbit of the point $x \in \mathbb{R}^N$, $\#Gx$ is its cardinality, and $G_x := \{g \in G : gx = x\}$ is its isotropy subgroup. M_τ is the middle ground between the valleys of V and the peaks of K weighted by the size of the G-orbits. Observe that the points in M_τ are not necessarily local minima of V or local maxima of K. Assumption (1.1) implies that M_τ is G-invariant. For $\rho > 0$ we set

$$B_\rho M_\tau := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M_\tau) \leq \rho\}$$

and write $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ for the embedding of the G-orbit space of M_τ in that of $B_\rho M_\tau$. We will show that this embedding has an effect on the number of solutions of $(\wp_\varepsilon)$ for ε small enough.

Let $c_{\mathbb{R}^N}$ denote the least energy of a nontrivial solution to the real-valued problem

$$\left\{ \begin{array}{c} -\Delta u + u = |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N, \mathbb{R}). \end{array} \right. \tag{1.3}$$

We shall prove the following.

Theorem 1.1. *Assume there exists $\alpha > 0$ such that the set*

$$\{x \in \mathbb{R}^N : (\#Gx)\frac{V^q(x)}{K^{2/(p-2)}(x)} \leq \ell_G + \alpha\} \tag{1.4}$$

is compact. Then, given $\rho > 0$ and $\delta \in (0, \alpha)$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon > 0$ one of the following two assertions holds:
(a) J_ε has a nonisolated τ-intertwining critical $\mathbb{S}^1$-orbit in $J_\varepsilon^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$.
(b) J_ε has finitely many τ-intertwining critical $\mathbb{S}^1$-orbits $\mathbb{S}^1 u_1, ..., \mathbb{S}^1 u_m$ in $J_\varepsilon^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$. They satisfy

$$\sum_{j=1}^{m} C^k_{\mathbb{S}^1}(J_\varepsilon, \mathbb{S}^1 u_j) \geq \operatorname{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G))$$

for every $k \geq 0$.

In particular, if every τ-intertwining critical $\mathbb{S}^1$-orbit of J_ε in $J_\varepsilon^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$ is nondegenerate then, for every $k \geq 0$, there are at least

$$\operatorname{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G))$$

of them having Morse index k for every $k \geq 0$.

An immediate consequence of this result is the following.

Corollary 1.2. *If assumption* (1.4) *holds then, given $\rho > 0$ and $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon > 0$ problem $(\wp_\varepsilon)$ has at least*

$$\sum_{k=0}^{\infty} \operatorname{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G))$$

geometrically different solutions in $J_\varepsilon^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$, counted with their multiplicity.

Symmetries often occur in this type of problems. As an example we consider the standard magnetic field $B(x_1, x_2, x_3) := (0, 0, 2)$ in $\mathbb{R}^3$ and the standard magnetic potential $A(x_1, x_2, x_3) := (-x_2, x_1, 0)$ associated to it. It is convenient to identify $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ and to write $A(z, t) = (iz, 0)$, $z = x_1 + ix_2$. Note that A satisfies $A(e^{i\theta}z, t) = e^{i\theta}A(z, t)$ for every $\theta \in \mathbb{R}$. Given $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we look for solutions to problem $(\wp_\varepsilon)$ which satisfy

$$u(e^{2\pi ik/m}z, t) = e^{2\pi ink/m}u(z, t) \quad \forall k = 1, ..., m, \ x \in \mathbb{R}^N. \tag{1.5}$$

Solutions of this type are produced in a natural way by gauge invariance in some problems where the magnetic potential is singular, see [16]. We assume that V and K satisfy:

(S$_1$) $V, K : \mathbb{R}^N \to \mathbb{R}$ are bounded continuous functions such that

$$0 < \inf_{\mathbb{R}^3} V < \liminf_{|x|\to\infty} V(x), \qquad 0 < \inf_{\mathbb{R}^3} K, \qquad \limsup_{|x|\to\infty} K < \sup_{\mathbb{R}^3} K.$$

(S$_2$) There exists $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} m_0 \inf_{x\in\mathbb{R}^3} \frac{V^q(x)}{K^{2/(p-2)}(x)} &< \inf_{t\in\mathbb{R}} \frac{V^q(0,t)}{K^{2/(p-2)}(0,t)}, \\ V(e^{2\pi ik/m_0}z, t) &= V(z, t) \quad \text{and} \\ K(e^{2\pi ik/m_0}z, t) &= K(z, t) \quad \forall k = 1, ..., m_0, \ (z, t) \in \mathbb{C} \times \mathbb{R}. \end{aligned}$$

For each $m \in \mathbb{N}$ which divides m_0 (writen $m|m_0$) we consider the group $G_m := \{e^{2\pi ik/m} : k = 1, ..., m\}$ acting by multiplication on the z-coordinate of each point $(z, t) \in \mathbb{C} \times \mathbb{R}$. Then A, V and K satisfy the assumptions at the beginning of this section for each $G = G_m$ and assumption (1.4) as well. For any homomorphism $\tau : G_m \to \mathbb{S}^1$ we have that $M_\tau = M$ where

$$M := \left\{ x \in \mathbb{R}^N : \frac{V^q(x)}{K^{2/(p-2)}(x)} = \inf_{x\in\mathbb{R}^3} \frac{V^q(x)}{K^{2/(p-2)}(x)} \right\}.$$

Given $n \in \mathbb{Z}$ we consider the homomorphism $\tau(e^{2\pi ik/m}) := e^{2\pi ink/m}$. Then u is τ-intertwining if it satisfies (1.5). Theorem 1.1 applies. In particular, given $\rho, \delta > 0$, Corollary 1.2 asserts that there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ problem $(\wp_\varepsilon)$ has at least

$$\sum_{k=0}^{\infty} \operatorname{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M/G_m) \to \mathcal{H}^k(M/G_m))$$

geometrically distinct solutions, counted with their multiplicity, which satisfy (1.5) and

$$\left| \frac{p-2}{2p} \int |u|^p - \varepsilon^N m \inf_{x\in\mathbb{R}^3} \frac{V^q(x)}{K^{2/(p-2)}(x)} c_{\mathbb{R}^N} \right| < \varepsilon^N \delta.$$

So altogether, for ε small enough, problem $(\wp_\varepsilon)$ has at least

$$\sum_{m|m_0} \sum_{k=0}^{\infty} m \operatorname{rank}(i_\rho^* : \mathcal{H}^k(B_\rho M/G_m) \to \mathcal{H}^k(M/G_m))$$

geometrically distinct solutions with

$$\frac{p-2}{2p}\int |u|^p < \varepsilon^N \left(m_0 \inf_{x\in\mathbb{R}^3} \frac{V^q(x)}{K^{2/(p-2)}(x)} c_{\mathbb{R}^N} + \delta \right),$$

counted with their multiplicity.

This paper is organized as follows. In Section 2 we state the results from Morse theory that we need for our main results. In Section 3 we discuss the variational problem. Sections 4 and 5 are devoted to the construction of maps which will help us estimate the cohomological dimension of an appropriate sublevel set of J_ε. Finally, in Section 6 we prove our main result.

2. A brief review on equivariant Morse theory

We start by reviewing some well known facts on equivariant Morse theory. We refer the reader to [8, 28] for further details.

Let Γ be a compact Lie group and X be a Γ-space. The Γ-orbit of a point $x \in X$ is the set $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$. A subset A of X is said to be Γ-invariant if $\Gamma x \subset A$ for every $x \in A$. The Γ-orbit space of A is the set $A/\Gamma := \{\Gamma x : x \in A\}$ with the quotient space topology. X is called a free Γ-space if $\gamma x \neq x$ for every $\gamma \in \Gamma$, $x \in X$. A map $f : X \to Y$ between Γ-spaces is called Γ-invariant if f is constant on each Γ-orbit of X, and it is called Γ-equivariant if $f(\gamma x) = \gamma f(x)$ for every $\gamma \in \Gamma$, $x \in X$.

We fix a field $\mathbb{K}$ and denote by $\mathcal{H}^*(X, A)$ the Alexander-Spanier cohomology of the pair (X, A) with coefficients in $\mathbb{K}$. If X is a Γ-pair, i.e. if X is a Γ-space and A is a Γ-invariant subset of X, we write

$$\mathcal{H}^*_\Gamma(X, A) := \mathcal{H}^*(E\Gamma \times_\Gamma X, E\Gamma \times_\Gamma A)$$

for the Borel-cohomology that pair. $E\Gamma$ is the total space of the classifying Γ-bundle and $E\Gamma \times_\Gamma X$ is the orbit space $(E\Gamma \times X)/\Gamma$ (see e.g. [18, Chapter III]). If X is a free Γ-space, as will be the case in our application, then the projection $E\Gamma \times_\Gamma X \to X/\Gamma$ is a homotopy equivalence and it induces an isomorphism

$$\mathcal{H}^*_\Gamma(X, A) \cong \mathcal{H}^*(X/\Gamma, A/\Gamma).$$

$\mathcal{H}^k_\Gamma(X, A)$ is a vector space over $\mathbb{K}$. Its dimension is called the Γ-*equivariant k-th Betti number* of (X, A), denoted

$$\mathcal{B}^\Gamma_k(X, A) := \dim \mathcal{H}^k_\Gamma(X, A).$$

The Γ-*Poincaré series* of (X, A) is the formal power series

$$\mathcal{P}_t(X, A) = \sum_{k=0}^{\infty} \mathcal{B}^\Gamma_k(X, A) t^k.$$

Let $\mathcal{M}$ be a complete C^2-Hilbert manifold with a differentiable action of Γ. Then Γ induces an action on the tangent bundle $T\mathcal{M}$ in the obvious way. We assume further that $\mathcal{M}$ is a Riemannian manifold and that Γ acts isometrically on the tangent bundle of $\mathcal{M}$. Such a manifold will be called a Γ-*Riemannian manifold*.

If $f : \mathcal{M} \to \mathbb{R}$ is a Γ-invariant C^1-function then its gradient vector field $\nabla f : \mathcal{M} \to T\mathcal{M}$ is Γ-equivariant. So if x is a critical point of f then its whole Γ-orbit Γx consists of critical points. Γx is then called a *critical* Γ-*orbit* of f. If Γx is an isolated critical Γ-orbit of f the k-*th critical group of* Γx is defined as

$$C^k_\Gamma(f, \Gamma x) := \mathcal{H}^k_\Gamma(f^c \cap U, (f^c \smallsetminus \Gamma x) \cap U),$$

where U is a Γ-invariant neighborhood of Γx in $\mathcal{M}$, $c := f(x)$ and $f^c := \{y \in \mathcal{M} : f(y) \leq c\}$. The excision property of cohomology asserts that $C^k_\Gamma(f, \Gamma x)$ does not depend on the choice of U. Its total dimension

$$\mu(f, \Gamma x) := \sum_{k=0}^{\infty} \dim C^k_\Gamma(f, \Gamma x)$$

is called the *multiplicity of* Γx.

Recall that f is said to satisfy the *Palais-Smale condition* $(\mathrm{PS})_c$ at $c \in \mathbb{R}$ if every sequence (x_n) in $\mathcal{M}$ such that $J(x_n) \to c$ and $\|\nabla J(x_n)\| \to 0$ contains a convergent subsequence.

The following result may be found in [8, Theorem 7.6].

Theorem 2.1. *Let* $\mathcal{M}$ *be a* Γ-*Riemannian* C^2-*manifold and* $f : \mathcal{M} \to \mathbb{R}$ *be a* Γ-*invariant* C^1-*function which satisfies* $(\mathrm{PS})_c$ *at every* $c \in [a, b]$. *If* a *and* b *are regular values of* f *and if* f *has only finitely many critical* Γ-*orbits* $\Gamma x_1, ..., \Gamma x_m$ *in* $f^{-1}[a, b]$ *then*

$$\sum_{k=0}^{\infty} \left(\sum_{j=1}^{m} \dim C^k_\Gamma(f, \Gamma x_j) \right) t^k = \mathcal{P}_t(f^b, f^a) + (1 + t)\mathcal{Q}(t)$$

where $\mathcal{Q}(t)$ *is a formal power series with nonnegative possibly infinite coefficients.*

Equality of two formal power series means that the coefficients of t^k on both sides of the equality are the same.

In order to give a geometric meaning to this formula let us assume further that f is of class C^2. If Γx is a critical orbit of f then $T_x(\Gamma x) \subset \ker d^2 f(x)$ where $d^2 f : T\mathcal{M} \to T\mathcal{M}$ is the Hessian operator of f. Let $N(\Gamma x) \to \Gamma x$ be the normal bundle of Γx in $\mathcal{M}$. A critical Γ-orbit Γx of f is *nondegenerate* if $d^2 f(x) : N_x(\Gamma x) \to N_x(\Gamma x)$ has a bounded inverse. The dimension of the negative subspace $N^-_x(\Gamma x)$ corresponding to the spectral decomposition of $d^2 f(x)$ is called the *Morse index of*

Γx, denoted $\mathrm{ind}(f,\Gamma x)$. If we write $N^{-}(\Gamma x)\to\Gamma x$ for the bundle of negative subspaces and $DN^{-}(\Gamma x)$ and $SN^{-}(\Gamma x)$ for the total spaces of its disk and sphere bundles respectively we have that

$$C_{\Gamma}^{k}(f,\Gamma x_j)=\mathcal{H}_{\Gamma}^{k}(DN^{-}(\Gamma x),SN^{-}(\Gamma x)).$$

If we assume further that $\gamma x\neq x$ for all $\gamma\in\Gamma$ and that $\lambda:=\mathrm{ind}(f,\Gamma x)<\infty$ then we have that

$$\mathcal{H}_{\Gamma}^{k}(DN^{-}(\Gamma x),SN^{-}(\Gamma x))=\mathcal{H}^{k}(\mathbb{B}^{\lambda},\mathbb{S}^{\lambda-1})=\begin{cases}\mathbb{K} & \text{if } k=\lambda,\\ 0 & \text{if } k\neq\lambda.\end{cases}$$

So as a consequence of Theorem 2.1 we obtain the following.

Corollary 2.2. *Let $\mathcal{M}$ be a free Γ-Riemannian C^2-manifold and $f:\mathcal{M}\to\mathbb{R}$ be a Γ-invariant C^2-function which satisfies $(\mathrm{PS})_c$ at every $c\in[a,b]$. If a and b are regular values of f and if f has only finitely many critical Γ-orbits in $f^{-1}[a,b]$ and they are nondegenerate and have finite Morse index, then f has at least $\mathcal{B}_k^{\Gamma}(f^b,f^a)$ critical Γ-orbits in $f^{-1}[a,b]$ with Morse index k for every $k\in\mathbb{N}$.*

3. The variational problem

Let $\nabla_{\varepsilon,A}u:=\varepsilon\nabla u+iAu$, and consider the real Hilbert space

$$H_{\varepsilon,A}^{1}(\mathbb{R}^N,\mathbb{C}):=\{u\in L^2(\mathbb{R}^N,\mathbb{C}):\nabla_{\varepsilon,A}u\in L^2(\mathbb{R}^N,\mathbb{C}^N)\}$$

with scalar product

$$\langle u,v\rangle_{\varepsilon,A,V}:=\mathrm{Re}\int\left(\nabla_{\varepsilon,A}u\cdot\overline{\nabla_{\varepsilon,A}v}+V(x)u\overline{v}\right).$$

We write

$$\|u\|_{\varepsilon,A,V}:=\left(\int\left(|\nabla_{\varepsilon,A}u|^2+V(x)\,|u|^2\right)\right)^{1/2}$$

for the corresponding norm. If $u\in H_{\varepsilon,A}^{1}(\mathbb{R}^N,\mathbb{C})$, then $|u|\in H^1(\mathbb{R}^N)$ and

$$\varepsilon\,|\nabla\,|u|\,(x)|\leq|\varepsilon\nabla u(x)+iA(x)u(x)|\quad\text{for a.e. } x\in\mathbb{R}^N. \tag{3.1}$$

This is called the *diamagnetic inequality* (see e.g. [24]). Together with the Sobolev inequality, it yields

$$|u|_{p,K}:=\left(\int K(x)\,|u|^p\right)^{1/p}\leq C\,\|u\|_{\varepsilon,A}\qquad\forall u\in H_{\varepsilon,A}^{1}(\mathbb{R}^N,\mathbb{C})$$

for some constant $C>0$ if $p\in[2,2^*]$.

The action of G on $H_{\varepsilon,A}^{1}(\mathbb{R}^N,\mathbb{C})$ defined by

$$(g,u)\mapsto u_g,\qquad (u_g)(x):=\tau(g)u(g^{-1}x),$$

is an orthogonal action, that is,

$$\langle u_g, v_g\rangle_{\varepsilon,A,V} = \langle u, v\rangle_{\varepsilon,A,V} \qquad \forall g \in G,\ u, v \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C}).$$

It also satisfies

$$|u_g|_{p,K} = |u|_{p,K} \qquad \forall g \in G,\ u \in L^p(\mathbb{R}^N, \mathbb{C}).$$

Therefore, the energy functional,

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_{\varepsilon,A,V} - \frac{1}{p}|u|^p_{p,K}, \qquad u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C}),$$

is G-invariant and by the principle of symmetric criticality [25, 29] the critical points of the restriction of J_ε to the fixed point space of the G-action, defined as

$$\begin{aligned} H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau : &= \{u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C}) : u_g = u\} \\ &= \{u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C}) : u(gx) = \tau(g)u(x) \ \ \forall x \in \mathbb{R}^N,\ g \in G\}, \end{aligned}$$

are the solutions to problem

$$\begin{cases} (-\varepsilon i\nabla + A)^2 u + Vu = K|u|^{p-2}u, \\ u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C}), \\ u(gx) = \tau(g)u(x) \ \ \forall x \in \mathbb{R}^N,\ g \in G. \end{cases} \tag{3.2}$$

The nontrivial solutions to (3.2) are the critical points of the restriction

$$J_\varepsilon : \mathcal{N}^\tau_\varepsilon \to \mathbb{R}$$

of J_ε to the Nehari manifold

$$\mathcal{N}^\tau_\varepsilon := \{u \in H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau : u \neq 0,\ \ \|u\|^2_{\varepsilon,A,V} = |u|^p_{p,K}\},$$

which is a C^2-submanifold of $H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau$, radially diffeomorphic to the unit sphere.

There is a natural action the group $\mathbb{S}^1$ of unit complex numbers on the space $H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau$ given by scalar multiplication $(\gamma, u) \mapsto \gamma u$. The Nehari manifold $\mathcal{N}^\tau_\varepsilon$ and the functional J_ε are $\mathbb{S}^1$-invariant and, since this action is orthogonal, $\mathcal{N}^\tau_\varepsilon$ is an $\mathbb{S}^1$-Riemannian manifold.

For every $u \in \mathcal{N}^\tau_\varepsilon$ we have that

$$J_\varepsilon(u) = \frac{p-2}{2p}\|u\|^2_{\varepsilon,A,V} = \frac{p-2}{2p}|u|^p_{p,K}.$$

We denote by

$$\pi_\varepsilon : H^1_{\varepsilon,A}(\mathbb{R}^N, \mathbb{C})^\tau \smallsetminus \{0\} \to \mathcal{N}^\tau_\varepsilon$$

the radial projection, given by

$$\pi_\varepsilon(u) := \left(\frac{\|u\|^2_{\varepsilon,A,V}}{|u|^p_{p,K}}\right)^{1/(p-2)} u. \tag{3.3}$$

Then

$$J_\varepsilon(\pi_\varepsilon(u)) = \frac{p-2}{2p}\left(\frac{\|u\|^2_{\varepsilon,A,V}}{|u|^2_{p,K}}\right)^{p/(p-2)} \qquad \forall u \in H^1_{\varepsilon,A}(\mathbb{R}^N,\mathbb{C})^\tau \smallsetminus \{0\}.$$

Let

$$c^\tau_{\varepsilon,A,V,K} := \inf_{\mathcal{N}^\tau_\varepsilon} J_\varepsilon.$$

Set $q := \frac{p}{p-2} - \frac{N}{2}$. Note that $q > 0$, because $p < 2^*$. The following holds.

Proposition 3.1. *For every* $\varepsilon > 0$*, the functional* $J_\varepsilon : \mathcal{N}^\tau_\varepsilon \to \mathbb{R}$ *satisfies the Palais-Smale condition* $(\mathrm{PS})_c$ *at each level*

$$c < \varepsilon^N \min_{x\in\mathbb{R}^N\smallsetminus\{0\}} (\#Gx)\frac{V^q_\infty}{K^{2/(p-2)}_\infty} c_{\mathbb{R}^N},$$

where $V_\infty := \liminf_{|x|\to\infty} V(x)$ *and* $K_\infty := \limsup_{|x|\to\infty} K(x)$*.*

Proof. The proof is completely analogous to that of Proposition 5 in [10]. □

4. The entrance map

Let

$$\ell_G := \min_{x\in\mathbb{R}^N} (\#Gx)\frac{V^q(x)}{K^{2/(p-2)}(x)} \leq \frac{V^q(0)}{K^{2/(p-2)}(0)},$$

and

$$M := \{y \in \mathbb{R}^N : (\#Gy)\frac{V^q(y)}{K^{2/(p-2)}(y)} = \ell_G\}.$$

From now on we assume that there exists $\alpha > 0$ such that

$$\{y \in \mathbb{R}^N : (\#Gy)\frac{V^q(y)}{K^{2/(p-2)}(y)} \leq \ell_G + \alpha\} \text{ is compact.} \tag{4.1}$$

Then, M is a compact G-invariant set and all G-orbits in M are finite. We split M according to the orbit type of its elements as follows: We choose subgroups $G_1, ..., G_m$ of G such that the isotropy subgroup G_x of every point $x \in M$ is conjugate to precisely one of the G_i's, and we set

$$M_i := \{y \in M : G_y = gG_ig^{-1} \text{ for some } g \in G\}.$$

Since isotropy subgroups satisfy $G_{gx} = gG_xg^{-1}$, the sets M_i are G-invariant and, since V and K are continuous, they are closed and pairwise disjoint, and

$$M = M_1 \cup \cdots \cup M_m.$$

We have that

$$|G/G_i|\,\frac{V^q(y)}{K^{2/(p-2)}(y)} = (\#Gy)\frac{V^q(y)}{K^{2/(p-2)}(y)} = \ell_G \qquad \forall y \in M_i,$$

where $|G/G_i|$ denotes the index of G_i in G. It follows that the quotient $\frac{V^q(y)}{K^{2/(p-2)}(y)}$ is constant on each M_i. We denote by m_i its value on M_i.

For each $\xi \in M_i$ let $\upsilon_{\varepsilon,\xi}$ be the positive least energy solution to the real-valued problem

$$\begin{cases} -\Delta u + V(\xi)u = K(\xi)\,|u|^{p-2}\,u, \\ u \in H_0^1(B(0,1/\sqrt{\varepsilon}),\mathbb{R}), \end{cases}$$

where $B(\xi,r) := \{x \in \mathbb{R}^N : |x-\xi| < r\}$. It is well known that $\upsilon_{\varepsilon,\xi}$ is radial, and it is easy to derive from Lemma 3.2 in [5] that

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{p-2}{2p}\left(\frac{\int_{B(0,1/\sqrt{\varepsilon})}\left(|\nabla \upsilon_{\varepsilon,\xi}|^2 + V(\xi)\,|\upsilon_{\varepsilon,\xi}|^2\right)}{\left(\int_{B(0,1/\sqrt{\varepsilon})} K(\xi)\,|\upsilon_{\varepsilon,\xi}|^p\right)^{2/p}}\right)^{p/(p-2)} &= \frac{V(\xi)^q}{K(\xi)^{2/(p-2)}}c_{\mathbb{R}^N} \\ &= m_i c_{\mathbb{R}^N}. \end{aligned} \tag{4.2}$$

Following [9] we set

$$\phi_{\varepsilon,\xi}(x) := \upsilon_{\varepsilon,\xi}\left(\frac{x-\xi}{\varepsilon}\right)e^{-iA(\xi)\cdot\left(\frac{x-\xi}{\varepsilon}\right)}.$$

The following holds.

Lemma 4.1. *Uniformly in $\xi \in M_i$, we have that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon\left[\pi_\varepsilon(\phi_{\varepsilon,\xi})\right] = m_i c_{\mathbb{R}^N},$$

where π_ε stands for the radial projection onto the Nehari manifold defined in (3.3).

Proof. Straightforward computations yield

$$\begin{aligned} \int_{B(\xi,\sqrt{\varepsilon})} |\varepsilon\nabla\phi_{\varepsilon,\xi} + iA\phi_{\varepsilon,\xi}|^2 &= \varepsilon^N\left(\int_{B(0,1/\sqrt{\varepsilon})} |\nabla\upsilon_{\varepsilon.\xi}|^2 + I_1\right), \\ \int_{B(\xi,\sqrt{\varepsilon})} V(x)\,|\phi_{\varepsilon,\xi}(x)|^2\,dx &= \varepsilon^N\left(\int_{B(0,1/\sqrt{\varepsilon})} V(\xi)\,|\upsilon_{\varepsilon,\xi}|^2 + I_2\right), \\ \int_{B(\xi,\sqrt{\varepsilon})} K(x)\,|\phi_{\varepsilon,\xi}(x)|^p\,dx &= \varepsilon^N\left(\int_{B(0,1/\sqrt{\varepsilon})} K(\xi)\,|\upsilon_{\varepsilon,\xi}|^p + I_3\right), \end{aligned}$$

where

$$I_1 = \int_{B(0,1/\sqrt{\varepsilon})} |(A(\varepsilon y+\xi) - A(\xi))\upsilon_{\varepsilon,\xi}(y)|^2\, dy,$$
$$I_2 = \int_{B(0,1/\sqrt{\varepsilon})} (V(\varepsilon y+\xi) - V(\xi))\, |\upsilon_{\varepsilon,\xi}(y)|^2\, dy,$$
$$I_3 = \int_{B(0,1/\sqrt{\varepsilon})} (K(\varepsilon y+\xi) - K(\xi))\, |\upsilon_{\varepsilon,\xi}(y)|^p\, dy$$

(cf. Lemma 1 in [10]). It follows that

$$\varepsilon^{-N} J_\varepsilon\left[\pi_\varepsilon\left(\phi_{\varepsilon,\xi}\right)\right] = \frac{p-2}{2p}\varepsilon^{-N}\left(\frac{\|\phi_{\varepsilon,\xi}\|^2_{\varepsilon,A,V}}{|\phi_{\varepsilon,\xi}|^2_{p,K}}\right)^{p/(p-2)}$$
$$= \frac{p-2}{2p}\left(\frac{\int_{B(0,1/\sqrt{\varepsilon})}\left(|\nabla\upsilon_{\varepsilon,\xi}|^2 + V(\xi)\,|\upsilon_{\varepsilon,\xi}|^2\right)}{\left(\int_{B(0,1/\sqrt{\varepsilon})} K(\xi)\,|\upsilon_{\varepsilon,\xi}|^p\right)^{2/p}} + o_\varepsilon(1)\right)^{p/(p-2)},$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ uniformly in $\xi \in M_i$. Using (4.2) we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon\left[\pi_\varepsilon\left(\phi_{\varepsilon,\xi}\right)\right] = m_i c_{\mathbb{R}^N},$$

as claimed. □

The map

$$G/G_\xi \to G\xi, \qquad gG_\xi \mapsto g\xi,$$

is a homeomorphism (see e.g. [18]). So, if $G_i \subset \ker\tau$ and $\xi \in M_i$, then the map

$$G\xi \to \mathbb{S}^1, \qquad g\xi \mapsto \tau(g),$$

is well defined and continuous. Define

$$\psi_{\varepsilon,\xi}(x) := \sum_{g\xi\in G\xi} \tau(g)\upsilon_{\varepsilon,\xi}\left(\frac{x-g\xi}{\varepsilon}\right) e^{-iA(g\xi)\cdot\left(\frac{x-g\xi}{\varepsilon}\right)}.$$

Lemma 4.2. *Assume that $G_i \subset \ker\tau$.*
(a) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\psi_{\varepsilon,\xi}(gx) = \tau(g)\psi_{\varepsilon,\xi}(x) \qquad \forall g \in G,\ x \in \mathbb{R}^N.$$

(b) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\tau(g)\psi_{\varepsilon,g\xi}(x) = \psi_{\varepsilon,\xi}(x) \qquad \forall g \in G,\ x \in \mathbb{R}^N.$$

(c) One has that

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon\left[\pi_\varepsilon(\psi_{\varepsilon,\xi})\right] = \ell_G c_{\mathbb{R}^N}$$

uniformly in $\xi \in M_i$.

Proof. The proof is completely analogous to that of Lemma 2 in [10]. □

Let

$$M_\tau := \{y \in M : G_y \subset \ker \tau\} = \bigcup_{G_i \subset \ker \tau} M_i.$$

The following holds.

Proposition 4.3. *The map $\widehat{\iota}_\varepsilon : M_\tau \to \mathcal{N}_\varepsilon^\tau$ given by*

$$\widehat{\iota}_\varepsilon(\xi) := \pi_\varepsilon(\psi_{\varepsilon,\xi})$$

is well defined and continuous, and satisfies

$$\tau(g)\widehat{\iota}_\varepsilon(g\xi) = \widehat{\iota}_\varepsilon(\xi) \qquad \forall \xi \in M_\tau,\ g \in G.$$

Moreover, given $d > \ell_G c_{\mathbb{R}^N}$, there exists $\varepsilon_d > 0$ such that

$$J_\varepsilon(\widehat{\iota}_\varepsilon(\xi)) \leq \varepsilon^N d \qquad \forall \xi \in M_\tau,\ \varepsilon \in (0, \varepsilon_d).$$

Proof. This is an immediate consequence of Lemma 4.2 □

5. A local baryorbit map

Next, we consider the real-valued problem

$$\left\{ \begin{array}{l} -\varepsilon^2 \Delta v + V v = K |v|^{p-2} v, \\ v \in H^1(\mathbb{R}^N, \mathbb{R}), \\ v(gx) = v(x) \ \ \forall x \in \mathbb{R}^N,\ g \in G. \end{array} \right. \tag{5.1}$$

We write

$$\langle v, w \rangle_{\varepsilon,V} := \int \varepsilon \nabla v \cdot \nabla w + V v w, \quad \|v\|_{\varepsilon,V}^2 := \int \left(|\varepsilon \nabla v|^2 + V |v|^2 \right),$$

and set

$$H^1(\mathbb{R}^N, \mathbb{R})^G := \{v \in H^1(\mathbb{R}^N, \mathbb{R}) : v(gx) = v(x) \ \forall x \in \mathbb{R}^N,\ g \in G\}.$$

The nontrivial solutions of (5.1) are the critical points of the energy functional

$$F_\varepsilon(v) = \frac{1}{2} \|v\|_{\varepsilon,V}^2 - \frac{1}{p} |v|_{p,K}^p$$

restricted to the Nehari manifold

$$\mathcal{M}_\varepsilon^G := \{v \in H^1(\mathbb{R}^N, \mathbb{R})^G : v \neq 0,\ \ \|v\|_{\varepsilon,V}^2 = |v|_{p,K}^p\}.$$

Let

$$c_{\varepsilon,V,K}^G := \inf_{\mathcal{M}_\varepsilon^G} F_\varepsilon = \inf_{\substack{v \in H^1(\mathbb{R}^N, \mathbb{R})^G \\ v \neq 0}} \frac{p-2}{2p} \left(\frac{\|v\|_{\varepsilon,V}^2}{|v|_{p,K}^2} \right)^{p/(p-2)}. \tag{5.2}$$

Lemma 5.1. $0 < \frac{V_0^q}{K_0^{2/(p-2)}} c_{\mathbb{R}^N} \leq \varepsilon^{-N} c^G_{\varepsilon,V,K}$ *for every* $\varepsilon > 0$, *and*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} c^G_{\varepsilon,V,K} \leq \ell_G c_{\mathbb{R}^N}.$$

where $V_0 := \inf_{x\in\mathbb{R}^N} V(x)$ *and* $K_0 := \sup_{x\in\mathbb{R}^N} K(x)$.

Proof. Set $v_\varepsilon(x) := v(\varepsilon x)$. Then $\|v_\varepsilon\|^2_{1,V_0} = \varepsilon^{-N} \|v\|^2_{\varepsilon,V_0}$ and $|v_\varepsilon|^p_{p,K_0} = \varepsilon^{-N} |v|^p_{p,K_0}$. It follows immediately from (5.2) that

$$\frac{V_0^q}{K_0^{2/(p-2)}} c_{\mathbb{R}^N} \leq c^G_{1,V_0,K_0} = \varepsilon^{-N} c^G_{\varepsilon,V_0,K_0} \leq \varepsilon^{-N} c^G_{\varepsilon,V,K}.$$

To prove the second inequality, take $\xi \in \mathbb{R}^N$ such that its G-orbit is finite. Write $G\xi := \{\xi_1, ..., \xi_m\}$. Fix $0 < \rho < \frac{1}{2}\min_{i\neq j} |\xi_i - \xi_j|$, and let $V_\rho := \sup_{B(\xi_1,\rho)} V$, $K_\rho := \inf_{B(\xi_1,\rho)} K$. Let $\upsilon_{\rho,\varepsilon}$ be the positive least energy solution to problem

$$\begin{cases} -\Delta v + V_\rho v = K_\rho |v|^{p-2} v, \\ v \in H^1_0(B(0, 1/\sqrt{\varepsilon}), \mathbb{R}). \end{cases}$$

Define

$$w_{\rho,\varepsilon}(x) := \sum_{i=1}^m \upsilon_{\rho,\varepsilon}\left(\frac{x-\xi_i}{\varepsilon}\right).$$

If $\sqrt{\varepsilon} \leq \rho$, then $\operatorname{supp}(w_{\rho,\varepsilon}) \subset \cup_{i=1}^m B(\xi_i, \rho)$ and, therefore,

$$\varepsilon^{-N} |w_{\rho,\varepsilon}|^p_{p,K_\rho} = m |\upsilon_{\rho,\varepsilon}|^p_{p,K_\rho} = m \|\upsilon_{\rho,\varepsilon}\|^2_{1,V_\rho} = \varepsilon^{-N} \|w_{\rho,\varepsilon}\|^2_{\varepsilon,V_\rho}.$$

Hence,

$$\begin{aligned} \varepsilon^{-N} c^G_{\varepsilon,V,K} &\leq \frac{p-2}{2p} \varepsilon^{-N} \left(\frac{\|w_{\rho,\varepsilon}\|^2_{\varepsilon,V}}{|w_{\rho,\varepsilon}|^2_{p,K}}\right)^{p/(p-2)} \\ &\leq \frac{p-2}{2p} \varepsilon^{-N} \left(\frac{\|w_{\rho,\varepsilon}\|^2_{\varepsilon,V_\rho}}{|w_{\rho,\varepsilon}|^2_{p,K_\rho}}\right)^{p/(p-2)} = m\frac{p-2}{2p} |\upsilon_{\rho,\varepsilon}|^p_{p,K_\rho}. \end{aligned}$$

Using Lemma 3.2 in [5] we obtain that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} c^G_{\varepsilon,V,K} \leq m \lim_{\varepsilon\to 0} \frac{p-2}{2p} |\upsilon_{\rho,\varepsilon}|^p_{p,K} = (\#G\xi) \frac{V_\rho^q}{K_\rho^{2/(p-2)}} c_{\mathbb{R}^N}$$

and, letting $\rho \to 0$, we conclude that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} c^G_{\varepsilon,V,K} \leq (\#G\xi) \frac{V^q(\xi)}{K^{2/(p-2)}(\xi)} c_{\mathbb{R}^N}.$$

Therefore, $\limsup_{\varepsilon\to 0} \varepsilon^{-N} c^G_{\varepsilon,V,K} \leq \ell_G c_{\mathbb{R}^N}$, as claimed. □

Let M, G_i and M_i be as in Section 4. Fix $\bar{\rho} > 0$ such that

$$|y - gy| > 2\bar{\rho} \quad \text{if} \quad gy \neq y \in M,$$
$$\text{dist}(M_i, M_j) > 2\bar{\rho} \quad \text{if} \quad i \neq j,$$

For $\rho \in (0, \bar{\rho})$, let

$$M_i^\rho := \{y \in \mathbb{R}^N : \text{dist}(y, M_i) \leq \rho, \quad G_y = gG_ig^{-1} \text{ for some } g \in G\}.$$

For every $\xi \in M_i^\rho$ and $\varepsilon > 0$ define

$$\theta_{\varepsilon,\xi} := \sum_{g\xi \in G\xi} \omega_\xi \left(\frac{x - g\xi}{\varepsilon} \right),$$

where ω_ξ is the positive radial solution to problem

$$\begin{cases} -\Delta w + V(\xi)w = K(\xi)\, |w|^{p-2}\, w, \\ w \in H^1(\mathbb{R}^N, \mathbb{R}). \end{cases} \tag{5.3}$$

This solution is unique. Hence

$$\omega_{\xi'}(y) = \left(\frac{K(\xi)V(\xi')}{K(\xi')V(\xi)} \right)^{1/(p-2)} \omega_\xi \left(\left(\frac{V(\xi')}{V(\xi)} \right)^{1/2} y \right). \tag{5.4}$$

It is well known that

$$\lim_{|x| \to \infty} |D^\nu \omega_\xi(x)|\, |x|^{\frac{N-1}{2}} \exp |x| = a_\nu > 0, \qquad \nu = 0, 1, \tag{5.5}$$

(see [21, 6]). Let

$$\Theta_{\rho,\varepsilon} := \{\theta_{\varepsilon,\xi} : \xi \in M_1^\rho \cup \cdots \cup M_m^\rho\}.$$

We continue to assume that (4.1) holds.

Proposition 5.2. *Let $\varepsilon_n > 0$ and $v_n \in H^1(\mathbb{R}^N, \mathbb{R})^G$ be such that*

$$\varepsilon_n \to 0, \qquad \varepsilon_n^{-N} F_{\varepsilon_n}(v_n) \to \bar{c}, \qquad \varepsilon_n^{-N} \left\| \nabla_{\varepsilon_n} F_{\varepsilon_n}(v_n) \right\|_{\varepsilon_n,V}^2 \to 0,$$

where $\bar{c} := \liminf_{\varepsilon \to 0} \varepsilon^{-N} c_{\varepsilon,V,K}^G$ and $\nabla_{\varepsilon_n} F_{\varepsilon_n}$ is the gradient of F_{ε_n} with respect to the scalar product $\langle \cdot, \cdot \rangle_{\varepsilon_n,V}$. Then, up to a subsequence, there exist an $i \in \{1, ..., m\}$ and a sequence (ξ_n) in $\mathbb{R}^N$ such that
(i) $G_{\xi_n} = G_i$,
(ii) $\xi_n \to \xi \in M_i$,
(iii) $\varepsilon_n^{-N} \left| |v_n| - \theta_{\varepsilon_n,\xi_n} \right|_{p,K}^p \to 0$,
(iv) $\bar{c} = \lim_{\varepsilon \to 0} \varepsilon^{-N} c_{\varepsilon,V,K}^G = \ell_G c_{\mathbb{R}^N}$.

Proof. Let $\widetilde{v}_n \in H^1(\mathbb{R}^N, \mathbb{R})^G$ be given by $\widetilde{v}_n(z) := v_n(\varepsilon_n z)$ and set $\widetilde{K}_n(z) := K(\varepsilon_n z)$ Then,

$$|\widetilde{v}_n|_{p,\widetilde{K}_n}^p = \varepsilon_n^{-N} |v_n|_{p,K}^p \to \frac{2p}{p-2} \bar{c} > 0.$$

Fix $R>0$ and let

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \widetilde{K}_n \left|\widetilde{v}_n\right|^p .$$

Lions' lemma [29, Lemma 1.21] applied to the sequence $(\widetilde{K}_n^{1/p}\widetilde{v}_n)$ implies that $\delta>0$. Choose $z_n\in\mathbb{R}^N$ such that

$$\int_{B(z_n,R)} \widetilde{K}_n \left|\widetilde{v}_n\right|^p \geq \frac{\delta}{2}.$$

Passing to a subsequence and replacing z_n by some point in Gz_n if necessary, we may assume that all isotropy groups G_{z_n} are the same. Let

$$H := \{g\in G : (gz_n - z_n) \text{ is bounded}\}.$$

Then H is a closed subgroup of G which contains G_{z_n}. Let ζ_n denote the orthogonal projection of z_n onto

$$(\mathbb{R}^N)^H := \left\{z\in\mathbb{R}^N : gz = z \ \ \forall g\in H\right\}.$$

One can easily verify that $(z_n-\zeta_n)$ is bounded, $G_{\zeta_n}=H$ for all $n\in\mathbb{N}$ and $(g\zeta_n - g'\zeta_n)$ is unbounded if $g^{-1}g'\notin H$ (cf. Proposition 3 in [10]). Let $\overline{v}_n(z) := \widetilde{v}_n(z+\zeta_n)$. Since $(\overline{v}_n)$ is bounded in $H^1(\mathbb{R}^N,\mathbb{R})$, passing to a subsequence,

$$\begin{aligned} \overline{v}_n &\rightharpoonup \overline{v} \ \text{ weakly in } H^1(\mathbb{R}^N,\mathbb{R}),\\ \overline{v}_n(x) &\to \overline{v}(x) \ \text{ a.e. on } \mathbb{R}^N,\\ \overline{v}_n &\to \overline{v} \ \text{ in } L^p_{loc}(\mathbb{R}^N,\mathbb{R}). \end{aligned}$$

Choosing $C\geq |\zeta_n - z_n|$ for all n, we obtain

$$\sup_{\mathbb{R}^N} K \int_{B(0,C+R)} \left|\overline{v}_n\right|^p \geq \int_{B(\zeta_n,C+R)} \widetilde{K}_n \left|\widetilde{v}_n\right|^p \geq \int_{B(z_n,R)} \widetilde{K}_n \left|\widetilde{v}_n\right|^p \geq \frac{\delta}{2}.$$

Therefore, $\overline{v}\neq 0$. Set $\xi_n := \varepsilon_n\zeta_n$. After passing to a subsequence,

$$\lim_{n\to\infty} V(\xi_n) =: \hat{V} \quad \text{and} \quad \lim_{n\to\infty} K(\xi_n) =: \hat{K}.$$

It is easy to see that $\overline{v}$ is a solution to problem

$$\begin{cases} -\Delta v + \hat{V}v = \hat{K}\left|v\right|^{p-2} v, \\ v\in H^1(\mathbb{R}^N,\mathbb{R}). \end{cases}$$

Note that, since v_n is G-invariant, $v_n(\varepsilon_n x + g\xi_n) = v_n(\varepsilon_n g^{-1}x + \xi_n) = \overline{v}_n(g^{-1}x)$. Fix $g_1,\dots,g_k\in G$ such that $k=|G/H|$ and $g_i^{-1}g_j\notin H$ if $i\neq j$. Then, since $(g_i\zeta_n - g_j\zeta_n)$ is unbounded for $i\neq j$, we have that

$$\overline{v}_n g_j^{-1} - \sum_{i=j+1}^{k} v g_i^{-1}(\cdot - g_i\zeta_n + g_j\zeta_n) \rightharpoonup \overline{v} g_j^{-1}$$

weakly in $H^1(\mathbb{R}^N,\mathbb{R})$. A straightforward computation gives

$$\lim_{n\to\infty}\varepsilon_n^{-N}\left|v_n\right|_{p,K}^p=\lim_{n\to\infty}\varepsilon_n^{-N}\left|v_n-\sum_{i=1}^k\bar{v}g_i^{-1}\left(\frac{\cdot-g_i\xi_n}{\varepsilon_n}\right)\right|_{p,K}^p+\left|G/H\right|\left|\bar{v}\right|_{p,\hat{K}}^p.$$

Therefore,

$$\begin{aligned}\ell_G c_{\mathbb{R}^N}&\leq\lim_{n\to\infty}(\#G\xi_n)\frac{V^q(\xi_n)}{K^{2/(p-2)}(\xi_n)}c_{\mathbb{R}^N}=\left|G/H\right|\frac{\hat{V}^q}{\hat{K}^{2/(p-2)}}c_{\mathbb{R}^N}\\&\leq\left|G/H\right|\frac{p-2}{2p}\left|\bar{v}\right|_{p,\hat{K}}^p\leq\bar{c}\leq\limsup_{\varepsilon\to0}\varepsilon^{-N}c_{\varepsilon,V,K}^G\leq\ell_G c_{\mathbb{R}^N}.\end{aligned}$$

This proves that $\bar{c}=\lim_{\varepsilon\to0}\varepsilon^{-N}c_{\varepsilon,V,K}^G=\ell_G c_{\mathbb{R}^N}$ and that

$$\lim_{n\to\infty}\varepsilon_n^{-N}\left|v_n-\sum_{i=1}^k\bar{v}g_i^{-1}\left(\frac{\cdot-g_i\xi_n}{\varepsilon_n}\right)\right|_{p,K}^p=0. \tag{5.6}$$

Moreover, $(\#G\xi_n)\frac{V^q(\xi_n)}{K^{2/(p-2)}(\xi_n)}\leq\ell_G+\alpha$ for n large enough. Thus, assumption (4.1) implies that, after passing to a subsequence, $\xi_n\to\xi$. It follows that $V(\xi)=\hat{V}$, $K(\xi)=\hat{K}$ and

$$(\#G\xi)\frac{V^q(\xi)}{K^{2/(p-2)}(\xi)}=\ell_G.$$

We conclude that $\xi\in M_i$ for some $i=1,...,m$ and that $H=G_\xi=gG_ig^{-1}$ for some $g\in G$. Moreover, $\overline{v}$ solves problem (5.3) and

$$\frac{p-2}{2p}\left|\bar{v}\right|_{p,\hat{K}}^p=\frac{V^q(\xi)}{K^{2/(p-2)}(\xi)}c_{\mathbb{R}^N}.$$

Hence, $\overline{v}(z)=\pm\omega_\xi(z-z_0)$ for some $z_0\in\mathbb{R}^N$. Observe that $\overline{v}$ is H-invariant because

$$\overline{v}_n(gz)=\widetilde{v}_n(gz+\zeta_n)=\widetilde{v}_n(z+g^{-1}\zeta_n)=\widetilde{v}_n(z+\zeta_n)=\overline{v}_n(z)\quad\forall g\in H.$$

So if H is nontrivial then $z_0=0$ and, since ω_ξ is radial, equation (5.6) yields

$$\lim_{n\to\infty}\varepsilon_n^{-N}\left|v_n\pm\sum_{i=1}^k\omega_\xi\left(\frac{\cdot-g_i\xi_n}{\varepsilon_n}\right)\right|_{p,K}^p=0. \tag{5.7}$$

If H is the trivial group an easy argument shows that this equality holds after replacing ξ_n by $\xi_n+\varepsilon_n z_0$. By (5.4) we have that $\omega_{\xi_n}(y)=$

$b_n\omega_\xi(a_n y)$ with $b_n \to 1$ and $a_n \to 1$. Using (5.5) it is easy to show that $|\omega_{\xi_n} - \omega_\xi|_{p,1}^p \to 0$. Hence,

$$\lim_{n\to\infty} \varepsilon_n^{-N} \left| \sum_{i=1}^k \omega_\xi \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) - \sum_{i=1}^k \omega_{\xi_n} \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) \right|_{p,K}^p = 0.$$

This, together with (5.7) yields $\varepsilon_n^{-N} \left| |v_n| - \theta_{\varepsilon_n,\xi_n} \right|_{p,K}^p \to 0$. □

The proof of the following proposition is more delicate than that of Proposition 5.5 in [15] because here ω_ξ depends on ξ.

Proposition 5.3. *Given $\rho \in (0,\bar\rho)$ there exist $d_\rho > \ell_G c_{\mathbb{R}^N}$ and $\varepsilon_\rho > 0$ with the following property: For every $\varepsilon \in (0,\varepsilon_\rho)$ and every $v \in \mathcal{M}_\varepsilon^G$ with $F_\varepsilon(v) \le \varepsilon^N d_\rho$ there exists precisely one G-orbit $G\xi_{\varepsilon,v}$ with $\xi_{\varepsilon,v} \in M_1^\rho \cup \cdots \cup M_m^\rho$ such that*

$$\varepsilon^{-N} \left\| |v| - \theta_{\varepsilon,\xi_{\varepsilon,v}} \right\|_{\varepsilon,V}^2 = \min_{\theta\in\Theta_{\rho,\varepsilon}} \left\| |v| - \theta \right\|_{\varepsilon,V}^2 .$$

Proof. Let $\eta > 0$ be given. By Proposition 5.2 there exist $\varepsilon_0 > 0$ and $d > \ell_G c_{\mathbb{R}^N}$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and $v \in \mathcal{M}_\varepsilon^G$ with $F_\varepsilon(v) \le \varepsilon^N d$ there is a $\xi \in M_1^\rho \cup \cdots \cup M_m^\rho$ such that

$$\varepsilon^{-N} \left\| |v| - \theta_{\varepsilon,\xi} \right\|_{\varepsilon,V}^2 = \min_{\theta\in\Theta_{\rho,\varepsilon}} \left\| |v| - \theta \right\|_{\varepsilon,V}^2 < \eta. \tag{5.8}$$

Next we show that $G\xi$ is unique if ε_0 and d are small enough. To simplify notation we assume $|v| = v$. Using (5.4) and (5.5) we conclude that, if η is chosen small enough and $\xi, \xi' \in M_1^\rho \cup \cdots \cup M_m^\rho$ are such that (5.8) holds, then $\varepsilon^{-1}\text{dist}(G\xi, G\xi') < C$ where C is some positive constant depending on η (cf. Lemma 5.6 in [15]). Therefore if ε_0 is chosen small enough then ξ and ξ' lie in the same M_i^ρ and have the same isotropy group. So changing G_i within its conjugacy class if necessary, we may assume that ξ and ξ' lie in $M_i^\rho \cap L_i$ where $L_i := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G_i\}$. Note that $M_i^\rho \cap L_i$ is an open bounded subset of the linear subspace L_i of $\mathbb{R}^N$. Our problem reduces to showing that any two minima of the function

$$f_{\varepsilon,v}(\xi) := \varepsilon^{-N} \left\| v - \theta_{\varepsilon,\xi} \right\|_{\varepsilon,V}^2 = \left\| \tilde v - \tilde\theta_{\varepsilon,\xi} \right\|_{1,V_\varepsilon}^2, \qquad \xi \in M_i^\rho \cap L_i,$$

are in the same G-orbit. Here $V_\varepsilon(z) := V(\varepsilon z)$, $\tilde v(z) := v(\varepsilon z)$ and $\tilde\theta_{\varepsilon,\xi}(z) := \theta_{\varepsilon,\xi}(\varepsilon z) = \sum_{g\xi\in G\xi} \omega_\xi(z - \varepsilon^{-1} g\xi)$. For $\xi, h \in M_i^\rho \cap L_i$ we have

$$D^2 f_{\varepsilon,v}(\xi)(h,h) = 2 \left\| D_\xi \tilde\theta_{\varepsilon,\xi}(\xi) h \right\|_{1,V_\varepsilon}^2 - 2 \left\langle \tilde v - \tilde\theta_{\varepsilon,\xi}, D_\xi^2 \tilde\theta_{\varepsilon,\xi}(\xi)(h,h) \right\rangle_{1,V_\varepsilon}.$$

We express ω_ξ in terms of the positive radial solution ω to problem (1.3) and set

$$W_g(\xi)(z) := \omega_\xi(z - \varepsilon^{-1} g\xi) = b(\xi)\omega(a(\xi)(z - \varepsilon^{-1} g\xi))$$

where $a(\xi) := (V(\xi))^{1/2}$ and $b(\xi) := (V(\xi)/K(\xi))^{1/(p-2)}$. Then

$$\begin{aligned}(DW_g(\xi)h)(z) &= (\nabla b(\xi) \cdot h)\, \omega(a(\xi)(z - \varepsilon^{-1} g\xi)) \\ &\quad + b(\xi)\,(\nabla a(\xi) \cdot h)\,\big(\nabla\omega(a(\xi)(z - \varepsilon^{-1} g\xi)) \cdot (z - \varepsilon^{-1} g\xi)\big) \\ &\quad - \varepsilon^{-1} b(\xi) a(\xi) \nabla\omega(a(\xi)(z - \varepsilon^{-1} g\xi)) \cdot gh.\end{aligned}$$

Hence, there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $C_1 > 0$ such that

$$\left\| D_\xi \tilde{\theta}_{\varepsilon,\xi}(\xi) h \right\|^2_{1,V_\varepsilon} \geq C_1 \varepsilon^{-2} |h|^2 \qquad \forall \varepsilon \in (0, \varepsilon_1).$$

Similarly,

$$\left\| D^2_\xi \tilde{\theta}_{\varepsilon,\xi}(\xi)(h, h) \right\|_{1,V_\varepsilon} \leq C_2 \varepsilon^{-2} |h|^2 \qquad \forall \varepsilon \in (0, \varepsilon_2).$$

Therefore,

$$D^2 f_{\varepsilon,v}(\xi)(h, h) \geq 2\varepsilon^{-2} |h|^2 \left(C_1 - C_2 \left\| \tilde{v} - \tilde{\theta}_{\varepsilon,\xi} \right\|_{1,V_\varepsilon} \right).$$

The constants C_1 and C_2 do not depend on v, ε, ξ or h. So choosing $\sqrt{\eta} < C_1/2C_2$ and setting $\varepsilon_3 := \min\{\varepsilon_1, \varepsilon_2\}$ we have that

$$D^2 f_{\varepsilon,v}(\xi)(h, h) \geq C_3 \varepsilon^{-2} |h|^2 \quad \text{if } f_{\varepsilon,v}(\xi) < \eta \text{ and } \varepsilon \in (0, \varepsilon_3).$$

If ξ and ξ' are minima of $f_{\varepsilon,v}$ in $M_i^\rho \cap L_i$ and $h := \xi' - \xi$ then Taylor's theorem gives

$$0 = f_{\varepsilon,v}(\xi') - f_{\varepsilon,v}(\xi) = \frac{1}{2} D^2 f_{\varepsilon,v}(\xi)(h, h) + r_{\varepsilon,v}(h) \geq C_3 \varepsilon^{-2} |h|^2 + r_{\varepsilon,v}(h).$$

If $h \neq 0$ direct computation shows that $\varepsilon^2 |h|^{-2} r_{\varepsilon,v}(h) \to 0$ as $\varepsilon^{-1} |h| \to 0$. It follows that there exists $R > 0$ such that

$$\varepsilon^{-1} |\xi - \xi'| \geq R. \tag{5.9}$$

Now we argue by contradiction. Assume there are sequences $\varepsilon_n \to 0$, $d_n \to \ell_G c_{\mathbb{R}^N}$ and $v_n \in F_{\varepsilon_n}^{\varepsilon_n^N d_n}$ such that f_{ε_n, v_n} has at least two minima ξ_n and ξ_n' in $M_i^\rho \cap L_i$ with $G\xi_n \neq G\xi_n'$. Then Proposition 5.2 asserts that $\varepsilon_n^{-N} \|v_n - \theta_{\varepsilon_n,\xi_n}\|^2_{\varepsilon_n,V} \to 0$ and $\varepsilon_n^{-N} \|v_n - \theta_{\varepsilon_n,\xi_n'}\|^2_{\varepsilon_n,V} \to 0$. Hence,

$$\varepsilon_n^{-N} \left\|\theta_{\varepsilon_n,\xi_n} - \theta_{\varepsilon_n,\xi_n'}\right\|^2_{\varepsilon_n,V} = \left\| \tilde{\theta}_{\varepsilon_n,\xi_n} - \tilde{\theta}_{\varepsilon_n,\xi_n'} \right\|^2_{1,V_{\varepsilon_n}} \to 0.$$

But this implies that $\varepsilon_n^{-1} \operatorname{dist}(G\xi_n, G\xi_n') \to 0$, contradicting (5.9). □

For every $c \in \mathbb{R}$ set

$$F_\varepsilon^c := \{v \in \mathcal{M}_\varepsilon^G : F_\varepsilon(v) \leq c\}.$$

Proposition 5.3 allows us to define, for each $\rho \in (0, \bar{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, a local baryorbit map

$$\widehat{\beta}_{\rho,\varepsilon,0} : F_\varepsilon^{\varepsilon^N d_\rho} \longrightarrow (M_1^\rho \cup \cdots \cup M_m^\rho)/G$$

by taking

$$\widehat{\beta}_{\rho,\varepsilon,0}(v) := G\xi_{\varepsilon,v},$$

where $G\xi_{\varepsilon,v}$ is the unique G-orbit given by the previous proposition.

Coming back to our original problem, for every $c \in \mathbb{R}$ set

$$J_\varepsilon^c := \{u \in \mathcal{N}_\varepsilon^\tau : J_\varepsilon(u) \leq c\}.$$

The following holds.

Corollary 5.4. *For each $\rho \in (0, \bar{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, the local baryorbit map*

$$\widehat{\beta}_{\rho,\varepsilon} : J_\varepsilon^{\varepsilon^N d_\rho} \longrightarrow (M_1^\rho \cup \cdots \cup M_m^\rho)/G,$$

given by

$$\widehat{\beta}_{\rho,\varepsilon}(u) := \widehat{\beta}_{\rho,\varepsilon,0}(\hat{\pi}_\varepsilon(|u|)),$$

where $\hat{\pi}_\varepsilon : H^1(\mathbb{R}^N, \mathbb{R})^G \smallsetminus \{0\} \to \mathcal{M}_\varepsilon^G$ is the radial projection, is well defined and continuous. It satisfies

$$\widehat{\beta}_{\rho,\varepsilon}(\gamma u) = \widehat{\beta}_{\rho,\varepsilon}(u) \qquad \forall \gamma \in \mathbb{S}^1,$$

$$\widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi)) = \xi \qquad \forall \xi \in M_\tau \text{ with } J_\varepsilon(\iota_\varepsilon(\xi)) \leq \varepsilon^N d_\rho,$$

where $\widehat{\iota}_\varepsilon$ is the map defined in Proposition 4.3.

Proof. If $u \in \mathcal{N}_\varepsilon^\tau$ then $\hat{\pi}_\varepsilon(|u|) \in \mathcal{M}_\varepsilon^G$. The diamagnetic inequality yields

$$F_\varepsilon(\hat{\pi}_\varepsilon(|u|)) \leq J_\varepsilon(u). \tag{5.10}$$

So if $J_\varepsilon(u) \leq \varepsilon^N d_\rho$ then $\widehat{\beta}_{\rho,\varepsilon}(u)$ is well defined. It is straightforward to verify that it has the desired properties. □

We conclude this section with the following observation.

Corollary 5.5. *If $M_\tau \neq \emptyset$ then*

$$\lim_{\varepsilon \to \infty} \varepsilon^{-N} c_{\varepsilon,A,V,K}^\tau = \ell_G c_{\mathbb{R}^N}.$$

where $c_{\varepsilon,A,V,K}^\tau := \inf_{\mathcal{N}_\varepsilon^\tau} J_\varepsilon$.

Proof. Inequality (5.10) yields $c_{\varepsilon,V,K}^G := \inf_{\mathcal{M}_\varepsilon^G} F_\varepsilon \leq \inf_{\mathcal{N}_\varepsilon^\tau} J_\varepsilon =: c_{\varepsilon,A,V,K}^\tau$. The result follows from Proposition 5.2 and Lemma 4.2. □

6. Proof of the main theorem

Let $\mathcal{H}^*$ be Alexander-Spanier cohomology with coefficients in a field $\mathbb{K}$ and for every $c \in \mathbb{R}$ set

$$J_\varepsilon^c := \{u \in \mathcal{N}_\varepsilon^\tau : J_\varepsilon(u) \le c\}.$$

Lemma 6.1. *For every $\rho \in (0, \bar{\rho})$ and $d \in (\ell_G c_{\mathbb{R}^N}, d_\rho]$, with d_ρ as in* Proposition 5.3*, there exists $\varepsilon_{\rho,d} > 0$ such that*

$$\dim \mathcal{H}^k(J_\varepsilon^{\varepsilon^N d}/\mathbb{S}^1) \ge \operatorname{rank}\left(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G)\right)$$

for every $\varepsilon \in (0, \varepsilon_{\rho,d})$ and $k \ge 0$, where $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ is the inclusion map.

Proof. Let $\varepsilon_{\rho,d} := \min\{\varepsilon_d, \varepsilon_\rho\}$ where ε_ρ is as in Proposition 5.3 and ε_d is as in Proposition 4.3. Fix $\varepsilon \in (0, \varepsilon_{\rho,d})$. Then,

$$J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \le \varepsilon^N d \quad \text{and} \quad \widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi)) = \xi \qquad \forall \xi \in M_\tau.$$

By Proposition 4.3 and Corollary 5.4 the maps

$$M_\tau/G \xrightarrow{\iota_\varepsilon} J_\varepsilon^{\varepsilon^N d}/\mathbb{S}^1 \xrightarrow{\beta_{\rho,\varepsilon}} B_\rho M/G$$

given by $\iota_\varepsilon(G\xi) := \widehat{\iota}_\varepsilon(\xi)$ and $\beta_{\rho,\varepsilon}(\mathbb{S}^1 u) := \widehat{\beta}_{\rho,\varepsilon}(u)$ are well defined and satisfy $\beta_{\rho,\varepsilon}(\iota_\varepsilon(G\xi)) = G\xi$ for all $\xi \in M_\tau$. Note that $M_\tau = \bigcup\{M_i : G_i \subset \ker \tau\}$ is the union of some connected components of M. Moreover, our choice of $\bar{\rho}$ implies that $B_\rho M_\tau \cap B_\rho (M \smallsetminus M_\tau) = \emptyset$. Therefore the inclusion $i_{\tau,\rho} : B_\rho M_\tau/G \hookrightarrow B_\rho M/G$ induces an epimorphism in cohomology. Since $\beta_{\rho,\varepsilon} \circ \iota_\varepsilon = i_{\tau,\rho} \circ i_\rho$ we conclude that

$$\begin{aligned} \dim \mathcal{H}^k(J_\varepsilon^{\varepsilon^N d}/\mathbb{S}^1) &\ge \operatorname{rank}(\iota_\varepsilon^* : \mathcal{H}^k(J_\varepsilon^{\varepsilon^N d}/\mathbb{S}^1) \to \mathcal{H}_k(M_\tau/G)) \\ &\ge \operatorname{rank}((\beta_{\rho,\varepsilon} \circ \iota_\varepsilon)^* : \mathcal{H}^k(B_\rho M/G) \to \mathcal{H}_k(M_\tau/G)) \\ &= \operatorname{rank}\left(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G)\right), \end{aligned}$$

as claimed. □

We are ready to prove our main theorem.

Proof of Theorem 1.1. Assume $M_\tau \ne \emptyset$ and let $\rho > 0$ and $\delta \in (0, \alpha c_{\mathbb{R}^N})$ be given. Without loss of generality we may assume that $\rho \in (0, \bar{\rho})$. Assumption (4.1) implies that

$$\ell_G + \alpha \le \min_{x \in \mathbb{R}^N \smallsetminus \{0\}} (\#Gx) \frac{V_\infty^q}{K_\infty^{2/(p-2)}}$$

where $V_\infty := \liminf_{|x|\to\infty} V(x)$ and $K_\infty := \limsup_{|x|\to\infty} K(x)$. By Proposition 3.1 the functional $J_\varepsilon : \mathcal{N}_\varepsilon^\tau \to \mathbb{R}$ satisfies $(\mathrm{PS})_c$ at each level

$c \leq \ell_G c_{\mathbb{R}^N} + \delta$ for every $\varepsilon > 0$. By Corollary 5.5 there exists $\varepsilon_0 > 0$ such that

$$\ell_G c_{\mathbb{R}^N} - \delta < \varepsilon^{-N} \inf_{u \in \mathcal{N}_\varepsilon^\tau} J_\varepsilon \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Let $d \in (\ell_G c_{\mathbb{R}^N}, \min\{d_\rho, \ell_G c_{\mathbb{R}^N} + \delta\}]$ with d_ρ as in Proposition 5.3, and $\overline{\varepsilon} = \min\{\varepsilon_0, \varepsilon_{\rho,d}\}$ with $\varepsilon_{\rho,d}$ as in Lemma 6.1. Fix $\varepsilon \in (0, \overline{\varepsilon})$ and for $u \in \mathcal{N}_\varepsilon^\tau$ with $J_\varepsilon(u) = c$ set

$$C^k_{\mathbb{S}^1}(J_\varepsilon, \mathbb{S}^1 u) := \mathcal{H}^k((J_\varepsilon^c \cap U)/\mathbb{S}^1, ((J_\varepsilon^c \smallsetminus \mathbb{S}^1 u) \cap U)/\mathbb{S}^1).$$

Assume that every critical $\mathbb{S}^1$-orbit of J_ε in $J_\varepsilon^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$ is isolated. Since $J_\varepsilon : \mathcal{N}_\varepsilon^\tau \to \mathbb{R}$ satisfies $(\mathrm{PS})_c$ at each $c \leq \ell_G c_{\mathbb{R}^N} + \delta$ there are only finitely many of them. Let $\mathbb{S}^1 u_1, \ldots, \mathbb{S}^1 u_m$ be those critical $\mathbb{S}^1$-orbits of J_ε in $\mathcal{N}_\varepsilon^\tau$ which satisfy $J_\varepsilon(u_i) < \varepsilon^N d$. Applying Theorem 2.1 to $J_\varepsilon : \mathcal{N}_\varepsilon^\tau \to \mathbb{R}$ with $a := \varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta)$ and $b := \varepsilon^N d$ and Lemma 6.1 we obtain that

$$\begin{aligned}\sum_{j=1}^m C^k_{\mathbb{S}^1}(J_\varepsilon, \mathbb{S}^1 u_i) &\geq \mathcal{B}_k^{\mathbb{S}^1}(J_\varepsilon^{\varepsilon^N d}) = \dim \mathcal{H}^k(J_\varepsilon^{\varepsilon^N d}/\mathbb{S}^1) \\ &\geq \operatorname{rank}\left(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G)\right)\end{aligned}$$

for every $k \geq 0$, as claimed. The last assertion of Theorem 1.1 is an immediate consequence of Corollary 2.2. □

If the inclusion $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ is a homotopy equivalence then

$$\operatorname{rank}\left(i_\rho^* : \mathcal{H}^k(B_\rho M_\tau/G) \to \mathcal{H}^k(M_\tau/G)\right) = \dim \mathcal{H}^k(M_\tau/G).$$

Below we give and example for which this equality does not hold. The argument used to prove Theorem 2 in [10] yields also the following.

Theorem 6.2. *If assumption* (1.4) *holds then, given $\rho, \delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ problem $(\wp_\varepsilon)$ has at least*

$$\operatorname{cat}_{B_\rho M_\tau/G}(M_\tau/G)$$

τ-intertwining geometrically distinct solutions in $J^{-1}[\varepsilon^N(\ell_G c_{\mathbb{R}^N} - \delta), \varepsilon^N(\ell_G c_{\mathbb{R}^N} + \delta)]$.

Here $\operatorname{cat}_{B_\rho M_\tau/G}(M_\tau/G)$ stands for the Lusternik-Schnirelmann category of M_τ/G in $B_\rho M_\tau/G$. There are cases where Theorem 1.1 yields a much better result than Theorem 6.2, as the following example shows.

Example 1. *Let M be the union of all spheres*

$$S_n := \{(x_1, ..., x_N) \in \mathbb{R}^N : (x_1 - \frac{1}{n})^2 + x_2^2 + \cdots + x_N^2 = (\frac{1}{n})^2\}, \quad n \geq 1.$$

Then $\mathrm{cat}M = 2$ *whereas*

$$\lim_{\rho\to 0} \mathrm{rank}\left(i_\rho^* : \mathcal{H}^{N-1}(B_\rho M) \to \mathcal{H}^{N-1}(M)\right) = +\infty.$$

Proof. Indeed, the sets $M^+ := \{(x_1, ..., x_N) \in S_n : x_N \geq 0\}$ and $M^- := \{(x_1, ..., x_N) \in S_n : x_N \leq 0\}$ are contractible to the origin and satisfy $M = M^+ \cup M^-$. So $\mathrm{cat}M = 2$. On the other hand, for each $\rho > 0$ small enough there exists an $n_\rho \in \mathbb{N}$ such that $B_\rho M$ is homotopy equivalent to $S_1 \cup \cdots \cup S_{n_\rho}$. Therefore,

$$\mathrm{rank}\left(i_\rho^* : \mathcal{H}^{N-1}(B_\rho M) \to \mathcal{H}^{N-1}(M)\right) = n_\rho.$$

Clearly, $n_\rho \to \infty$ as $\rho \to 0$. □

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