A POSITIVE SOLUTION TO THE PURE CRITICAL EXPONENT PROBLEM IN UNBOUNDED DOMAINS

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ABSTRACT. We prove existence of a positive solution to problem

$$-\Delta u = |u|^{2^*-2} u, \quad u \in D_0^{1,2}(\Omega),$$

where $\Omega := \{(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : a < |y| < b\}$, $N \geq 3$, $2 \leq k \leq N$, $0 < a < b < \infty$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. This solution is radial in each of the variables y and z.

1. INTRODUCTION

We consider the problem

$$\left\{ \begin{array}{c} -\Delta u = |u|^{2^*-2} u, \\ u \in D_0^{1,2}(\Omega), \end{array} \right. \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$, $N \geq 3$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

It is well known that the existence of a solution depends on the domain. Pohožaev [25] showed that (1.1) does not have a nontrivial solution if $\Omega \subsetneq \mathbb{R}^N$ is smooth and strictly starshaped. A considerable amount of research has been done for bounded domains. The most remarkable result was obtained by Bahri and Coron [2] who showed that problem (1.1) has at least one positive solution in every bounded domain Ω having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients. We refer to [9, 10, 12, 23, 27, 16, 24, 7, 17, 20, 21, 8, 6, 5, 13] for further results in bounded domains.

For $\Omega = \mathbb{R}^N$ all positive solutions to problem (1.1) are known [1, 3, 28]. They are the so-called *standard bubbles* or *instantons* which are the least energy nontrivial solutions to (1.1). Existence of infinitely many sign changing solutions for $\Omega = \mathbb{R}^N$ was shown in [11].

Concerning other unbounded domains little seems to be known. If Ω is an exterior domain (i.e. its complement is bounded), one may use the invariance of this problem under Möbius transformations to obtain results by translating known ones for bounded domains to Ω, via inversion with respect to some appropriate sphere. To our knowledge there seem to be no results for other unbounded domains.

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We prove the following.

Theorem 1.1. *If* $\Omega := \{(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : a < |y| < b\}$ *with* $2 \le k \le N$ *and* $0 < a < b < \infty$ *then problem* (1.1) *has a positive solution* u *with double cylindrical symmetry, i.e.* $u = u(|y|, |z|)$.

Note that if $k = N$ then Ω is an annulus. Kazdan and Warner [15] showed that problem (1.1) has infinitely many radial solutions in this case.

2. Proof of Theorem 1.1

Let

$$\|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2} \quad \text{and} \quad |u|_{2^*} := \left(\int_\Omega |u|^{2^*}\right)^{1/2^*}$$

be the norms in $D_0^{1,2}(\Omega)$ and $L^{2^*}(\Omega)$ respectively. The nontrivial solutions to problem (1.1) are in one-to-one correspondence with the critical points of the functional

$$I(u) := \|u\|^2$$

on the manifold

$$\Sigma := \{u \in D_0^{1,2}(\Omega) : |u|_{2^*} = 1\}.$$

Namely, $u \in \Sigma$ is a critical point of I iff $I(u)^{\frac{1}{2^*-2}} u$ is a nontrivial solution of (1.1). It is well-known that

$$S := \inf_{u \in \Sigma} I(u) = \inf_{\substack{u \in D_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\|u\|^2}{|u|_{2^*}^2}$$

does not depend on Ω and that it is not attained if $\overline{\Omega} \neq \mathbb{R}^N$.

Let G be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$. The G-orbit of a point $x \in \mathbb{R}^N$ is the set $Gx := \{gx : g \in G\}$. A subset X of $\mathbb{R}^N$ is said to be G-invariant if $Gx \subset X$ for every $x \in X$, and a function $u : X \to \mathbb{R}$ is said to be G-invariant if it is constant on Gx for every $x \in X$.

Let Ω be an open G-invariant subset of $\mathbb{R}^N$. Then G acts on the Sobolev space $D_0^{1,2}(\Omega)$ by $(g,u) \mapsto u_g$, where

$$u_g(x) := u(g^{-1}x). \tag{2.1}$$

This action satisfies $\|u_g\| = \|u\|$ and $|u_g|_{2^*} = |u|_{2^*}$ for all $g \in G$, $u \in D_0^{1,2}(\Omega)$. Set

$$D_0^{1,2}(\Omega)^G := \{u \in D_0^{1,2}(\Omega) : u_g = u \ \forall g \in G\},$$
$$\Sigma^G := \Sigma \cap D_0^{1,2}(\Omega)^G.$$

By the principle of symmetric criticality [22] (see also Theorem 1.28 in [30]) the critical points of the restriction of I to Σ^G correspond to the nontrivial

G-invariant solutions to problem (1.1). Set

$$S_\Omega^G := \inf_{u\in\Sigma^G} I(u) = \inf_{\substack{u\in D_0^{1,2}(\Omega)^G \\ u\neq 0}} \frac{\|u\|^2}{|u|_{2^*}^2}.$$

Consider the space $\mathcal{M}(\mathbb{R}^N)$ of finite measures in $\mathbb{R}^N$. A positive measure $\nu \in \mathcal{M}(\mathbb{R}^N)$ *is concentrated at a single G-orbit in* $\mathbb{R}^N$ if there exists $\zeta \in \mathbb{R}^N$ such that $\nu(G\zeta) = \|\nu\|$. Note that ν is concentrated at a single G-orbit in $\mathbb{R}^N$ iff $\nu(X) \in \{0, \|\nu\|\}$ for every G-invariant open subset X of $\mathbb{R}^N$. Since $u \in D_0^{1,2}(\Omega)$ may be extended by 0 outside Ω, we consider u as an element of $D^{1,2}(\mathbb{R}^N)$ whenever convenient. The following lemma is a slight modification of Lemma 4.3 in [29]. It goes back to P.L. Lions' work on the concentration-compactness principle [18, 19], see also [30] and the references therein.

Lemma 2.1 (Concentration-compactness)**.** *Let (u_n) be a sequence in $D_0^{1,2}(\Omega)^G$ such that*

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } D_0^{1,2}(\Omega),\\ |\nabla(u_n-u)|^2 &\rightharpoonup \mu && \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\ |u_n-u|^{2^*} &\rightharpoonup \nu && \text{weakly in } \mathcal{M}(\mathbb{R}^N),\\ u_n(x) &\to u(x) && \text{a.e. in } \mathbb{R}^N, \end{aligned}$$

and let

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla u_n|^2, \qquad \nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{2^*}.$$

Then, the following hold:
(a) $S_\Omega^G \|\nu\|^{2/2^*} \leq \|\mu\|$.
(b) $S_\Omega^G (\nu_\infty)^{2/2^*} \leq \mu_\infty$.
(c) $\limsup_{n\to\infty} \|u_n\|^2 = \|u\|^2 + \|\mu\| + \mu_\infty$.
(d) $\limsup_{n\to\infty} |u_n|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + \|\nu\| + \nu_\infty$.
(e) If $u = 0$ and $S_\Omega^G \|\nu\|^{2/2^*} = \|\mu\|$ then μ and ν are concentrated at a single finite G-orbit in $\mathbb{R}^N$ or are zero.

Proof. Assertions *(a)-(d)* are part of Lemma 4.3 in [29] (see also Lemma 1.40 in [30]). That lemma also asserts that if $u = 0$ and $S_\Omega^G \|\nu\|^{2/2^*} = \|\mu\|$ then μ and ν are concentrated at a single G-orbit $G\zeta$ in $\mathbb{R}^N$ unless they are zero. Next we show that $G\zeta$ is finite. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ we have that

$$\begin{aligned} S\left(\int_{\mathbb{R}^N} |\varphi u_n|^{2^*}\right)^{2/2^*} &\leq \int_{\mathbb{R}^N} |\nabla(\varphi u_n)|^2 = \int_{\mathbb{R}^N} |\varphi\nabla u_n + u_n\nabla\varphi|^2 \\ &\leq 2\int_{\mathbb{R}^N} |\varphi\nabla u_n|^2 + 2\int_{\operatorname{supp}(\varphi)} |\nabla\varphi|^2 u_n^2 \\ &= 2\int_{\mathbb{R}^N} |\varphi\nabla u_n|^2 + o(1). \end{aligned}$$

Passing to the limit as $n \to \infty$ we obtain that

$$S\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}\, d\nu\right)^{2/2^*} \le 2\int_{\mathbb{R}^N} \varphi^2\, d\mu \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

By Proposition 4.2 in [29] there exist an at most countable set $\{x_j : j \in J\}$ of points in $\mathbb{R}^N$ and numbers $\nu_j \in (0,\infty)$ such that

$$\nu = \textstyle\sum_{j\in J} \nu_j \delta_{x_j}.$$

Since $\nu(\{gx\}) = \nu(\{x\})$ for every $g \in G$ it follows that $G\zeta = \{x_j : j \in J\}$. Therefore $G\zeta$ must be finite. □

Proof of Theorem 1.1. The group $G := O(k)$ of linear isometries of $\mathbb{R}^k$ acts on $\mathbb{R}^N \equiv \mathbb{R}^k \times \mathbb{R}^{N-k}$ in the obvious way, i.e. $g(y,z) := (gy, z)$ if $g \in G$, $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. The domain $\Omega := \{(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : a < |y| < b\}$ is invariant under this action. We will apply Lemma 2.1 to show that S_Ω^G is attained. Set $B_r(\xi) := \{x \in \mathbb{R}^N : |x - \xi| \le r\}$.

Let (u_n) be a sequence in Σ^G such that $\|u_n\|^2 \to S_\Omega^G$. Since $b < \infty$, Poincaré's inequality holds. Hence (u_n) is bounded in $H_0^1(\Omega)$. So if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |u_n|^{2^*} = 0,$$

by Lemma 2.1 in [26] we would have that $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$, which is impossible because $|u_n|_{2^*} = 1$. Therefore there exist $\delta > 0$ and $\xi_n = (y_n, z_n) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that, after passing to a subsequence,

$$\int_{B_1(\xi_n)} |u_n|^{2^*} \ge \delta.$$

Replacing $u_n(x)$ by $u_n(x + (0, z_n))$ we may assume that (ξ_n) is bounded and hence that

$$\int_{B_R(0)} |u_n|^{2^*} \ge \delta \tag{2.2}$$

for $R > 0$ sufficiently large.

Passing to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $D_0^{1,2}(\Omega)^G$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, $|\nabla(u_n - u)|^2 \rightharpoonup \mu$ and $|u_n - u|^{2^*} \rightharpoonup \nu$ weakly in $\mathcal{M}(\mathbb{R}^N)$. Since

$$\lim_{n\to\infty} \|u_n\|^2 = S_\Omega^G = S_\Omega^G \lim_{n\to\infty} |u_n|_{2^*}^2,$$

using Lemma 2.1 and the definition of S_Ω^G we obtain

$$\begin{aligned}\|u\|^2 + \|\mu\| + \mu_\infty &= S_\Omega^G \left(|u|_{2^*}^{2^*} + \|\nu\| + \nu_\infty\right)^{2/2^*} \\ &\le S_\Omega^G \left(|u|_{2^*}^2 + \|\nu\|^{2/2^*} + \nu_\infty^{2/2^*}\right) \le \|u\|^2 + \|\mu\| + \mu_\infty.\end{aligned}$$

Hence

$$\left(|u|_{2^*}^{2^*} + \|\nu\| + \nu_\infty\right)^{2/2^*} = |u|_{2^*}^2 + \|\nu\|^{2/2^*} + \nu_\infty^{2/2^*}.$$

It follows that exactly one of the quantities $|u|_{2^*}$, $\|\nu\|$, ν_∞ is 1 and the other two are 0. Inequality (2.2) implies that $\nu_\infty \leq 1-\delta$, hence $\nu_\infty = 0$. Assume $\|\nu\| = 1$. Then $u = 0$ and $S_\Omega^G \|\nu\|^{2/2^*} = \|\mu\|$, so by Lemma 2.1 ν is concentrated at a finite G-orbit. But since $a > 0$, $\overline{\Omega}$ does not contain finite G-orbits. Therefore $\|\nu\| = 0$. Consequently $|u|_{2^*} = 1$ and $\|u\|^2 = S_\Omega^G$. Since $I(u) = I(|u|)$ and $u \in \Sigma^G$ implies $|u| \in \Sigma^G$, we may assume $u \geq 0$. But then $u > 0$ by Harnack's inequality.
Since $u \in D_0^{1,2}(\Omega)^G$, u is radial in y. Applying the moving plane method [14] (see also Appendix C in [30]) we conclude that u is also radial in z. □

Note that the proof of Theorem 1.1 carries over to domains of the form $\Omega = U \times \mathbb{R}^{N-k}$ where U is an open bounded subset of $\mathbb{R}^k$ which is G-invariant for some closed subgroup G of $O(k)$ such that Gy is infinite for every $y \in \overline{U}$. So in this kind of domains problem (1.1) has a positive solution u satisfying $u(gy, \gamma z) = u(y,z)$ for every $g \in G$, $\gamma \in O(N-k)$. In the particular case where $k = N$ it is known that there are infinitely many solutions satisfying $u(gy) = u(y)$ for every $g \in G$, cf. [4].

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