Alternating sign multibump solutions of nonlinear elliptic equations in expanding tubular domains

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Let Γ denote a smooth, connected, compact submanifold of dimension one of $\mathbb{R}^N$, $N \geq 2$, possibly with boundary. Denote by Ω_R the open normal tubular neighborhood of radius 1 of the expanded curve $R\Gamma := \{Rx \mid x \in \Gamma \smallsetminus \partial\Gamma\}$. Consider the superlinear problem $-\Delta u + \lambda u = f(u)$ on the domains Ω_R, as $R \to \infty$, with homogeneous Dirichlet boundary conditions. We prove the existence of multibump solutions with bumps lined up along $R\Gamma$ and with alternating signs. The function f is superlinear at 0 and at ∞, but it is not assumed to be odd.

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1 Introduction

Let $\gamma \in C^3([0,1],\mathbb{R}^N)$, $N \geq 2$, be a regular curve without self-intersections except possibly for $\gamma(0)=\gamma(1)$. If $\gamma(0)=\gamma(1)$ then we also require $\dot\gamma(0)=\dot\gamma(1)$. For $R>0$ define

$$\Omega_R := \operatorname{int}\Bigg(\bigcup_{t\in[0,1]} \{R\gamma(t)+v \mid v\in\mathbb{R}^N,\ |v|<1,\ \dot\gamma(t)\cdot v=0\}\Bigg), \tag{1.1}$$

where $\operatorname{int}(X)$ denotes the interior of X in $\mathbb{R}^N$. Thus, for R large enough, Ω_R is the C^1-tubular neighborhood of radius 1 of the 1-dimensional submanifold Γ_R of $\mathbb{R}^N$ defined as

$$\Gamma_R := \begin{cases} \{R\gamma(t) \mid t\in[0,1]\}, & \text{if } \gamma(0)=\gamma(1),\\ \{R\gamma(t) \mid t\in(0,1)\}, & \text{if } \gamma(0)\neq\gamma(1). \end{cases}$$

We are interested in finding solutions to the problem

$$\begin{cases} -\Delta u+\lambda u = f(u) & \text{in } \Omega_R,\\ u=0 & \text{on } \partial\Omega_R, \end{cases} \tag{1.2}$$

for R large enough. Let

$$F(u) := \int_0^u f(s)\,\mathrm{d}s \qquad \text{if } u\in\mathbb{R}$$

and denote by $\lambda_{1,1}$ the first eigenvalue of the Laplace operator $-\Delta_{N-1}$ on B_1^{N-1} with Dirichlet boundary conditions. Here $B_r^{N-1} := \{\eta\in\mathbb{R}^{N-1} \mid |\eta|<r\}$.

We assume the following hypotheses:

(H1) $\lambda > -\lambda_{1,1}$.

(H2) $f\in C^1(\mathbb{R})\cap C^2(\mathbb{R}\backslash\{0\})$.

(H3) $f(0)=f'(0)=0$.

(H4) There are $a>0$ and $p_1,p_2>1$, $p_1,p_2<(N+2)/(N-2))$ if $N\geq 3$, such that $p_1\leq p_2$ and

$$|f''(u)| \leq a(|u|^{p_1-2}+|u|^{p_2-2})$$

for all $u\neq 0$.

(H5) $f(u)u>0$ for all $u\neq 0$.

(H6) There are $b>0$ and $\alpha\in(1/2,1]$ such that for $|u|,|v|$ small

$$|f(u+v)-f(u)-f(v)| \leq b|uv|^\alpha$$

and

$$|F(u+v)-F(u)-F(v)-f(u)v-f(v)u| \leq b|uv|^{2\alpha}.$$

Let $\Omega \subseteq \mathbb{R}^N$. For the Dirichlet problem $-\Delta u + \lambda u = f(u)$ stated on Ω the variational (or energy) functional is given by

$$J_\Omega(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x, \qquad u \in H_0^1(\Omega).$$

We write a point in $\mathbb{R}^N$ as (ξ, η), where $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}^{N-1}$, and denote by

$$\mathbb{L} := \{(\xi, \eta) \in \mathbb{R}^N \mid |\eta| < 1\}$$

the cylinder in $\mathbb{R}^N$ of radius 1 around the ξ-axis. Locally, $\mathbb{L}$ is the limit domain of Ω_R as $R \to \infty$. Consider the limit problem

$$\begin{cases} -\Delta u + \lambda u = f(u), \\ \qquad u \in H_0^1(\mathbb{L}). \end{cases} \tag{1.3}$$

By Lemma 2.5 below, the operator $-\Delta + \lambda$ with Dirichlet boundary conditions in $L^2(\mathbb{L})$ has positive spectrum. Under an Ambrosetti-Rabinowitz type condition the mountain pass theorem (or the Nehari manifold method), together with translation invariance in the ξ-direction and concentration compactness, yield a positive and a negative solution of (1.3) in $H_0^1(\mathbb{L})$, energy minimal in their respective cones. We shall impose the following hypothesis on the limit problem:

(H7) Problem (1.3) has a positive solution U^+ and a negative solution U^- that are non-degenerate, in the sense that the solution space of the problem

$$-\Delta u + \lambda u = f'(U^\pm)u, \qquad u \in H_0^1(\mathbb{L}),$$

has dimension one (i.e., is minimal).

Some examples where this hypothesis is satisfied are given in [10].

By [5, Theorem 1.2] the solutions $U^\pm$ are radially symmetric in η and decreasing in $|\eta|$. By [6, Theorem 6.2] we may assume that they are also even in ξ, after a translation in the ξ-direction, and strictly decreasing in ξ for $\xi > 0$. It follows that they have a unique extremal point, which we may assume to be 0.

We fix a map $[0,1] \to O(N)$, $t \mapsto A_t$, such that

$$A_t\left(\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}\right) = (1, 0, \dots, 0) \qquad \text{for all } t \in [0,1]. \tag{1.4}$$

Extending $U^\pm$ to all of $\mathbb{R}^N$ by 0, for each $R > 0$ and $x \in \Gamma_R$ we choose $t \in [0,1]$ with $x = R\gamma(t)$ and define $U^\pm_{x,R}$ as follows:

$$U^\pm_{x,R}(y) := U^\pm(A_t(y - x)) \qquad \text{for all } y \in \mathbb{R}^N.$$

Note that by the radial symmetry of $U^{\pm}$ in η, $U^{\pm}_{x,R}$ is independent of the choice of A_t, as long as (1.4) is satisfied.

If $n \in \mathbb{N}$ and $\gamma(0) \neq \gamma(1)$ then we say that $(x_1, x_2, \dots, x_n) \in (\Gamma_R)^n$ is an *n-chain* if there are $0 \leq t_1 < t_2 < \dots < t_n < 1$ such that

$$x_i = R\gamma(t_i) \qquad \text{for } i = 1, 2, \dots, n. \tag{1.5}$$

If $\gamma(0) = \gamma(1)$ then we only require that (1.5) holds after some circular shift of the tuple $(x_1, x_2, \dots, x_n)$.

Theorem 1.1. *Assume that* $\gamma(0) = \gamma(1)$. *Suppose also that* (H1)–(H7) *hold. For each* $k \in \mathbb{N}$ *there exists* $R_k > 0$ *such that for every* $R \geq R_k$ *there are a* $2k$*-chain* $(x_{R,1}, x_{R,2}, \dots, x_{R,2k})$ *in* $(\Gamma_R)^{2k}$ *and a solution* u_R *of* (1.2) *such that*

$$u_R = \sum_{i=1}^{k} (U^{+}_{x_{R,2i-1},R} + U^{-}_{x_{R,2i},R}) + o(1) \tag{1.6}$$

in $H^1(\mathbb{R}^N)$ *as* $R \to \infty$. *Moreover,* $|x_{R,i} - x_{R,j}| \to \infty$ *as* $R \to \infty$, *if* $i \neq j$.

Theorem 1.2. *Assume that* $\gamma(0) \neq \gamma(1)$. *Suppose also that* (H1)–(H7) *hold. For each* $n \in \mathbb{N}$, $n \geq 2$, *there exists* $R_n > 0$ *such that for every* $R \geq R_n$ *there are an* n*-chain* $(x_{R,1}, x_{R,2}, \dots, x_{R,n})$ *in* $(\Gamma_R)^n$ *and a solution* u_R *of* (1.2) *such that, setting* $k := \lfloor n/2 \rfloor$,

$$u_R = \sum_{i=1}^{k} (U^{+}_{x_{R,2i-1},R} + U^{-}_{x_{R,2i},R}) + (n - 2k)U^{+}_{x_{R,n},R} + o(1) \tag{1.7}$$

in $H^1(\mathbb{R}^N)$ *as* $R \to \infty$. *Moreover, as* $R \to \infty$, $|x_{R,i} - x_{R,j}| \to \infty$ *if* $i \neq j$, *and* $\operatorname{dist}(x_{R,i}, \partial\Gamma_R) \to \infty$ *for all* i.

All solutions constructed in Theorems 1.1 and 1.2 change sign. If γ is a closed curve these solutions have an even number of bumps with alternating signs along the curve, whereas in the open-end case $\gamma(0) \neq \gamma(1)$ the number of alternating bumps may be odd or even. Note that the term $(n-2k)$ in Theorem 1.2 is 0 if n is even, and it is 1 if n is odd. In the former case we have a positive and a negative bump at the ends of the chain, and in the latter case we have positive bumps at both ends of the chain. Of course, applying Theorem 1.2 with $f(u)$ replaced by $-f(-u)$ and then multiplying the obtained multibump solution by -1, one can reverse the order of the signs of the bumps.

Observe that in the open-end case the domains Ω_R are contractible, and they are even convex if Γ is a segment. This means that to get multiplicity of sign changing solutions neither topological nor particular geometrical assumptions are needed. This contrasts with the case of positive solutions where it has been conjectured that for some power-type nonlinearities only one positive solution exists in any convex domain [8], as holds true for a ball. Of course this difference between multiplicity of positive and sign changing solutions can be easily understood by looking at odd nonlinearities. In fact, if f is odd (as for example when $f(u) = |u|^{p-1}u$, $1 < p < (N+2)/(N-2)$) it is well known that infinitely many sign changing solutions exist in any bounded domain. Our results *do not assume that f is odd*, therefore multiplicity of sign changing solutions is not so obvious. In fact, if f is not odd only few multiplicity results are available, see e.g. [3, 4, 12].

For the singularly perturbed problem D'Aprile and Pistoia [12] recently showed existence of sign changing solutions with at most six peaks in any bounded smooth domain. Moreover, Dancer showed the existence of positive solutions with multiple bumps for "dumbbell shaped domains", see [8, 9]. This idea could also be used to construct, for any k, a domain such that sign changing solutions exist with up to k bumps, by implementing sufficiently many "very narrow" regions in the domain. On the other hand, our results exhibit, for each fixed $k \in \mathbb{N}$, *convex* domains in which problem (1.2) has at least k nodal solutions with up to $k+1$ peaks, not assuming that f is odd (see Theorem 1.2 in the case that Γ is a segment). We believe this is the first result of this type.

After a finite dimensional reduction Theorems 1.1 and 1.2 follow from a minimization argument. A crucial role is played by the interaction between a positive and a negative bump, which increases the value of the energy functional. This allows us to find a solution by minimization of an associated reduced functional in the closed tube case (Theorem 1.1). It also explains why the number of bumps must be even and why they should be placed along the tube with alternating signs. In the open-end case (Theorem 1.2) the energy increases as the bumps approach the ends of the tube. Therefore the minimization procedure goes through in the same way.

It is harder to prove similar results when Γ is a higher dimensional manifold, instead of a curve. For positive solutions some results have been obtained by Dancer and Yan [11] when Γ is the boundary of a convex domain. Our argument, which relies on a finite dimensional reduction as in [11], could also be used to construct positive multibump solutions in a tubular neighborhood of an expanding compact manifold. We treat this question in the forthcoming article [2]. Concerning sign changing solutions the problem is more subtle and requires minimax arguments.

The outline of the paper is as follows: In Section 2 we have collected some tools, and results about the linear problem. Section 3 contains the essential energy estimates, while in section 4 we describe the finite dimensional reduction and prove our main results.

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1.1 Notation

We set

$$\mu := \sqrt{\lambda + \lambda_{1,1}}.$$

For $r > 0$ denote by $\lambda_{1,r}$ the smallest eigenvalue of $-\Delta_{N-1}$ on the open ball B_r^{N-1} of radius r in $\mathbb{R}^{N-1}$, and denote by $\vartheta_{1,r}$ the positive eigenfunction corresponding to $\lambda_{1,r}$, normalized such that $\|\vartheta_{1,r}\|_{L^2} = 1$.

2 Preliminaries

2.1 Algebraic and Geometric Tools

Lemma 2.1. *Suppose that $\mu_k > \bar{\mu} \geq 0$ for $k = 1, 2, 3$. Then there is $C > 0$ such that the following inequalities hold for all $x_1, x_2, x_3 \in \mathbb{R}^N$:*

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,\mathrm{d}x \leq C\mathrm{e}^{-\bar{\mu}|x_1-x_2|} \tag{2.1}$$

and

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\mathrm{e}^{-\mu_3|x-x_3|}\,\mathrm{d}x \leq C\exp\Big(-\bar{\mu}\min_{x\in\mathbb{R}^N}\sum_{k=1}^{3}|x-x_k|\Big). \tag{2.2}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,\mathrm{d}x &\leq \int_{\mathbb{R}^N} \mathrm{e}^{-\bar{\mu}(|x-x_1|+|x-x_2|)}\mathrm{e}^{-(\mu_2-\bar{\mu})|x-x_2|}\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^N} \mathrm{e}^{-\bar{\mu}|x_1-x_2|}\mathrm{e}^{-(\mu_2-\bar{\mu})|x-x_2|}\,\mathrm{d}x\\ &= C\mathrm{e}^{-\bar{\mu}|x_1-x_2|}\end{aligned}$$

with

$$C := \int_{\mathbb{R}^N} \mathrm{e}^{-(\mu_2-\bar{\mu})|x-x_2|}\,\mathrm{d}x.$$

The proof of the other inequality is similar. □

Recall the constants α and b from condition (H6).

Lemma 2.2. *Given $\widetilde{C}_1 \geq 1$ and $n \in \mathbb{N}$ there is a constant $\widetilde{C}_2 = \widetilde{C}_2(\alpha, n, \widetilde{C}_1) > 0$ such that for $u_1, u_2, \ldots, u_n \in \mathbb{R}$ with $|u_i| \leq \widetilde{C}_1$ the following inequalities hold:*

$$\left|f\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} f(u_i)\right| \leq b\sum_{i<j}|u_iu_j|^{\alpha} \tag{2.3}$$

and

$$\left|F\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} F(u_i) - \sum_{i\neq j} f(u_i)u_j\right| \leq \widetilde{C}_2\Big(\sum_{i<j}|u_iu_j|^{2\alpha} + \sum_{i<j<k}|u_iu_ju_k|^{2/3}\Big). \tag{2.4}$$

Proof. We prove these inequalities by induction on n. If $n = 2$, inequality (2.3) is part of assumption (H6). Assume (2.3) holds for $n-1$. Then by (H6)

$$\begin{aligned}
\Big|f\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} f(u_i)\Big| &\leq \Big|f\Big(\sum_{i=1}^{n} u_i\Big) - f\Big(\sum_{i=1}^{n-1} u_i\Big) - f(u_n)\Big| \\
&\quad + \Big|f\Big(\sum_{i=1}^{n-1} u_i\Big) - \sum_{i=1}^{n-1} f(u_i)\Big| \\
&\leq b\Big|\sum_{i=1}^{n-1} u_i u_n\Big|^{\alpha} + b \sum_{1\leq i<j\leq n-1} |u_i u_j|^{\alpha} \\
&\leq b\Big(\sum_{i=1}^{n-1} |u_i u_n|\Big)^{\alpha} + b \sum_{1\leq i<j\leq n-1} |u_i u_j|^{\alpha} \\
&\leq b \sum_{i<j} |u_i u_j|^{\alpha}.
\end{aligned}$$

This proves (2.3). Similarly, if $n = 2$, inequality (2.4) is part of assumption (H6). Assume (2.4) holds for $n-1$. After relabeling we may assume that $|u_n| \leq |u_i| \leq \widetilde{C}_1$ for all $i = 1, \ldots, n-1$. Then, $|u_i u_j|^{\alpha} |u_n| \leq \widetilde{C}_1^{2\alpha} |u_i u_j u_n|^{2/3}$. Using (H6) and (2.3) we obtain

$$\begin{aligned}
&\Big|F\Big(\sum_{i=1}^{n} u_i\Big) - \sum_{i=1}^{n} F(u_i) - \sum_{i\neq j} f(u_i) u_j\Big| \\
&\quad \leq \Big|F\Big(\sum_{i=1}^{n} u_i\Big) - F\Big(\sum_{i=1}^{n-1} u_i\Big) - F(u_n) - f\Big(\sum_{i=1}^{n-1} u_i\Big) u_n - \sum_{i=1}^{n-1} u_i f(u_n)\Big| \\
&\qquad + \Big|F\Big(\sum_{i=1}^{n-1} u_i\Big) - \sum_{i=1}^{n-1} F(u_i) - \sum_{i\neq j\in\{1,\ldots,n-1\}} f(u_i) u_j\Big| \\
&\qquad + \Big|f\Big(\sum_{i=1}^{n-1} u_i\Big) u_n - \sum_{i=1}^{n-1} f(u_i) u_n\Big| \\
&\quad \leq b\Big|\sum_{i=1}^{n-1} u_i u_n\Big|^{2\alpha} + \widetilde{C}_2\Big(\sum_{1\leq i<j\leq n-1} |u_i u_j|^{2\alpha} + \sum_{1\leq i<j<k\leq n-1} |u_i u_j u_k|^{2/3}\Big) \\
&\qquad + b \sum_{1\leq i<j\leq n-1} |u_i u_j|^{\alpha} |u_n| \\
&\quad \leq \widetilde{C}_2\Big(\sum_{i<j} |u_i u_j|^{2\alpha} + \sum_{i<j<k} |u_i u_j u_k|^{2/3}\Big)
\end{aligned}$$

with a suitably chosen constant $\widetilde{C}_2$. This proves (2.4). □

Lemma 2.3. *Consider a triangle in $\mathbb{R}^N$, with vertices $x_1, x_2, x_3 \in \mathbb{R}^N$ and side lengths $w \le v \le u$. Denote $s := \min_{x\in\mathbb{R}^N} \sum_{k=1}^{3} |x - x_k|$. Then the following hold:*

(a) *If one of the interior angles is larger than or equal to $2\pi/3$, then $s = v + w$.*

(b) *In any case, $s \ge (w + v + u)/2$.*

Proof. For the following facts from triangle geometry see for example [18]. The minimum s is achieved in a unique point x_0 in $\mathbb{R}^N$. In case (a) that point is the vertex of the triangle with the largest interior angle, so the claim follows immediately.

To prove (b), observe that adding up the inequalities $|x_i - x_0| + |x_j - x_0| \ge |x_i - x_j|$, $i \ne j$, yields

$$2s = 2\sum_{k=1}^{3} |x_0 - x_k| \ge w + v + u \qquad \forall x \in \mathbb{R}^N.$$

□

Lemma 2.4. *For $n \in \mathbb{N}$ there is a constant $C = C(n)$ such that if $x_1, x_2 \in \mathbb{R}^n$ satisfy $|x_1 - x_2| < 1$ and if $r \in [1, |x_1 - x_2| + 1]$ then*

$$|B_r(x_2)\backslash B_1(x_1)|_n \le C\big(|x_1 - x_2| + r - 1\big), \tag{2.5}$$

$$\sup_{x\in\partial B_r(x_2)} \operatorname{dist}(x, \partial B_1(x_1)) \le |x_1 - x_2| + r - 1, \tag{2.6}$$

$$\sup_{x\in\partial B_1(x_1)} \operatorname{dist}(x, \partial B_r(x_2)) \le |x_1 - x_2| + r - 1. \tag{2.7}$$

Here $|\cdot|_n$ denotes n-dimensional Lebesgue measure.

Proof. For $k \in \mathbb{N}$ let ω_k denote the volume of the unit ball in $\mathbb{R}^k$. Set $c := |x_1 - x_2|$. Without loss of generality we may suppose that $x_1 = 0$ and $x_2 = (c, 0, \dots, 0)$. Setting $B_1 := B_1(0)$ and using that $r \in [1, 2]$ we obtain

$$\begin{aligned} m_n(B_r(x_2)\backslash B_1) &\le m_n(B_1(x_2)\backslash B_1) + m_n(B_r(x_2)\backslash B_1(x_2)) \\ &= m_n(B_1\backslash B_1(x_2)) + \omega_n(r^n - 1) \\ &\le m_n(B_1\backslash B_1(x_2)) + \omega_n(2^n - 1)(r - 1). \end{aligned}$$

Write $x = (y, y') \in \mathbb{R}^n$ with $y' \in \mathbb{R}^{n-1}$. Set $B^{\pm} := \{(y, y') \in B_1 \mid \pm y \ge c/2\}$. Then

$$m_n(B_1\backslash B_1(x_2)) = \omega_n - 2m_n(B^+) = m_n(B_1\backslash(B^+ \cup B^-)) \le \omega_{n-1}c.$$

Together with the previous estimate this proves (2.5). An obvious geometric argument proves (2.6) and (2.7). □

2.2 Analysis of Linear Operators and the Limit Problem

The following lemma motivates the use of $-\lambda_{1,1}$ as a lower bound for λ in (1.2).

Lemma 2.5. *If $\lambda_1(\mathbb{L})$ denotes the bottom of the spectrum of $-\Delta$ in $L^2(\mathbb{L})$ with Dirichlet boundary conditions, then it holds that $\lambda_1(\mathbb{L}) = \lambda_{1,1}$.*

Proof. Set $u(x) := \vartheta_{1,1}(\eta)$ for $x = (\xi, \eta) \in \mathbb{L}$. Then $u > 0$ in $\mathbb{L}$, u is bounded, and $-\Delta u = \lambda_{1,1} u$. A variation of the second proof of [17, Thm. C.4.1] shows that $\lambda_{1,1} \in \sigma(-\Delta)$. On the other hand, the proof of [17, Thm. C.8.1] applies in our present situation (see also the proof of Corollary 2.7 below), and therefore the existence of the function u constructed above yields $\inf \sigma(-\Delta) \geq \lambda_{1,1}$. □

We will prove a partial analogue of this result that is sufficient for our needs, for the expanding domains Ω_R: The bottom of the spectrum of $-\Delta$ has an inferior limit larger than or equal to $\lambda_{1,1}$ as $R \to \infty$. But first we prove the existence of positive superharmonic functions for $-\Delta + \lambda$ on these domains. This will imply the spectral estimate as well as a maximum principle for $-\Delta + \lambda$.

Lemma 2.6. *There exists a superharmonic function for $-\Delta + \lambda$ in $C^2(\mathbb{L}) \cap C(\overline{\mathbb{L}})$ that is positive on $\overline{\mathbb{L}}$. If R is large enough then there exists a superharmonic function for $-\Delta + \lambda$ in $C^2(\Omega_R) \cap C(\overline{\Omega_R})$ that is positive on $\overline{\Omega_R}$.*

Proof. Recall that $\lambda_{1,1} + \lambda > 0$. We fix $r > 1$ close enough to 1 such that $\lambda_{1,r} + \lambda > 0$. Then $W(\xi, \eta) := \vartheta_{1,r}(\eta)$ satisfies

$$(-\Delta + \lambda)W = (\lambda_{1,r} + \lambda)W > 0$$

in $\mathbb{L}$ and

$$\min_{\overline{\mathbb{L}}} W > 0.$$

It therefore yields the desired positive superharmonic function on $\mathbb{L}$.

To construct a positive superharmonic function for $-\Delta + \lambda$ in Ω_R note first that for $R \geq 1$ large enough the set $\{x \in \mathbb{R}^N \mid \operatorname{dist}(x, \Gamma_R) < 2\}$ is a tubular neighborhood of Γ_R. Hence Ω_R is a C^1-domain, with bounded local geometric data as $R \to \infty$. Denote by π the corresponding tubular projection onto Γ_R.

Since $\vartheta_{1,r}$ only depends on $|\eta|$ we will abuse notation and write $\vartheta_{1,r}(s) := \vartheta_{1,r}(\eta)$ for $s \in [0, r]$, using any $\eta \in \overline{B_r^{N-1}}$ that satisfies $|\eta| = s$.

With these provisions we define $W(y) := \vartheta_{1,r}(|y - \pi(y)|)$ for $y \in \overline{\Omega_R}$. It is obvious that

$$\liminf_{R\to\infty} \min_{\overline{\Omega_R}} W > 0.$$

We claim that

$$W \in C^2(\Omega_R) \cap C(\overline{\Omega_R}) \tag{2.8}$$

and

$$\liminf_{R\to\infty} \min_{\Omega_R}\big((-\Delta+\lambda)W\big) > 0. \tag{2.9}$$

In proving the claims we denote by z^ξ and z^η the ξ- and η-components of $z \in \mathbb{R}\times\mathbb{R}^{N-1}$. Fix $y_0 \in \Omega_R$. We will prove the claims near y_0 using an adapted coordinate system centered at $\pi(y_0) \in \Gamma_R$. After translating and rotating we may assume that $\pi(y_0) = 0$ and $T_0\Gamma_R = \mathbb{R}\times\{0\}^{N-1}$. For some small $\varepsilon > 0$ denote by $\tau\colon (-\varepsilon,\varepsilon) \to \mathbb{R}^N$ a parametrization by arc length of the patch of Γ_R centered at 0, imposing $\tau(0) = 0$ and $\dot\tau(0) = \dot\gamma_R(t_0)/|\gamma_R(t_0)|$, where $t_0 = \gamma_R^{-1}(0)$. Note that $|\dot\tau(\xi)| = 1$ for all $\xi \in (-\varepsilon,\varepsilon)$ and

$$\ddot\tau(0) \to 0, \qquad \dddot\tau(0) \to 0 \qquad \text{as } R\to\infty, \tag{2.10}$$

independently of y_0.

Now define

$$h(\xi,\eta) := (0,\eta) - (\eta\cdot\dot\tau^\eta(\xi))\dot\tau(\xi)$$

for $\xi \in (-\varepsilon,\varepsilon)$ and $\eta \in B_1^{N-1}$, and

$$\Phi(\xi,\eta) := \tau(\xi) + \frac{h(\xi,\eta)}{|h(\xi,\eta)|}|\eta|.$$

In effect, $h(\xi,\eta)$ is the orthogonal projection of $(0,\eta)$ onto $(T_{\tau(\xi)}\Gamma_R)^\perp$. Since $\dot\tau^\eta(0) = 0$ it holds that $\Phi(0,\eta) = (0,\eta)$. Moreover,

$$\mathrm{D}\Phi(0,\eta) = \begin{pmatrix} 1-\eta\cdot\ddot\tau^\eta(0) & 0 \\ 0 & I_{N-1}\end{pmatrix}. \tag{2.11}$$

Hence Φ has a twice continuously differentiable inverse Ψ, a chart of Ω_R, defined on a neighborhood of $\overline{0\times B_1^{N-1}}$, if R is large enough.

We now represent W in these curved coordinates, first observing that $\pi\Phi(\xi,\eta) = \tau(\xi)$. If y has the representation $\Phi(\xi,\eta)$ then

$$|y-\pi(y)| = |\eta| = |\Psi^\eta(y)|. \tag{2.12}$$

Therefore

$$W(y) = \vartheta_{1,r}(|y-\pi(y)|) = \vartheta_{1,r}(\Psi^\eta(y)).$$

This identity proves (2.8).

Recall that presently $\pi(y_0) = 0$. To prove (2.9) it is sufficient to prove that

$$(-\Delta+\lambda)W(0,\eta) \ge C > 0 \tag{2.13}$$

for $\eta \in B_1^{N-1}$ and large R, where C is independent of y_0 and R.

From (2.11) we obtain

$$\mathrm{D}\Psi(0,\eta) = \begin{pmatrix} \frac{1}{1-\eta\cdot\ddot{\tau}^{\eta}(0)} & 0 \\ 0 & I_{N-1} \end{pmatrix}.$$

Moreover, $\Psi(0,\eta) = (0,\eta)$. Using these facts a straightforward calculation yields

$$(-\Delta + \lambda)W(0,\eta) = (\lambda_{1,r} + \lambda)\vartheta_{1,r}(\eta) + O(|\mathrm{D}^2\Phi(0,\eta)|),$$

independently of y_0. A tedious but elementary calculation using (2.10) shows that

$$\mathrm{D}^2\Phi(0,\eta) \to 0 \qquad \text{as } R \to \infty, \text{ independently of } y_0 \text{ and } \eta. \tag{2.14}$$

Since $\vartheta_{1,r}$ is positive and continuous on $\overline{B_1^{N-1}}$ we can set

$$C := \frac{\lambda_{1,r} + \lambda}{2} \min_{|\eta|\leq 1} \vartheta_{1,r}(\eta) > 0$$

and obtain (2.13). □

Corollary 2.7. *If $\lambda_1(\Omega_R)$ denotes the bottom of the spectrum of $-\Delta$ in $L^2(\Omega_R)$ with Dirichlet boundary conditions, then it holds that*

$$\liminf_{R\to\infty} \lambda_1(\Omega_R) \geq \lambda_{1,1}.$$

Proof. Let R be large enough such that Lemma 2.6 provides us with a positive superharmonic function W for $-\Delta + \lambda$ on Ω_R. It follows that

$$\frac{\Delta W}{W} \leq \lambda \qquad \text{on } \Omega_R. \tag{2.15}$$

For any $\varphi \in C_c^\infty(\Omega_R)$ (2.15) implies after two partial integrations

$$\begin{aligned}\int_{\Omega_R} (-\Delta\varphi + \lambda\varphi)\varphi \,\mathrm{d}x &\geq \int_{\Omega_R} \left(|\nabla\varphi|^2 + \varphi^2 \frac{\Delta W}{W} \right) \mathrm{d}x \\ &= \int_{\Omega_R} \left(|\nabla\varphi|^2 - \nabla W \cdot \left(\frac{2W\varphi\nabla\varphi - \varphi^2\nabla W}{W^2} \right) \right) \mathrm{d}x \\ &= \int_{\Omega_R} W^2 \left| \nabla \left(\frac{\varphi}{W} \right) \right|^2 \mathrm{d}x \\ &\geq 0.\end{aligned}$$

Since $C_c^\infty(\Omega_R)$ is dense in the domain of $-\Delta + \lambda$ with Dirichlet boundary conditions we obtain $-\Delta + \lambda \geq 0$ for R large enough, that is,

$$\liminf_{R\to\infty} \lambda_1(\Omega_R) \geq -\lambda.$$

The observation that the same argument applies to all $\lambda > -\lambda_{1,1}$ proves the claim. □

Remark 2.8. By Lemma 2.6 and [19, Theorem 1] the operator $-\Delta + \lambda$ satisfies the strong maximum principle on any subdomain of $\mathbb{L}$, and of Ω_R if R is large enough. This fact will be used throughout the rest of the paper without further comment.

We shall need the following decay estimates for the solutions $U^\pm$ of the limit problem (1.3), which is an immediate consequence of [6, Proposition 4.2]:

Lemma 2.9. *There are constants $C_1, C_2 > 0$ such that*

$$C_1 \mathrm{e}^{-\mu|\xi|}\vartheta_{1,1}(\eta) \le |U^\pm(\xi,\eta)| \le C_2 \mathrm{e}^{-\mu|\xi|}\vartheta_{1,1}(\eta) \qquad \text{for all } (\xi,\eta) \in \mathbb{L}.$$

Remark 2.10. The previous lemma implies that all the quantities $|U^\pm(x)|$, $|\mathrm{D}U^\pm(x)|$, and $|\mathrm{D}^2U^\pm(x)|$ are bounded by $C\mathrm{e}^{-\mu|x|}$ with a suitable constant $C > 0$.

3 Asymptotics of the Energy and its Gradient

Recall the definition of A_t and $U^\pm_{x,R}$ in Section 1.

For $R > 0$ large enough and $x \in \Gamma_R$ denote by $V^\pm_{x,R}$ the unique solution of the problem

$$\begin{cases} -\Delta u + \lambda u = f(U^\pm_{x,R}) & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases}$$

customarily called the projection of $U^\pm_{x,R}$ into Ω_R. It is well defined because of (H1) and Corollary 2.7. We consider $V^\pm_{x,R}$ to be extended by 0 to all of $\mathbb{R}^N$.

3.1 The Closed Tube Case

In this subsection we assume that $\gamma(0) = \gamma(1)$.

Lemma 3.1. *There are positive constants C_3, C_4, and R_0, independent of $x \in \Gamma_R$, such that the quantities $|V^\pm_{x,R}(y)|$ and $|\mathrm{D}V^\pm_{x,R}(y)|$ are bounded by $C_3\mathrm{e}^{-C_4|y-x|}$ for all $R \ge R_0$ and almost all $y \in \mathbb{R}^N$. Moreover, $|\mathrm{D}^2V^\pm_{x,R}(y)|$ is bounded uniformly on Ω_R, independently of $R \ge R_0$.*

Proof. The proof of these uniform decay estimates is straightforward if $\lambda > 0$. In that case $\mathrm{e}^{-\nu|x|}$ is an exponentially decaying superharmonic function for $-\Delta + \lambda$ on $\mathbb{R}^N \backslash B_1^N$ if $\nu > 0$ is small enough. The maximum principle can then be used to bound $|V^\pm_{x,R}(y)|$ by a multiple of $\mathrm{e}^{-\nu|y-x|}$. This simple approach does not apply if $\lambda \le 0$ and explains why a detailed proof is required.

For fixed $x \in \Gamma_R$ and an appropriate $T > 0$ let $\gamma_R\colon [0,T] \to \mathbb{R}^N$ denote a parametrization of Γ_R by arc length, such that $\gamma_R(0) = \gamma_R(T) = x$. Define $d^\pm\colon \Gamma_R \to \mathbb{R}_0^+$ by setting $d^\pm(x) := 0$, and $d^+(y) := \gamma_R^{-1}(y) \in (0,T)$ as well as $d^-(y) := T - \gamma_R^{-1}(y) \in (0,T)$ for

$y \neq x$. Hence $d^+(y)$ and $d^-(y)$ measure the length of the path from x to y in the natural Riemannian metric of Γ_R, in one and in the other direction.

Fix $r > 1$ as in the proof of Lemma 2.6 such that $\lambda_{1,r} + \lambda > 0$, denote by $\nu > 0$ a constant to be fixed later and define $W^\pm \colon \Omega_R \to \mathbb{R}$ by

$$W^\pm(y) := \mathrm{e}^{-\nu d^\pm(\pi(y))} \vartheta_{1,r}(|y - \pi(y)|).$$

We claim that there is $\nu_0 > 0$ such that for $\nu \in (0, \nu_0)$ there are $\widetilde{C}_0 > 0$ and R_0 such that

$$(-\Delta + \lambda) W^\pm(y) \geq \widetilde{C}_0 \mathrm{e}^{-\nu d^\pm(\pi(y))} \tag{3.1}$$

for all $R \geq R_0$ and $y \in \Omega_R \backslash \pi^{-1}(x)$. Moreover, r, ν_0, $\widetilde{C}_0$ and R_0 do not depend on x.

Fix $y_0 \in \Omega_R \backslash \pi^{-1}(x)$. We will prove the claim in y_0 using an adapted coordinate system centered at $\pi(y_0) \in \Gamma_R$. After translating and rotating we may assume that $\pi(y_0) = 0$ and $T_0\Gamma_R = \mathbb{R} \times \{0\}^{N-1}$. Define ε, τ, Φ and Ψ as in the proof of Lemma 2.6. We represent $W^\pm$ using the chart Ψ at $\pi(y_0)$. Since τ is a parametrization by arc length we have

$$\tau(d^+(y) - d^+(0)) = y \qquad \text{for all } y \in \tau((-\varepsilon, \varepsilon)).$$

Therefore $\Phi(d^+(y) - d^+(0), 0) = y$ and thus

$$\Psi^\xi(y) = d^+(y) - d^+(0) \qquad \text{for all } y \in \tau((-\varepsilon, \varepsilon)). \tag{3.2}$$

From (3.2) and (2.12) it follows that

$$\begin{aligned} W^+(y) &= \mathrm{e}^{-\nu d^+(\pi(y))} \vartheta_{1,r}(|y - \pi(y)|) \\ &= \mathrm{e}^{-\nu d^+(0)} \mathrm{e}^{-\nu(d^+(\pi(y)) - d^+(0))} \vartheta_{1,r}(|y - \pi(y)|) \\ &= \mathrm{e}^{-\nu d^+(0)} \mathrm{e}^{-\nu \Psi^\xi(y)} \vartheta_{1,r}(\Psi^\eta(y)). \end{aligned}$$

Recall that presently $\pi(y_0) = 0$. To prove the claim it is therefore sufficient to prove for

$$\widetilde{W}(y) := \mathrm{e}^{-\nu \Psi^\xi(y)} \vartheta_{1,r}(\Psi^\eta(y))$$

that

$$(-\Delta + \lambda)\widetilde{W}(0, \eta) \geq C > 0 \tag{3.3}$$

for $\eta \in B_1^{N-1}$ and large R, where C is independent of x, y_0 and R.

As earlier one calculates

$$\begin{aligned} &(-\Delta + \lambda)\widetilde{W}(0, \eta) \\ &\qquad = \left(-\left(\frac{\nu}{1 - \eta \cdot \ddot{\tau}^\eta(0)} \right)^2 + \lambda_{1,r} + \lambda \right) \vartheta_{1,r}(\eta) + O(|\mathrm{D}^2\Phi(0, \eta)|), \end{aligned} \tag{3.4}$$

independently of x and y_0. We set $\nu_0 := \sqrt{\lambda_{1,r} + \lambda}$. For any $\nu \in (0, \nu_0)$ we can find the desired constants C in (3.3) and R_0 by (2.10), (2.14), (3.4), and by the fact that $\vartheta_{1,r}$ is bounded below on $\overline{B_1^{N-1}}$ by a positive constant. The function W^- is treated analogously, and together these facts prove the claim.

Geometric considerations imply the existence of a constant $\widetilde{C}_1 > 0$, independent of R and x, such that

$$\begin{aligned} d^+(\gamma_R(t)) &\le \widetilde{C}_1 |\gamma_R(t) - x|, \qquad t \in [0, T/2], \\ d^-(\gamma_R(t)) &\le \widetilde{C}_1 |\gamma_R(t) - x|, \qquad t \in [T/2, T]. \end{aligned}$$

We fix $\nu > 0$ small enough such that (3.1) is satisfied for certain R_0 and $\widetilde{C}_0$ and such that

$$\nu \widetilde{C}_1 < p_1 \mu,$$

and we choose $\widetilde{C}_2 > 0$ large enough such that

$$e^{-\nu \widetilde{C}_1} \widetilde{C}_0 \widetilde{C}_2 e^{-\nu \widetilde{C}_1 |y-x|} \ge |f(U^+_{x,R}(y))| \tag{3.5}$$

for all $y \in \mathbb{R}^N$ and $R \ge R_0$. This is possible since

$$|f(U^+_{x,R}(y))| \le C e^{-p_1 \mu |y-x|}$$

by Lemma 2.9 and condition (H4).

Now define the function $W := W^+ + W^-$. If $y \in \Omega_R \backslash \pi^{-1}(x)$ is such that $\pi(y) = \gamma_R(t)$ for some $t \in [0, T/2]$ then

$$\begin{aligned} (-\Delta + \lambda)\widetilde{C}_2 W(y) \ge \widetilde{C}_2 \widetilde{C}_0 e^{-\nu d^+(\gamma_R(t))} &\ge \widetilde{C}_2 \widetilde{C}_0 e^{-\nu \widetilde{C}_1 |\gamma_R(t) - x|} \\ &\ge \widetilde{C}_2 \widetilde{C}_0 e^{-\nu \widetilde{C}_1 (|y-x|+1)} = e^{-\nu \widetilde{C}_1} \widetilde{C}_2 \widetilde{C}_0 e^{-\nu \widetilde{C}_1 |y-x|}. \end{aligned}$$

Similarly, if $t \in [T/2, T]$, then

$$(-\Delta + \lambda)\widetilde{C}_2 W(y) \ge \widetilde{C}_2 \widetilde{C}_0 e^{-\nu d^-(\gamma_R(t))} \ge e^{-\nu \widetilde{C}_1} \widetilde{C}_2 \widetilde{C}_0 e^{-\nu \widetilde{C}_1 |y-x|}.$$

All in all we obtain from (3.5) that

$$(-\Delta + \lambda)\widetilde{C}_2 W \ge f(U^+_{x,R}) \qquad \text{in } \Omega_R \backslash \pi^{-1}(x). \tag{3.6}$$

Using regularity estimates it is easy to see that we have a uniform L^∞-bound for $V^+_{x,R}$ as $R \to \infty$. Taking $\widetilde{C}_2$ even larger if necessary and recalling that $V^+_{x,R}$ vanishes on $\mathbb{R}^N \backslash \Omega_R$ we can achieve $\widetilde{C}_2 W \ge V^+_{x,R}$ on $\partial(\Omega_R \backslash \pi^{-1}(x))$. Hence (3.6) and the maximum principle (see Remark 2.8) imply that $\widetilde{C}_2 W \ge V^+_{x,R} \ge 0$ in $\mathbb{R}^N$. On the other hand,

$$W(y) \le C e^{-\nu |\pi(y) - x|} \le e^{\nu} C e^{-\nu |y-x|}.$$

Hence we have proved that $|V_{x,R}^+(y)| \le \widetilde{C}_3 \mathrm{e}^{-\widetilde{C}_4|y-x|}$ for all $R \ge R_0$ and all $y \in \mathbb{R}^N$, with positive constants $\widetilde{C}_3, \widetilde{C}_4$ that are independent of x. In a similar way one obtains this bound for $V_{x,R}^-$, and the remaining estimates follow from regularity theory and the fact that the boundary of Ω_R has local geometric data that remains bounded uniformly as $R \to \infty$. □

Lemma 3.2. *If $p > 0$ then we have the following asymptotic estimates as $R \to \infty$, independently of $x \in \Gamma_R$:*

$$\int_{\mathbb{R}^N} |V_{x,R}^\pm - U_{x,R}^\pm|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}), \tag{3.7}$$

$$\int_{\mathbb{R}^N} |\nabla V_{x,R}^\pm - \nabla U_{x,R}^\pm|^2 \,\mathrm{d}y = O(R^{-1}), \tag{3.8}$$

$$\int_{\mathbb{R}^N} |F(V_{x,R}^\pm) - F(U_{x,R}^\pm)| \,\mathrm{d}y = O(R^{-1}), \tag{3.9}$$

$$\int_{\mathbb{R}^N} |f(V_{x,R}^\pm) - f(U_{x,R}^\pm)|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}). \tag{3.10}$$

Proof. There exist $\delta, \rho > 0$ such that for every $x \in \Gamma$ the following is true: Assuming after a translation and a rotation that $x = 0$ and that Γ is tangent to $\mathbb{R} \times \{0\}$ in x, there is a C^3-map $h\colon (-\rho, \rho) \to B_\delta^{N-1}$, the ball of radius δ and center 0 in $\mathbb{R}^{N-1}$, such that

$$\Gamma \cap \big((-\rho,\rho) \times B_\delta^{N-1}\big) = \{(t, h(t)) \mid t \in (-\rho,\rho)\}.$$

It follows that $h(0) = \dot h(0) = 0$. Moreover, by taking ρ sufficiently small, we can assume that $|\ddot h(t)|$ and $|\dddot h(t)|$ are bounded, independently of $t \in (-\rho, \rho)$ and x. Similarly, setting $h_R(t) := Rh(t/R)$, it holds that

$$\Gamma_R \cap \big((-\rho R,\rho R) \times B_{\delta R}^{N-1}\big) = \{(t, h_R(t)) \mid t \in (-\rho R,\rho R)\}.$$

We claim that there is a constant C, independent of x, such that

$$|h_R(t)| \le \frac{Ct^2}{R} \tag{3.11}$$

for all $t \in (-\rho R, \rho R)$, and such that

$$\{t\} \times B_1^{N-1}(h_R(t)) \subseteq \big(\{t\} \times \mathbb{R}^{N-1}\big) \cap \Omega_R \subseteq \{t\} \times B_{1+C(1+t^2)/R^2}^{N-1}(h_R(t)) \tag{3.12}$$

for all $t \in (-\rho R + 1, \rho R - 1)$ and R large enough, independently of x.

To prove the claim we first note that

$$h_R^{(k)}(t) = h^{(k)}(t/R)/R^{k-1} \qquad \text{for } k = 0,1,2,3. \tag{3.13}$$

It follows that

$$|h_R(t)| = \left| h_R(0) + \dot{h}_R(0)t + t^2 \int_0^1 \ddot{h}_R(st)(1-s)\,\mathrm{d}s \right|$$
$$\leq \frac{t^2}{R} \int_0^1 |\ddot{h}(st/R)|(1-s)\,\mathrm{d}s$$

and hence (3.11). We also need an estimate for the derivative of h_R:

$$|\dot{h}_R(t)| = |\dot{h}(t/R)| = \left| \int_0^{t/R} \ddot{h}(s)\,\mathrm{d}s \right| \leq \frac{C|t|}{R}. \tag{3.14}$$

It is clear that the first inclusion of (3.12) holds. Suppose now that $t \in (-\rho R+1, \rho R-1)$ and $y \in \mathbb{R}^{N-1}$ such that $(t,y) \in \Omega_R$. We need to show that there is a constant $C > 0$, independent of x, t and R, such that

$$|y - h_R(t)| \leq 1 + \frac{C(1+t^2)}{R^2}. \tag{3.15}$$

Because of $\operatorname{dist}((t,y),\Gamma_R) < 1$ there is a unique pair $t_1 \in (-\rho R, \rho R)$ and $v = (v_\xi, v_\eta) \in \mathbb{R} \times \mathbb{R}^{N-1}$ such that $v \perp T_{(t_1,h_R(t_1))}\Gamma_R$ and $(t,y) = (t_1, h_R(t_1)) + v$. It follows that $|v| < 1$. Since $T_{(t_1,h_R(t_1))}\Gamma_R$ is the subspace of $\mathbb{R}^N$ spanned by $(1, \dot{h}_R(t_1))$, it also holds that

$$v_\xi + v_\eta \cdot \dot{h}_R(t_1) = 0.$$

Therefore we find $|v_\eta| \leq 1$ and

$$|v_\xi| \leq |v_\eta|\,|\dot{h}_R(t_1)| \leq \frac{C|t_1|}{R},$$

by (3.14). Using $t = t_1 + v_\xi$, $|v_\xi| \leq 1$ and $y = h_R(t_1) + v_\eta$ this yields

$$|v_\xi| \leq \frac{C(1+|t|)}{R},$$

$$|y - h_R(t)| = \left| v_\eta - \dot{h}_R(t)(t_1 - t) - (t_1 - t)^2 \int_0^1 \ddot{h}_R(t + s(t_1 - t))(1-s)\,\mathrm{d}s \right|$$
$$\leq |v_\eta| + |\dot{h}_R(t)|\,|v_\xi| + |v_\xi|^2 \int_0^1 |\ddot{h}_R(t - sv_\xi)|(1-s)\,\mathrm{d}s \leq 1 + \frac{C(1+t^2)}{R^2},$$

and hence (3.15). This finishes the proof of (3.11) and (3.12).

Let us denote

$$Q_R := (-R^{1/4}, R^{1/4}) \times B^{N-1}_{R^{1/4}}(0).$$

There are positive constants $\widetilde{C}_1, \widetilde{C}_2$ such that

$$|V_{x,R}^{\pm}(\xi,\eta) - U_{x,R}^{\pm}(\xi,\eta)|^p \leq \widetilde{C}_1 \mathrm{e}^{-\widetilde{C}_2(|\xi|+|\eta|)} \tag{3.16}$$

for all $(\xi,\eta) \in \mathbb{R} \times \mathbb{R}^{N-1}$, by Lemma 3.1. From this it follows that

$$\int_{\mathbb{R}^N \setminus Q_R} |V_{x,R}^{\pm} - U_{x,R}^{\pm}|^p \, \mathrm{d}y = O(R^{-1}) \tag{3.17}$$

as $R \to \infty$. We consider the partition of $Q_R \cap (\Omega_R \cup \mathbb{L})$ into the three sets

$$Q_R \cap (\Omega_R \setminus \mathbb{L}), \; Q_R \cap (\mathbb{L} \setminus \Omega_R), \; Q_R \cap \mathbb{L} \cap \Omega_R \tag{3.18}$$

and estimate the respective integrals individually. Note that the integral over $Q_R \setminus (\Omega_R \cup \mathbb{L})$ vanishes. First we have for large R by (3.16), (3.12), Lemma 2.4 and (3.11) that

$$\begin{aligned}
&\int_{Q_R \cap (\Omega_R \setminus \mathbb{L})} |V_{x,R}^{\pm} - U_{x,R}^{\pm}|^p \, \mathrm{d}y \\
&\quad \leq \widetilde{C}_1 \int_{-R^{1/4}}^{R^{1/4}} \mathrm{e}^{-\widetilde{C}_2|\xi|} \left| B_{1+C(1+\xi^2)/R^2}^{N-1}(h_R(\xi)) \setminus B_1^{N-1}(0) \right|_{N-1} \mathrm{d}\xi \\
&\quad \leq C \int_{-R^{1/4}}^{R^{1/4}} \mathrm{e}^{-\widetilde{C}_2|\xi|} \left(|h_R(\xi)| + C(1+\xi^2)/R^2 \right) \mathrm{d}\xi \\
&\quad \leq \frac{C}{R} \int_{-\infty}^{\infty} \mathrm{e}^{-\widetilde{C}_2|\xi|} (1+\xi^2) \, \mathrm{d}\xi \\
&\quad = O(R^{-1})
\end{aligned} \tag{3.19}$$

as $R \to \infty$ (recall that $|\cdot|_{N-1}$ denotes $(N-1)$-dimensional Lebesgue measure). Using the first inclusion in (3.12) it follows similarly that

$$\int_{Q_R \cap (\mathbb{L} \setminus \Omega_R)} |V_{x,R}^{\pm} - U_{x,R}^{\pm}|^p \, \mathrm{d}y = O(R^{-1}) \tag{3.20}$$

as $R \to \infty$.

To estimate the integral over the third set in (3.18) we set $D_R := Q_R \cap \mathbb{L} \cap \Omega_R$ and claim that

$$|V_{x,R}^{\pm}(\xi,\eta) - U_{x,R}^{\pm}(\xi,\eta)| \leq C \mathrm{e}^{-C_4|\xi|} \, \frac{1+\xi^2}{R} \tag{3.21}$$

for all $(\xi,\eta) \in \partial D_R$, with a suitable constant C that is independent of x and R. Suppose that $(\xi,\eta) \in \partial\mathbb{L}$ is such that $\xi \in (-R^{1/4}, R^{1/4})$. Considering the geometry in the hyperplane $\{\xi\} \times \mathbb{R}^{N-1}$ an application of Lemma 2.4, (3.12), and (3.11) yields

$$\operatorname{dist}((\xi,\eta), \partial\Omega_R) \leq |h_R(\xi)| + C\frac{1+\xi^2}{R^2} \leq C\frac{1+\xi^2}{R}. \tag{3.22}$$

Similarly, if $(\xi,\eta)\in\partial\Omega_R$ is such that $\xi\in(-R^{1/4},R^{1/4})$ then

$$\operatorname{dist}((\xi,\eta),\partial\mathbb{L})\le C\frac{1+\xi^2}{R}.$$

Recalling that $U^{\pm}_{x,R}$ vanishes on $\partial\mathbb{L}$ and $V^{\pm}_{x,R}$ vanishes on $\partial\Omega_R$, the gradient estimates in Lemma 3.1 imply that (3.21) holds on $\partial D_R\cap Q_R$. Finally, the bounds on $U^{\pm}_{x,R}$ and $V^{\pm}_{x,R}$ given in Lemma 3.1 show that (3.21) also holds on $\partial D_R\cap\partial Q_R$, proving the claim.

By definition, $V^{\pm}_{x,R}-U^{\pm}_{x,R}$ is harmonic for $-\Delta+\lambda$ in D_R. Recall the definitions of $\lambda_{1,r}$ and $\vartheta_{1,r}$ from the proof of Lemma 3.1. Choose $r>1$ and $\nu_0>0$ as in that proof, and fix a positive $\nu<\min\{\nu_0,C_4\}$. Then $W(\xi,\eta):=\mathrm{e}^{-\nu\xi}\vartheta_{1,r}(\eta)$ is superharmonic for $-\Delta+\lambda$ in $\mathbb{L}$, and W satisfies

$$W(\xi,\eta)\ge C\mathrm{e}^{-\nu\xi}\qquad\text{on }\overline{\mathbb{L}}.$$

This, (3.21) and the choice of ν imply the existence of $C>0$, independent of R, such that

$$|V^{\pm}_{x,R}-U^{\pm}_{x,R}|\le\frac{C}{R}W$$

on ∂D_R. Hence the maximum principle implies that

$$|V^{\pm}_{x,R}(\xi,\eta)-U^{\pm}_{x,R}(\xi,\eta)|\le\frac{C}{R}W(\xi,\eta)=\frac{C}{R}\mathrm{e}^{-\nu\xi}\vartheta_{1,r}(\eta)$$

holds on D_R, with C independent of x and R. Similarly, setting $W(\xi,\eta):=\mathrm{e}^{\nu\xi}\vartheta_{1,r}(\eta)$ in the argument above, we get

$$|V^{\pm}_{x,R}(\xi,\eta)-U^{\pm}_{x,R}(\xi,\eta)|\le\frac{C}{R}\mathrm{e}^{\nu\xi}\vartheta_{1,r}(\eta)$$

on D_R. Therefore,

$$\int_{D_R}|V^{\pm}_{x,R}-U^{\pm}_{x,R}|^p\,\mathrm{d}y=O(R^{-p})\tag{3.23}$$

as $R\to\infty$. Combining (3.19), (3.20) and (3.23) we obtain (3.7).

To prove (3.8) we first note that

$$\int_{\mathbb{R}^N\setminus Q_R}|\nabla V^{\pm}_{x,R}-\nabla U^{\pm}_{x,R}|^2\,\mathrm{d}y=O(R^{-1}),\tag{3.24}$$

$$\int_{Q_R\cap(\Omega_R\setminus\mathbb{L})}|\nabla V^{\pm}_{x,R}-\nabla U^{\pm}_{x,R}|^2\,\mathrm{d}y=O(R^{-1})\tag{3.25}$$

and

$$\int_{Q_R\cap(\mathbb{L}\setminus\Omega_R)}|\nabla V^{\pm}_{x,R}-\nabla U^{\pm}_{x,R}|^2\,\mathrm{d}y=O(R^{-1})\tag{3.26}$$

as $R \to \infty$, by Lemma 3.1 and by the same arguments as in the proofs of (3.17), (3.19) and (3.20).

To treat the remaining integral over the set D_R we remark that $U^{\pm}_{x,R}$ and $V^{\pm}_{x,R}$ can be extended from functions on $\mathbb{L}$ and Ω_R to C^2-functions in neighborhoods of $\mathbb{L}$ and Ω_R, respectively (recall that we are assuming $x = 0$ and $T_x\Gamma_R = \mathbb{R} \times \{0\}$). Denote by Y_R the difference of these extensions, defined on a neighborhood of $\overline{D_R}$.

Note that the boundaries $\partial\Omega_R$ and $\partial\mathbb{L}$ are C^1-submanifolds of $\mathbb{R}^N$. By (3.11), (3.13), and (3.14) we have that

$$\sup_{t\in[-R^{1/4},R^{1/4}]} |h_R^{(k)}(t)| \to 0 \tag{3.27}$$

as $R \to \infty$, for $k = 0, 1, 2, 3$. Since the minimum of two continuously differentiable maps is Lipschitz continuous, it follows that D_R has Lipschitz boundary if R is large enough. Hence we can apply the Gauss-Green theorem, see e.g. [20, Theorem 5.8.2] and the remark immediately following it, and obtain from $(-\Delta + \lambda)Y_R = 0$ in D_R that

$$\int_{D_R} |\nabla Y_R|^2 \, \mathrm{d}x = \int_{\partial D_R} Y_R \, n_R(x) \cdot \nabla Y_R \, \mathrm{d}H_{N-1}(x) - \lambda \int_{D_R} Y_R^2 \, \mathrm{d}x \tag{3.28}$$

Here $n_R(x)$ denotes the measure theoretic exterior normal to ∂D_R at $x \in \partial D_R$, and H_{N-1} denotes $(N-1)$-dimensional Hausdorff measure. By Lemma 3.1 ∇Y_R is bounded uniformly, and independently of R. Hence (3.28) and (3.21) imply

$$\begin{aligned}
&\int_{D_R} |\nabla Y_R|^2 \, \mathrm{d}x \\
&\qquad \le \frac{C}{R} \left(\int_{\partial D_R} \mathrm{e}^{-C_4|\xi|} \, (1+|\xi|)^2 \, \mathrm{d}H_{N-1}(\xi,\eta) + \int_{D_R} \mathrm{e}^{-C_4|\xi|} \, (1+|\xi|)^2 \, \mathrm{d}(\xi,\eta) \right) \\
&\qquad = O(R^{-1}).
\end{aligned}$$

Together with (3.24)–(3.26) this proves (3.8).

Equations (3.9) and (3.10) follow easily from (3.7) since $U^{\pm}_{x,R}$ and $V^{\pm}_{x,R}$ are bounded uniformly as $R \to \infty$ by Lemma 3.1 and because F and f are continuously differentiable. □

Proposition 3.3.

$$\sup_{x\in\Gamma_R} \|V^{\pm}_{x,R} - U^{\pm}_{x,R}\|_{H^1(\mathbb{R}^N)} = O(R^{-1/2}), \tag{3.29}$$

$$\sup_{x\in\Gamma_R} |J_{\Omega_R}(V^{\pm}_{x,R}) - J_{\mathbb{L}}(U^{\pm})| = O(R^{-1}), \tag{3.30}$$

$$\sup_{x\in\Gamma_R} \|\nabla J_{\Omega_R}(V^{\pm}_{x,R})\|_{H_0^1(\Omega_R)} = O(R^{-1/2}), \tag{3.31}$$

as $R \to \infty$.

Proof. The first two asymptotic estimates (3.29) and (3.30) are obvious from (3.7) with $p = 2$, (3.8) and (3.9). To prove (3.31) fix R and $x \in \Gamma_R$, and consider $v \in H_0^1(\Omega_R)$ with $\|v\|_{H_0^1(\Omega_R)} = 1$. The definition of $V_{x,R}^{\pm}$ yields

$$|\mathrm{D}J_{\Omega_R}(V_{x,R}^{\pm})[v]| = \left|\int_{\Omega_R} (f(U_{x,R}^{\pm}) - f(V_{x,R}^{\pm}))v \,\mathrm{d}y\right| \leq \|f(U_{x,R}^{\pm}) - f(V_{x,R}^{\pm})\|_{L^2(\Omega_R)}.$$

The asymptotic estimate therefore follows from (3.10), using $p = 2$. □

For k as defined in the statement of Theorem 1.1 denote $n := 2k$,

$$E^{\pm} := J_{\mathbb{L}}(U^{\pm})$$

and

$$E_n := k(E^+ + E^-).$$

For $m = 1, 2$ consider functions $g_m\colon \mathbb{R}^+ \to \mathbb{R}^+$ (to be fixed later) satisfying

$$g_2 < g_1, \tag{3.32}$$

$$g_m(R) \to \infty \qquad \text{as } R \to \infty, \text{ for } m = 1, 2 \tag{3.33}$$

$$g_m(R) = o(R) \qquad \text{as } R \to \infty, \text{ for } m = 1, 2 \tag{3.34}$$

for $m = 1, 2$. Define $D_{m,R}$ as the set of points $(x_1, x_2, \ldots, x_n)$ in $(\Gamma_R)^n$ such that there are $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ and $|x_i - x_j| \leq g_m(R)$. Hence $D_{2,R} \subseteq D_{1,R}$. For $m = 1, 2$ put

$$\mathcal{U}_{m,R} := \{(x_1, x_2, \ldots, x_n) \in (\Gamma_R)^n \backslash D_{m,R} \mid (x_1, x_2, \ldots, x_n) \text{ is an } n\text{-chain}\}.$$

Then $\mathcal{U}_{1,R}$ and $\mathcal{U}_{2,R}$ are open subsets of $(\Gamma_R)^n$ such that $\overline{\mathcal{U}_{1,R}} \subset \mathcal{U}_{2,R}$.

Writing $X = (x_1, x_2, \ldots, x_n) \in \mathcal{U}_{2,R}$, define $\varphi_R\colon \mathcal{U}_{2,R} \to H_0^1(\Omega_R)$ by

$$\varphi_R(X) := \sum_{i=1}^{k} (V_{x_{2i-1},R}^+ + V_{x_{2i},R}^-).$$

Proposition 3.4. *Fix some $\alpha' \in (1/2, \alpha)$, where α is given in* (H6)*. Then*

$$\sup_{X \in \mathcal{U}_{2,R}} \|\nabla J_{\Omega_R}(\varphi_R(X))\|_{H_0^1(\Omega_R)} = O(\mathrm{e}^{-\alpha' \mu g_2(R)}) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. For fixed X we set

$$\overline{U}_i := \begin{cases} U_{x_i,R}^+ & \text{if } i \text{ is odd} \\ U_{x_i,R}^- & \text{if } i \text{ is even,} \end{cases} \tag{3.35}$$

and

$$\overline{V}_i := \begin{cases} V^+_{x_i,R} & \text{if } i \text{ is odd} \\ V^-_{x_i,R} & \text{if } i \text{ is even.} \end{cases} \tag{3.36}$$

Then

$$\varphi_R(X) = \sum_{i=1}^n \overline{V}_i.$$

If $v \in H^1_0(\Omega_R)$ satisfies $\|v\|_{H^1_0(\Omega_R)} = 1$ then we estimate, using (3.31), (2.3), (3.7) with $p = 2\alpha$ and the fact that the functions $\overline{U}_i$ and $\overline{V}_i$ are bounded uniformly by Lemma 3.1 and Lemma 2.9, and Lemma 2.1:

$$\begin{aligned}
&\big|\mathrm{D}J_{\Omega_R}(\varphi_R(X))[v]\big| \\
&\qquad = \left|\sum_{i=1}^n \mathrm{D}J_{\Omega_R}(\overline{V}_i)[v] + \int_{\Omega_R}\left(\sum_{i=1}^n f(\overline{V}_i) - f\Big(\sum_{i=1}^n \overline{V}_i\Big)\right) v\,\mathrm{d}y\right| \\
&\qquad \le \sum_{i=1}^n \|\nabla J_{\Omega_R}(\overline{V}_i)\|_{H^1_0(\Omega_R)} + \left(\int_{\Omega_R}\left|\sum_{i=1}^n f(\overline{V}_i) - f\Big(\sum_{i=1}^n \overline{V}_i\Big)\right|^2 \mathrm{d}y\right)^{\frac12} \\
&\qquad \le O(R^{-1/2}) + C\sum_{i<j}\left(\int_{\Omega_R} |\overline{V}_i\overline{V}_j|^{2\alpha}\,\mathrm{d}y\right)^{\frac12} \\
&\qquad = O(R^{-1/2}) + C\sum_{i<j}\left(\int_{\mathbb{R}^N} |\overline{U}_i\overline{U}_j|^{2\alpha}\,\mathrm{d}y\right)^{\frac12} \\
&\qquad = O(R^{-1/2}) + O(\mathrm{e}^{-\alpha'\mu g_2(R)}).
\end{aligned}$$

Clearly these asymptotics are independent of the choice of X. □

Proposition 3.5. *There is $\beta > 0$ such that*

$$\inf_{X\in\partial\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \ge E_n + \beta\mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R \to \infty$.

Proof. We define the n-circular index distance $d_n\colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$ as follows:

$$d_n(i, j) := \min\{|i - j|, |i - j + n|, |i - j - n|\}.$$

Consider $X = (x_1, x_2, \ldots, x_n) \in \partial\mathcal{U}_{1,R}$ and define $\overline{U}_i$ and $\overline{V}_i$ as in (3.35) and (3.36). We claim that

$$|x_i - x_j| \ge 4g_1(R)/3 \qquad \text{for large } R, \text{ if } d_n(i, j) \ge 2. \tag{3.37}$$

To see this, recall that $g_1(R) = o(R)$ as $R \to \infty$ and argue by contradiction: If points x_i, x_j satisfy $d_n(i, j) \ge 2$ and $|x_i - x_j| \le 4g_1(R)/3$, then they tend to line up, together with all x_ℓ

where $i < \ell < j$, along a straight line, a tangent to Γ_R, as $R \to \infty$ (possibly after a circular shift on X). There is always at least one intermediate point x_{ℓ_0} with $i < \ell_0 < j$, and the minimum distance between each pair of points from the set $\{x_\ell\}_{\ell=1}^n$ is $g_1(R)$. Hence we must have $|x_i - x_j| \geq 5g_1(R)/3$ for large R, a contradiction.

Condition (H5) implies that

$$f(\overline{U}_i)\overline{U}_j \leq 0 \qquad \text{if } d_n(i,j) = 1. \tag{3.38}$$

Since $X \in \partial\mathcal{U}_{1,R}$ and by (3.37) there are i_0, j_0 with $d_n(i_0, j_0) = 1$ and $|x_{i_0} - x_{j_0}| = g_1(R)$ if R is large. Moreover, from Lemma 2.9 it follows that there are $r, \varepsilon > 0$ such that $|f(\overline{U}_{i_0})| \geq \varepsilon$ and $|\overline{U}_{j_0}| \geq C_1 \mathrm{e}^{-\mu g_1(R)}$ in $B_r(x_{i_0})$, independently of the choice of $X \in \partial\mathcal{U}_{1,R}$ and for R large enough. To see this consider also (3.34) since it implies that $B_r(x_{i_0})$ is in the support of $\overline{U}_{j_0}$ for large R. Hence

$$\frac{1}{2}\int_{\mathbb{R}^N} |f(\overline{U}_{i_0})\overline{U}_{j_0}|\,\mathrm{d}x \geq \beta \mathrm{e}^{-\mu g_1(R)} \tag{3.39}$$

for some $\beta > 0$ and for large R. On the other hand, using (3.37) and Lemmas 2.1 and Lemma 2.9 we obtain

$$\int_{\mathbb{R}^N} |f(\overline{U}_i)\overline{U}_j|\,\mathrm{d}x = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{if } d_n(i,j) \geq 2 \tag{3.40}$$

as $R \to \infty$.

Fix any $i, j, \ell \in \{1, 2, \dots, n\}$ and set

$$s(R) := \min_{x \in \mathbb{R}^N} (|x - x_i| + |x - x_j| + |x - x_\ell|).$$

We claim that then

$$s(R) \geq 2g_1(R) \qquad \text{if } R \text{ is large enough.} \tag{3.41}$$

This can be seen as follows: If

$$\max\{|x_i - x_j|, |x_i - x_\ell|, |x_j - x_\ell|\} = o(R)$$

as $R \to \infty$, then since $x_i, x_j, x_\ell \in \Gamma_R$ Lemma 2.3(a) implies (3.41). On the other hand, if, say, $|x_j - x_\ell| \geq CR$ for some $C > 0$ as $R \to \infty$, then by (b) of Lemma 2.3 it follows that $s(R) \geq CR/2 \geq 2g_1(R)$ for large R, by (3.34).

Using Lemma 2.2, Equation (3.7) with $p = 2\alpha$ and $p = 2/3$, respectively, as well as the uniform boundedness of $\overline{U}_i$ and $\overline{V}_i$, Lemma 2.1 together with (3.41) and the fact that

$\alpha > 1/2$, we estimate one of the error terms that appears in the upcoming energy estimates:

$$\begin{aligned}
&\left|\int_{\Omega_R}\left(F\Big(\sum_i \overline{V}_i\Big) - \sum_i F(\overline{V}_i)\right)\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j\,\mathrm{d}x\right| \\
&\qquad \le C\sum_{i<j}\int_{\Omega_R}|\overline{V}_i\overline{V}_j|^{2\alpha}\,\mathrm{d}x + C\sum_{i<j<\ell}\int_{\Omega_R}|\overline{V}_i\overline{V}_j\overline{V}_\ell|^{2/3}\,\mathrm{d}x \\
&\qquad = C\sum_{i<j}\int_{\mathbb{R}^N}|\overline{U}_i\overline{U}_j|^{2\alpha}\,\mathrm{d}x + C\sum_{i<j<\ell}\int_{\mathbb{R}^N}|\overline{U}_i\overline{U}_j\overline{U}_\ell|^{2/3}\,\mathrm{d}x + O(R^{-2/3}) \\
&\qquad = o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).
\end{aligned} \tag{3.42}$$

In the following asymptotic estimate we make use of (3.30), the definition of E_n, a partial integration, (3.42), (3.7), (3.38), (3.39), and (3.40):

$$\begin{aligned}
&J_{\Omega_R}(\varphi_R(X)) \\
&\qquad = \sum_{i=1}^{n} J_{\Omega_R}(\overline{V}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R}(\nabla\overline{V}_i\cdot\nabla\overline{V}_j + \lambda\overline{V}_i\overline{V}_j)\,\mathrm{d}x \\
&\qquad\qquad - \int_{\Omega_R}\left(F\Big(\sum_i \overline{V}_i\Big) - \sum_i F(\overline{V}_i)\right)\mathrm{d}x \\
&\qquad = E_n + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{V}_j\,\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j\,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\qquad = E_n - \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\qquad \ge E_n + \frac{1}{2}\int_{\mathbb{R}^N}|f(\overline{U}_{i_0})\overline{U}_{j_0}|\,\mathrm{d}x - \frac{1}{2}\sum_{d_n(i,j)\ge 2}\int_{\mathbb{R}^N}|f(\overline{U}_i)\overline{U}_j|\,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\qquad \ge E_n + \beta\mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).
\end{aligned}$$

It is clear that this asymptotic estimate is independent of X. □

Proposition 3.6. *It holds that*

$$\inf_{X\in\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \le E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R\to\infty$.

Proof. Pick numbers t_i such that $0 < t_1 < t_2 < \cdots < t_n < 1$. Denote $x_{R,i} := R\gamma(t_i) \in \Gamma_R$ for $i = 1, 2, \ldots, n$, and set $X_R := (x_{R,1}, x_{R,2}, \ldots, x_{R,n})$. Then $X_R \in \mathcal{U}_{1,R}$ for large R

because of (3.34). As in the proof of Proposition 3.5, and using the same definition for $\overline{U}_i$, we obtain

$$J_{\Omega_R}(\varphi_R(X_R)) = E_n - \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).$$

Choose $\varepsilon \in (0,\mu)$ and $\delta \in \big(0, \min_{i\neq j}|\gamma(t_i)-\gamma(t_j)|\big)$. The claim follows from $|x_{R,i}-x_{R,j}| \geq \delta R$ for $i\neq j$, because Lemma 2.1 and Lemma 2.9 imply that

$$\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x = O(\mathrm{e}^{-(\mu-\varepsilon)\delta R}) = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{if } i\neq j.$$

□

3.2 The Open-End Tube Case

We now suppose that $\gamma(0)\neq\gamma(1)$.

Define for $r,s>0$ the finite cylinder

$$\mathbb{L}_{r,s} := (-r,s)\times B_1^{N-1}.$$

Recall the definitions of $U^{\pm}$ from (H7), of A_t and $U^{\pm}_{x,R}$ from Section 1, and of $V^{\pm}_{x,R}$ from the beginning of Section 3. We denote by $\widetilde{U}^{\pm}_{r,s}$ the unique solution of

$$\begin{cases} -\Delta u + \lambda u = f(U^{\pm}) & \text{in } \mathbb{L}_{r,s},\\ u = 0 & \text{on } \partial\mathbb{L}_{r,s}.\end{cases}$$

It is well defined because of (H1), and because Lemma 2.5 implies that $\lambda(\mathbb{L}_{r,s}) \geq \lambda_{1,1}$. Here $\lambda(\mathbb{L}_{r,s})$ denotes, as before, the bottom of the spectrum of $-\Delta$ with Dirichlet boundary conditions on $\mathbb{L}_{r,s}$.

Again $\widetilde{U}^{\pm}_{r,s}$ is considered as extended to $\mathbb{R}^N$ by 0. If $R>0$, $x\in\Gamma_R$ and $t\in[0,1]$ such that $x = R\gamma(t)$ then set

$$W^{\pm}_{x,R}(y) := \widetilde{U}^{\pm}_{|x-R\gamma(0)|,|x-R\gamma(1)|}(A_t(y-x))$$

for all $y\in\mathbb{R}^N$.

To estimate the error when replacing $W^{\pm}_{x,R}$ by $U^{\pm}_{x,R}$ we will need the following

Lemma 3.7. *With the constants C_2 from Lemma 2.9 it holds that*

$$0 \leq \pm\widetilde{U}^{\pm}_{r,s}(\xi,\eta) \leq \pm U^{\pm}(\xi,\eta) \tag{3.43}$$

and

$$|U^{\pm}(\xi,\eta) - \widetilde{U}^{\pm}_{r,s}(\xi,\eta)| \leq C_2\vartheta_{1,1}(\eta)\big(\mathrm{e}^{-\mu(r+|\xi+r|)} + \mathrm{e}^{-\mu(s+|\xi-s|)}\big) \tag{3.44}$$

for $(\xi,\eta)\in\mathbb{L}$. There are $C_5, C_6>0$ such that

$$C_5\mathrm{e}^{-\mu\min\{r,s\}} \leq \|U^{\pm}-\widetilde{U}^{\pm}_{r,s}\|_{H_0^1(\mathbb{L})} \leq C_6\mathrm{e}^{-\mu\min\{r,s\}}. \tag{3.45}$$

Proof. First we note that $U^\pm$ and $\widetilde{U}^\pm_{r,s}$ are in $C^2(\mathbb{L}_{r,s}) \cap C(\overline{\mathbb{L}_{r,s}})$. The interior regularity follows from standard regularity theory. For continuity up to the boundary see [14, 15].

Set $Y_{r,s} := U^\pm - \widetilde{U}^\pm_{r,s}$. Suppose that $r, s > 0$. We claim that

$$\begin{aligned} C_1 \vartheta_{1,1}(\eta) \max\bigl\{\mathrm{e}^{-\mu(r+|\xi+r|)}, \mathrm{e}^{-\mu(s+|\xi-s|)}\bigr\} &\le Y_{r,s} \\ &\le C_2 \vartheta_{1,1}(\eta) \bigl(\mathrm{e}^{-\mu(r+|\xi+r|)} + \mathrm{e}^{-\mu(s+|\xi-s|)}\bigr) \end{aligned} \tag{3.46}$$

holds for $(\xi, \eta) \in \mathbb{L}$, where C_1 and C_2 are as in Lemma 2.9. To see this, observe that on $\overline{\mathbb{L}} \backslash \mathbb{L}_{r,s}$ (3.46) holds trivially, since there $\widetilde{U}^\pm_{r,s} = 0$, and by Lemma 2.9. Consider the equalities

$$(-\Delta + \lambda)\vartheta_{1,1}(\eta)\mathrm{e}^{-\mu(r+|\xi+r|)} = 0,$$
$$(-\Delta + \lambda)\vartheta_{1,1}(\eta)\mathrm{e}^{-\mu(s+|\xi-s|)} = 0,$$

and

$$(-\Delta + \lambda) Y_{r,s} = 0$$

that hold in $\mathbb{L}_{r,s}$. Now the maximum principle, applied to the operator $-\Delta + \lambda$ on $\mathbb{L}_{r,s}$, implies that (3.46) also holds on $\mathbb{L}_{r,s}$. Here one treats the two exponentials on the left hand side separately.

To prove (3.43) observe that since $(-\Delta + \lambda)(\pm \widetilde{U}^\pm_{r,s}) = \pm f(U^\pm) \ge 0$ in $\mathbb{L}_{r,s}$, the maximum principle implies $\pm \widetilde{U}^\pm_{r,s} \ge 0$. The other inequality in (3.43) is a consequence of (3.46). This proves the claim.

Coming to the proof of (3.45), a straightforward calculation using (3.46) yields

$$\|Y_{r,s}\|_{L^2(\mathbb{L})} = O(\mathrm{e}^{-\mu \min\{r,s\}}) \tag{3.47}$$

as $r, s \to \infty$. For the moment we may consider $Y_{r,s}$ extended from $\mathbb{L}_{r,s}$ by odd reflection to the finite cylinder $\overline{\mathbb{L}_{r+1,s+1}}$. If $r + s \ge 1$ then this extension is a classical harmonic function for $-\Delta + \lambda$, continuous up to the boundary, and taking the value 0 on the boundary portion $T := [-r-1, s+1] \times (\partial B_1^{N-1})$. This can be seen as in the proof of [15, Lemma 9.12]. Using domains $\Omega_t := (t-1, t+1) \times B_1^{N-1}$ and $\Omega'_t := (t - 1/2, t + 1/2) \times B_1^{N-1}$ we have $\Omega'_t \subset\subset \Omega_t \cup T$ in the sense described on page 9 of [15]. Therefore Theorem 9.13 *loc. cit.* implies a bound

$$\|\nabla Y_{r,s}\|^2_{L^2(\Omega'_t)} \le C \|Y_{r,s}\|^2_{L^2(\Omega_t)}$$

with constant C that is independent of $t \in [-r, s]$. Summing up this inequality over appropriately chosen center points t_i such that the Ω'_{t_i} form a cover of $\mathbb{L}_{r+1/2,s+1/2}$ and using the exponential bound for $\nabla Y_{r,s}$ in $\mathbb{L} \backslash \mathbb{L}_{r,s}$ given by Lemma 3.1 we obtain

$$\|\nabla Y_{r,s}\|_{L^2(\mathbb{L})} = O(\mathrm{e}^{-\mu \min\{r,s\}}).$$

Together with (3.47) this gives the right half of (3.45).

To prove the left half of (3.45) it is sufficient to show that

$$\|Y_{r,s}\|_{L^2(\mathbb{L})} \geq C\mathrm{e}^{-\mu \min\{r,s\}}, \tag{3.48}$$

where C is independent of r and s. Therefore, consider the fact that

$$r + |\xi + r| \leq s + |\xi - s| \quad \text{if and only if} \quad \xi \leq s - r.$$

It implies via another straightforward calculation that

$$\int_{\mathbb{R}} \max\left\{\mathrm{e}^{-\mu(r+|\xi+r|)}, \mathrm{e}^{-\mu(s+|\xi-s|)}\right\} \mathrm{d}\xi = \frac{1}{\mu}(\mathrm{e}^{-2\mu r} - \mathrm{e}^{-2\mu(r+s)} + \mathrm{e}^{-2\mu s}). \tag{3.49}$$

Now the fact that $a - ab + b \geq \max\{a, b\}$ if $a, b \in [0, 1]$ yields that the right hand side of (3.49) is larger than or equal to

$$\frac{1}{\mu}\max\{\mathrm{e}^{-2\mu r}, \mathrm{e}^{-2\mu s}\} = \frac{1}{\mu}\mathrm{e}^{-2\mu \min\{r,s\}}.$$

Together with (3.46) we obtain (3.48), and the proof is complete. □

Lemma 3.8. *There are constants $C_3, C_4 > 0$, independent of $R \geq 1$ and $x \in \Gamma_R$, such that the quantities $|V^{\pm}_{x,R}(y)|$ and $|\mathrm{D}V^{\pm}_{x,R}(y)|$ are bounded by $C_3\mathrm{e}^{-C_4|y-x|}$ for almost all $y \in \mathbb{R}^N$. Moreover, $|\mathrm{D}^2V^{\pm}_{x,R}(y)|$ is bounded uniformly on Ω_R, independently of R.*

Proof. The proof is analogous to that of Lemma 3.1, with one notable change: If π denotes the tubular projection $\Omega_R \to \Gamma_R$ and if $d\colon \Gamma_R \to \mathbb{R}^+_0$ denotes, for fixed $x \in \Gamma_R$, the distance to x in the natural Riemannian metric of Γ_R, then one uses

$$W(y) := \mathrm{e}^{-\nu d(\pi(y))}\vartheta_{1,r}(|y - \pi(y)|),$$

as a supersolution on $\Omega_R \backslash \pi^{-1}(x)$, instead of $W^{\pm}$ as defined previously. Of course, ν and r have to be chosen appropriately in the same way as before. □

The error of replacing $V^{\pm}_{x,R}$ by $W^{\pm}_{x,R}$ will be estimated by

Lemma 3.9. *If $p > 0$ then we have the following asymptotic estimates as $R \to \infty$, independently of $x \in \Gamma_R$:*

$$\int_{\mathbb{R}^N} |V^{\pm}_{x,R} - W^{\pm}_{x,R}|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}), \tag{3.50}$$

$$\int_{\mathbb{R}^N} |\nabla V^{\pm}_{x,R} - \nabla W^{\pm}_{x,R}|^2 \,\mathrm{d}y = O(R^{-1}), \tag{3.51}$$

$$\int_{\mathbb{R}^N} |F(V^{\pm}_{x,R}) - F(W^{\pm}_{x,R})| \,\mathrm{d}y = O(R^{-1}), \tag{3.52}$$

$$\int_{\mathbb{R}^N} |f(V^{\pm}_{x,R}) - f(W^{\pm}_{x,R})|^p \,\mathrm{d}y = O(R^{-\min\{p,1\}}). \tag{3.53}$$

Proof. The proof is similar to the proof of Lemma 3.2 in many parts, but it requires some new geometric considerations that need to be explained in detail. To do this it seems clearer to work with sequences, as the location of x on Γ_R is more important now. Suppose therefore that $R_n \to \infty$ and $x_n \in \Gamma_{R_n}$. Passing to a subsequence it suffices to consider the following two cases:

a) $|x_n - R_n\gamma(0)| \geq 2R_n^{1/4}$ and $|x_n - R_n\gamma(1)| \geq 2R_n^{1/4}$ for all n. In view of Lemma 3.8 the proof of (3.50)–(3.53), replacing x by x_n and R by R_n, is the same as for Lemma 3.2.

b) For each n either $|x_n - R_n\gamma(0)| < 2R_n^{1/4}$ or $|x_n - R_n\gamma(1)| < 2R_n^{1/4}$ holds. We consider the second case, as the first one is analogous. Therefore we assume from now on that

$$|x_n - R_n\gamma(1)| < 2R_n^{1/4} \qquad \text{for all } n \in \mathbb{N}. \tag{3.54}$$

We fix $\varepsilon > 0$ such that there is a regular C^3-extension $\widetilde{\gamma}\colon [-\varepsilon, 1] \to \mathbb{R}^N$ of γ without self-intersection. If $t_n \in [0,1]$ is such that $x_n = R_n\gamma(t_n)$ then we set

$$\begin{aligned} \gamma_n(t) &:= A_{t_n}(\gamma(t) - \gamma(t_n)), && t \in [0,1],\\ \widetilde{\gamma}_n(t) &:= A_{t_n}(\widetilde{\gamma}(t) - \gamma(t_n)), && t \in [-\varepsilon,1], \end{aligned}$$

and

$$\begin{aligned} \Gamma_n &:= \{R_n\gamma_n(t) \mid t \in [0,1]\},\\ \widetilde{\Gamma}_n &:= \{R_n\widetilde{\gamma}_n(t) \mid t \in [-\varepsilon,1]\}. \end{aligned}$$

Hence Γ_n is obtained from Γ_{R_n} by translating x_n to 0 and transforming $\Gamma_{R_n} - x_n$ with A_{t_n} such that $T_0\Gamma_n = \mathbb{R} \times \{0\}^{N-1}$. Moreover, $\widetilde{\Gamma}_n$ contains Γ_n as a submanifold. We define Ω_n and $\widetilde{\Omega}_n$ as in (1.1) using, respectively, γ_n and $\widetilde{\gamma}_n$ in place of γ, and letting t range over $[-\varepsilon, 1]$ in the case of $\widetilde{\Omega}_n$. We also denote

$$\mathbb{L}_n := \mathbb{L}_{|R_n\gamma(0) - x_n|, |R_n\gamma(1) - x_n|}$$

and

$$\widetilde{Q}_n := (-R_n^{1/3}, R_n^{1/3}) \times B^{N-1}_{R_n^{1/3}}(0).$$

Let $y_n := (\xi_n, \eta_n) := R_n\gamma_n(1)$ and set $r_n := |y_n| = |x_n - R_n\gamma(1)|$. By the definition of γ_n we have $\xi_n \geq 0$. Denote by $h_n\colon [-R_n^{1/3}, R_n^{1/3}] \to \mathbb{R}^{N-1}$ the map such that $\widetilde{\Gamma}_n \cap \widetilde{Q}_n$ is the graph of h_n. Note that equations (3.11) and (3.14) still hold for h_n instead of h. Moreover, as in (3.27) it holds that

$$\sup_{t \in [-R^{1/3}, R^{1/3}]} |h_n^{(k)}(t)| \to 0 \tag{3.55}$$

as $n \to \infty$, for $k = 0, 1, 2, 3$.

From $0 \leq \xi_n \leq r_n$ and $r_n^2 - \xi_n^2 = \eta_n^2 = h_n(\xi_n)^2$ we obtain, using (3.11),

$$|r_n - \xi_n| = \frac{h_n(\xi_n)^2}{|r_n + \xi_n|} \leq \frac{C\xi_n}{2R_n^2} \leq C\frac{r_n}{R_n^2} \tag{3.56}$$

for all n. Note also that

$$r_n \le 2R_n^{1/4}$$

by (3.54).

Now we set

$$\widetilde{V}_n(y) := V^{\pm}_{x_n,R_n}(x_n + A_{t_n}^{-1}y) \qquad \text{and} \qquad \widetilde{W}_n(y) := W^{\pm}_{x_n,R_n}(x_n + A_{t_n}^{-1}y)$$

for $y \in \mathbb{R}^N$. To prove (3.50) we need to show that

$$\int_{\mathbb{R}^N} |\widetilde{V}_n - \widetilde{W}_n|^p \,\mathrm{d}y = O(R_n^{-1}) \tag{3.57}$$

as $n \to \infty$. From Lemma 3.8 it follows as in the proof of (3.17) that

$$\int_{\mathbb{R}^N \setminus \widetilde{Q}_n} |\widetilde{V}_n - \widetilde{W}_n|^p \,\mathrm{d}y = O(R_n^{-1}).$$

To take care of the remaining set $\widetilde{Q}_n \cap (\Omega_n \cup \mathbb{L}_n)$ where the integrand is nonzero we consider its partition into the sets

$$\begin{aligned}
D_n^1 &:= \widetilde{Q}_n \cap (\Omega_n \Delta \mathbb{L}_n) \cap (\widetilde{\Omega}_n \Delta \mathbb{L}) \\
D_n^2 &:= \widetilde{Q}_n \cap (\Omega_n \Delta \mathbb{L}_n) \cap (\widetilde{\Omega}_n \cap \mathbb{L}) \\
D_n^3 &:= \widetilde{Q}_n \cap \Omega_n \cap \mathbb{L}_n
\end{aligned}$$

and treat them individually.

Since $D_n^1 \subseteq \widetilde{Q}_n \cap (\widetilde{\Omega}_n \Delta \mathbb{L})$ we can apply the same argument as in the proof of (3.19) and (3.20) to obtain

$$\int_{D_n^1} |\widetilde{V}_n - \widetilde{W}_n|^p \,\mathrm{d}y = O(R_n^{-1}). \tag{3.58}$$

To handle D_n^2 let α_n denote the angle between $(T_{y_n}\widetilde{\Gamma}_n)^\perp$ and $\{0\} \times \mathbb{R}^{N-1}$. It holds that

$$\tan \alpha_n = |\dot{h}_n(\xi_n)| \le C\frac{r_n}{R_n} \tag{3.59}$$

by (3.14). Since $\operatorname{diam}(B_1^{N-1}) = 2$ it follows that

$$D_n^2 \subseteq [\xi_n - 2\tan\alpha_n, r_n + 2\tan\alpha_n] \times B_1^{N-1} \subseteq [r_n - s_n, r_n + s_n] \times B_1^{N-1} \tag{3.60}$$

where $s_n \ge 0$ satisfies

$$s_n \le C\frac{r_n}{R_n^2} + C\frac{r_n}{R_n} \le C\frac{r_n}{R_n}. \tag{3.61}$$

Here we have used (3.56) and (3.59). Hence

$$\begin{aligned}
&\int_{D_n^2} |\widetilde{V}_n - \widetilde{W}_n|^p \,\mathrm{d}y \\
&\qquad \le C \int_{[r_n - s_n, r_n + s_n] \times B_1^{N-1}} \mathrm{e}^{-pC_4|\xi|} \,\mathrm{d}(\xi,\eta) \qquad \text{by (3.60) and Lemma 3.8} \\
&\qquad = C \int_{r_n - s_n}^{r_n + s_n} \mathrm{e}^{-pC_4\xi} \,\mathrm{d}\xi \\
&\qquad = C\mathrm{e}^{-pC_4 r_n} \sinh(pC_4 s_n) \\
&\qquad \le C\mathrm{e}^{-pC_4 r_n} \frac{r_n}{R_n} \qquad \text{by (3.61)} \\
&\qquad = O(R_n^{-1}).
\end{aligned} \tag{3.62}$$

The integral over the set D_n^3 is taken up next. If $(\xi,\eta) \in \partial D_n^3 \cap \partial \mathbb{L}$ then

$$\operatorname{dist}((\xi,\eta), \partial\widetilde{\Omega}_n) \le C\frac{1+\xi^2}{R_n},$$

as in (3.22). Similarly, if $(\xi,\eta) \in \partial D_n^3 \cap \partial\widetilde{\Omega}_n$ then

$$\operatorname{dist}((\xi,\eta), \partial\mathbb{L}) \le C\frac{1+\xi^2}{R_n}.$$

If $(\xi,\eta) \in \partial D_n^3 \cap \partial\Omega_n \cap \widetilde{\Omega}_n$ then

$$\operatorname{dist}((\xi,\eta), \partial\mathbb{L}_n \cap \mathbb{L}) \le 2s_n \le C\frac{r_n}{R_n} \le C\frac{\xi + s_n}{R_n} \le C\frac{\xi + r_n/R_n}{R_n} \le C\frac{1+\xi^2}{R_n}.$$

Similarly, if $(\xi,\eta) \in \partial D_n^3 \cap \partial\mathbb{L}_n \cap \mathbb{L}$ then

$$\operatorname{dist}((\xi,\eta), \partial\Omega_n \cap \widetilde{\Omega}_n) \le C\frac{1+\xi^2}{R_n}.$$

Since $\widetilde{V}_n = 0$ in $\mathbb{R}^N \backslash \Omega_n$ and $\widetilde{W}_n = 0$ in $\mathbb{R}^N \backslash \mathbb{L}_n$ we find, using Lemma 3.8, that

$$|\widetilde{V}_n - \widetilde{W}_n| \le C\mathrm{e}^{-C_4|\xi|}\frac{1+\xi^2}{R_n} \qquad \text{on } \partial D_n^3. \tag{3.63}$$

From $D_n^3 \subseteq \mathbb{L}$ it follows as in the proof of (3.23) that

$$\int_{D_n^3} |\widetilde{V}_n - \widetilde{W}_n|^p \,\mathrm{d}y = O(R_n^{-p}). \tag{3.64}$$

Now (3.57) is a consequence of (3.58), (3.62) and (3.64). This finishes the proof of (3.50).

To show (3.51) we need to prove that

$$\int_{\mathbb{R}^N} |\nabla \widetilde{V}_n - \nabla \widetilde{W}_n|^2 \, \mathrm{d}y = O(R_n^{-1}).$$

By (3.55) the proof of this fact is analogous to the proof of (3.8), using the refined partition of $\widetilde{Q}_n \cap (\Omega_n \cup \mathbb{L}_n)$ introduced in the present setting, and (3.63). Equations (3.52) and (3.53) follow from (3.50) as in the proof of Lemma 3.2. □

The functions g_m, $m = 1, 2$, from Section 3.1 will also be used in the present setting. To account for the new boundary part near $\partial\Gamma_R$ we need to change the definition of $D_{m,R}$ in comparison with Section 3.1. Here we define $D_{m,R}$ as the set of points $(x_1, x_2, \dots, x_n)$ in $(\Gamma_R)^n$ such that either there are $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ and $|x_i - x_j| \leq g_m(R)$ or there is $i \in \{1, 2, \dots, n\}$ with $2\,\mathrm{dist}(x_i, \partial\Gamma_R) \leq g_m$. The manifolds $\mathcal{U}_{m,R}$ are then defined as before.

Proposition 3.10. *It holds that*

$$\sup_{x \in \Gamma_R} \|V^{\pm}_{x,R} - W^{\pm}_{x,R}\|_{H^1_0(\mathbb{R}^N)} = O(R^{-1/2}), \tag{3.65}$$

$$\sup_{x \in \Gamma_R} |J_{\Omega_R}(V^{\pm}_{x,R}) - J_{\mathbb{L}}(W^{\pm}_{x,R})| = O(R^{-1}), \tag{3.66}$$

$$\sup_{\substack{x \in \Gamma_R \\ \mathrm{dist}(x, \partial\Gamma_R) \geq g_2(R)/2}} \|\nabla J_{\Omega_R}(V^{\pm}_{x,R})\|_{H^1_0(\Omega_R)} = O(R^{-1/2}) + O(\mathrm{e}^{-\min\{p_1,2\}\mu g_2(R)/2}) \tag{3.67}$$

as $R \to \infty$.

Proof. The asymptotic estimates (3.65) and (3.66) are immediate from Lemma 3.9. To prove (3.67) we first observe that $|tU^{\pm} + (1-t)\widetilde{U}^{\pm}_{r,s}| \leq |U^{\pm}|$ for every $t \in [0, 1]$ by (3.43). Moreover, (H4) implies that $|f'(u)| \leq C|u|^{p_1 - 1}$, where C depends only on an upper bound for $|u|$. Therefore Lemma 2.9 and (3.44) imply

$$\begin{aligned}
\int_{\mathbb{L}} &|f(U^{\pm}) - f(\widetilde{U}^{\pm}_{r,s})|^2 \, \mathrm{d}x \\
&\leq \int_{\mathbb{L}} \left(\int_0^1 |f'(tU^{\pm} + (1-t)\widetilde{U}^{\pm}_{r,s})| \, \mathrm{d}t \right)^2 |U^{\pm} - \widetilde{U}^{\pm}_{r,s}|^2 \, \mathrm{d}x \\
&\leq C \int_{\mathbb{R}} \mathrm{e}^{-2(p_1 - 1)\mu} \big(\mathrm{e}^{-2\mu(r + |\xi + r|)} + \mathrm{e}^{-2\mu(s + |\xi - s|)} \big) \, \mathrm{d}\xi \\
&= O(\mathrm{e}^{-2\min\{p_1,2\}\mu \min\{r,s\}})
\end{aligned} \tag{3.68}$$

as $r, s \to \infty$. The last asymptotic estimate follows from a straightforward calculation.

Now consider any $v \in H^1_0(\Omega_R)$ such that $\|v\|_{H^1_0(\Omega_R)} = 1$. A partial integration and the uniform boundedness of $U^{\pm}_{x,R}$, $V^{\pm}_{x,R}$ and $W^{\pm}_{x,R}$, Lipschitz continuity of f on bounded subsets of $\mathbb{R}$, (3.68), and (3.50) imply

$$\begin{aligned}|\mathrm{D}J_{\Omega_R}(V^{\pm}_{x,R})[v]| &= \left|\int_{\Omega_R}(f(U^{\pm}_{x,R})-f(V^{\pm}_{x,R}))v\,\mathrm{d}y\right| \\ &\le \|f(U^{\pm}_{x,R})-f(W^{\pm}_{x,R})\|_{L^2(\Omega_R)}+\|f(W^{\pm}_{x,R})-f(V^{\pm}_{x,R})\|_{L^2(\Omega_R)} \\ &= O(\mathrm{e}^{-\min\{p_1,2\}\mu g_2(R)/2})+O(R^{-1/2}).\end{aligned}$$

□

For n and k as defined in the statement of Theorem 1.1 define $E^{\pm}$ as in Section 3.1 and put

$$E_n := k(E^+ + E^-) + (n-2k)E^+.$$

Define $\varphi_R\colon \mathcal{U}_{2,R} \to H^1_0(\Omega_R)$ by

$$\varphi_R(X) := \sum_{i=1}^{k}(V^+_{x_{2i-1},R} + V^-_{x_{2i},R}) + (n-2k)V^+_{x_n,R}.$$

Proposition 3.11. *Choose $\alpha' \in (1/2, \min\{\alpha, p_1/2, 1\})$. Then*

$$\sup_{X\in\mathcal{U}_{2,R}} \|\nabla J_{\Omega_R}(\varphi_R(X))\|_{H^1_0(\Omega_R)} = O(\mathrm{e}^{-\alpha'\mu g_2(R)}) + O(R^{-1/2})$$

as $R \to \infty$.

Proof. For fixed $X = (x_1, x_2, \dots, x_n) \in \partial\mathcal{U}_{2,R}$ we define $\overline{U}^{\pm}_i$ and $\overline{V}^{\pm}_i$ as in the proof of Proposition 3.4 and set

$$\overline{W}_i := \begin{cases} W^+_{x_i,R} & \text{if } i \text{ is odd} \\ W^-_{x_i,R} & \text{if } i \text{ is even.}\end{cases}$$

Similarly as before, using Lemma 2.2, (3.67), (3.50) with $p = 2\alpha$, (3.43), and Lemmas 2.9 and 2.1 we obtain

$$\begin{aligned}&\left|\mathrm{D}J_{\Omega_R}(\varphi_R(X))[v]\right| \\ &\quad\le \sum_{i=1}^{n}\|\nabla J_{\Omega_R}(\overline{V}_i)\|_{H^1_0(\Omega_R)} + C\sum_{i<j}\left(\int_{\Omega_R}|\overline{V}_i\overline{V}_j|^{2\alpha}\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &\quad= O(R^{-1/2}) + O(\mathrm{e}^{-\min\{p_1,2\}\mu g_2(R)/2}) + C\sum_{i<j}\left(\int_{\Omega_R}|\overline{W}_i\overline{W}_j|^{2\alpha}\,\mathrm{d}y\right)^{\frac{1}{2}} \\ &\quad= O(R^{-1/2}) + O(\mathrm{e}^{-\alpha'\mu g_2(R)}).\end{aligned}$$

□

Proposition 3.12. *There is $\beta > 0$ such that*

$$\inf_{X \in \partial \mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \geq E_n + \beta \mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R \to \infty$.

Proof. As $r, s \to \infty$ consider the functions

$$v_{r,s} := \frac{\widetilde{U}^{\pm}_{r,s} - U^{\pm}}{\|\widetilde{U}^{\pm}_{r,s} - U^{\pm}\|_{H^1_0(\mathbb{L})}}$$

in $H^1_0(\mathbb{L})$. By (3.44) and (3.45) we may estimate for $(\xi, \eta) \in \mathbb{L}$

$$|v_{r,s}(\xi, \eta)| \leq C\big(\mathrm{e}^{-\mu|\xi + r|} + \mathrm{e}^{-\mu|\xi - s|}\big).$$

Since $|f'(U^{\pm}(\xi, \eta))| \leq C\mathrm{e}^{-(p_1-1)\mu|\xi|}$ it follows easily that

$$\int_{\mathbb{L}} f'(U^{\pm}) v_{r,s}^2 \,\mathrm{d}x \to 0 \qquad \text{as } r, s \to \infty.$$

This implies, recalling that $J'_{\mathbb{L}}(U^{\pm}) = 0$ and using (3.45), the following behavior:

$$\begin{aligned} J_{\mathbb{L}}(\widetilde{U}^{\pm}_{r,s}) &= J_{\mathbb{L}}(U^{\pm}) + \frac{1}{2} J''_{\mathbb{L}}(U^{\pm})[v_{r,s}, v_{r,s}] \, \|\widetilde{U}^{\pm}_{r,s} - U^{\pm}\|^2_{H^1_0(\mathbb{L})} \\ &\qquad\qquad\qquad\qquad + o(\|\widetilde{U}^{\pm}_{r,s} - U^{\pm}\|^2_{H^1_0(\mathbb{L})}) \\ &= E^{\pm} + \frac{1}{2}\|\widetilde{U}^{\pm}_{r,s} - U^{\pm}\|^2_{H^1_0(\mathbb{L})} + o(\|\widetilde{U}^{\pm}_{r,s} - U^{\pm}\|^2_{H^1_0(\mathbb{L})}) \\ &\geq E^{\pm} + C\mathrm{e}^{-2\mu \min\{r,s\}} \end{aligned} \tag{3.69}$$

as $r, s \to \infty$.

Fix $X = (x_1, x_2, \ldots, x_n) \in \partial \mathcal{U}_{1,R}$ and reuse the notation $\overline{U}_i$, $\overline{V}_i$, and $\overline{W}_i$ from the proof of Proposition 3.11. Similarly as in the proof of Proposition 3.5 we obtain

$$\left| \int_{\Omega_R} \left(F\Big(\sum_i \overline{V}_i\Big) - \sum_i F(\overline{V}_i) - \right) \mathrm{d}x - \sum_{i \neq j} \int_{\Omega_R} f(\overline{V}_i)\overline{V}_j \,\mathrm{d}x \right| = o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).$$

Here we need to use $\overline{W}_i$ instead of $\overline{U}_i$ in the intermediate step, and we make use of the fact that by (3.43) the functions $\overline{W}_i$ decay with exponent μ, as the functions $\overline{U}_i$ do.

We need to provide some extra estimates for the replacement of $W^{\pm}_{x,R}$ by $U^{\pm}_{x,R}$. We claim that for $i \neq j$ it holds that

$$\int_{\mathbb{R}^N} \big(f(\overline{W}_i) - f(\overline{U}_i)\big)\overline{W}_j \,\mathrm{d}y = o(\mathrm{e}^{-\mu g_1(R)}) \tag{3.70}$$

$$\int_{\mathbb{R}^N} f(\overline{U}_i)(\overline{W}_j - \overline{U}_j) \,\mathrm{d}y = o(\mathrm{e}^{-\mu g_1(R)}) \tag{3.71}$$

as $R \to \infty$. We only prove (3.70), the proof of the second estimate is similar. For clarity we work with sequences. Suppose therefore that $R_\ell \to \infty$, $y_\ell, z_\ell \in \Gamma_{R_\ell}$, $\mathrm{dist}(y_\ell, \partial\Gamma_{R_\ell}) \geq g_1(R_\ell)/2$, $\mathrm{dist}(z_\ell, \partial\Gamma_{R_\ell}) \geq g_1(R_\ell)/2$, and $|y_\ell - z_\ell| \geq g_1(R_\ell)$. We need to show that

$$\int_{\mathbb{R}^N} \big(f(W^\pm_{y_\ell,R_\ell}) - f(U^\pm_{y_\ell,R_\ell})\big) W^\pm_{z_\ell,R_\ell}\, \mathrm{d}y = o(\mathrm{e}^{-\mu g_1(R_\ell)})$$

as $\ell \to \infty$. It is convenient to make this calculation in a rotated coordinate system. Therefore set $r_\ell := |y_\ell - R_\ell\gamma(0)|$, $s_\ell := |y_\ell - R_\ell\gamma(1)|$. We assume without loss of generality that $r_\ell = \mathrm{dist}(y_\ell, \partial\Gamma_{R_\ell})$ and hence that $s_\ell \geq CR_\ell$ for some $C > 0$ and all ℓ. Choose $t_\ell \in (0,1)$ such that $y_\ell = R_\ell\gamma(t_\ell)$ for all ℓ and set $x_\ell := A_{t_\ell}(z_\ell - y_\ell)$. Since $|W^\pm_{z_\ell,R_\ell}(x)| \leq C\mathrm{e}^{-\mu|x-z_\ell|}$ (see (3.43) and Lemma 2.9) it remains to show that

$$\int_{\mathbb{L}} |f(U^\pm(x)) - f(\widetilde{U}^\pm_{r_\ell,s_\ell}(x))|\, \mathrm{e}^{-\mu|x-x_\ell|}\, \mathrm{d}x = o(\mathrm{e}^{-\mu g_1(R_\ell)}) \tag{3.72}$$

as $\ell \to \infty$. Passing to a subsequence we have may encounter the following cases:

(a) $\inf_\ell(|x_\ell|/R_\ell) > 0$,

(b) $x_\ell = o(R_\ell)$ and, writing $x_\ell = (\xi_\ell, \eta_\ell)$, one of

- (i) $\xi_\ell < 0$ for all ℓ,
- (ii) $\xi_\ell > 0$ for all ℓ.

In case (a) we use $|f(u)| \leq C|u|^{p_1}$ on bounded subsets, and obtain by Lemma 2.1 and (3.34)

$$\begin{aligned}\int_{\mathbb{L}} |f(U^\pm(x)) - f(\widetilde{U}^\pm_{r_\ell,s_\ell}(x))|\, \mathrm{e}^{-\mu|x-x_\ell|}\, \mathrm{d}x &\leq C \int_{\mathbb{R}^N} \mathrm{e}^{-p_1\mu|x|}\mathrm{e}^{-\mu|x-x_\ell|}\, \mathrm{d}x \\ &= O(\mathrm{e}^{-CR_\ell}) = o(\mathrm{e}^{-\mu g_1(R_\ell)})\end{aligned}$$

as $\ell \to \infty$. In case (b) we can say in general that $\eta_\ell = o(\xi_\ell)$ because $z_\ell \in \Gamma_{R_\ell}$. Geometric considerations imply that always $-r_\ell < \xi_\ell$. We estimate, using the Lipschitz continuity of f on bounded sets and (3.44):

$$\begin{aligned}&\int_{\mathbb{L}} |f(U^\pm(x)) - f(\widetilde{U}^\pm_{r_\ell,s_\ell}(x))|\, \mathrm{e}^{-\mu|x-x_\ell|}\, \mathrm{d}x \\ &\qquad\leq C \int_{\mathbb{R}} (\mathrm{e}^{-\mu(r_\ell+|\xi+r_\ell|)} + \mathrm{e}^{-\mu(s_\ell+|\xi-s_\ell|)})\mathrm{e}^{-\mu|\xi-\xi_\ell|}\, \mathrm{d}\xi \\ &\qquad\leq C((r_\ell+\xi_\ell)\mathrm{e}^{-\mu(r_\ell+|\xi_\ell+r_\ell|)} + (s_\ell-\xi_\ell)\mathrm{e}^{-\mu(s_\ell+|\xi_\ell-s_\ell|)}) \\ &\qquad= C(r_\ell+\xi_\ell)\mathrm{e}^{-\mu(r_\ell+|\xi_\ell+r_\ell|)} + o(\mathrm{e}^{-\mu g_1(R_\ell)}).\end{aligned}$$

The last equality follows from $\xi_\ell = o(R_\ell)$, $s_\ell \geq CR_\ell$, and (3.34). Now in sub-case (b)(i) we have $r_\ell \geq \frac{3}{2}g_1(R_\ell)(1+o(1))$, and in sub-case (b)(ii) we have $\xi_\ell + r_\ell \geq \frac{3}{2}g_1(R_\ell)(1+o(1))$ as $\ell \to \infty$. This proves (3.72) and therefore (3.70).

It holds that

$$|x_i - x_j| \geq \frac{4}{3} g_1(R) \qquad \text{for large } R, \text{ if } |i-j| \geq 2. \tag{3.73}$$

Here we do not need to account for circular shifts and therefore use $|\cdot|$ instead of d_n to measure the distance of indices.

Following the proof of Proposition 3.11 we obtain the analogue of (3.42), replacing $\overline{U}_i$ by $\overline{W}_i$ and using (3.43) and Lemma 2.9. Moreover, we estimate, using Lemma 3.9, Proposition 3.10, (3.70), (3.71), (3.69), Lemma 2.1, and Lemma 2.9:

$$\begin{aligned}
&J_{\Omega_R}(\varphi_R(X)) \\
&\quad = \sum_{i=1}^{n} J_{\Omega_R}(\overline{V}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\Omega_R} f(\overline{U}_i)\overline{V}_j \,\mathrm{d}x - \sum_{i\neq j}\int_{\Omega_R} f(\overline{V}_i)\overline{V}_j \,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\quad = \sum_{i=1}^{n} J_{x_i + A_t^{-1}\mathbb{L}}(\overline{W}_i) + \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{W}_j \,\mathrm{d}x - \sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{W}_i)\overline{W}_j \,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\quad = \sum_{i=1}^{n} J_{\mathbb{L}}\big(\widetilde{U}^{\pm}_{|x_i - R\gamma(0)|,|x_i - R\gamma(1)|}\big) - \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j \,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\
&\quad \geq E_n + C\sum_{i=1}^{n} \mathrm{e}^{-2\mu \operatorname{dist}(x_i,\partial\Gamma_R)} + \frac{1}{2}\sum_{|i-j|=1}\int_{\mathbb{R}^N} |f(\overline{U}_i)\overline{U}_j| \,\mathrm{d}x \\
&\qquad\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).
\end{aligned}$$

Since $X \in \partial\mathcal{U}_{1,R}$ and by (3.73), at least one of the following is true: There is i_0 such that $\operatorname{dist}(x_i, \partial\Gamma_R) = g_1(R)/2$ or there are i_0, j_0 such that $|i_0 - j_0| = 1$ and $|x_{i_0} - x_{j_0}| = g_1(R)$. Recalling the proof of Proposition 3.5, in any case the claim of the proposition follows. □

Proposition 3.13. *It holds that*

$$\inf_{X\in\mathcal{U}_{1,R}} J_{\Omega_R}(\varphi_R(X)) \leq E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3})$$

as $R \to \infty$.

Proof. Define $X_R := (x_{R,1}, x_{R,2}, \dots, x_{R,n})$ exactly as in the proof of Proposition 3.6. Reusing the definition of $\overline{U}_i$ we obtain from (3.45) and (3.69) as in the proof of Proposition 3.12 that

$$\begin{aligned}J_{\Omega_R}(\varphi_R(X)) &= \sum_{i=1}^{n} J_{\mathbb{L}}\big(\widetilde{U}^{\pm}_{|x_i-R\gamma(0)|,|x_i-R\gamma(1)|}\big) - \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N} f(\overline{U}_i)\overline{U}_j\,\mathrm{d}x \\ &\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) \\ &\le E_n + C\sum_{i=1}^{n}\mathrm{e}^{-2\mu\,\mathrm{dist}(x_{R,i},\partial\Gamma_R)} + \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^N}|f(\overline{U}_i)\overline{U}_j|\,\mathrm{d}x \\ &\qquad + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}).\end{aligned}$$

The claim follows as in the proof of Proposition 3.6, taking into account that $\mathrm{dist}(x_i,\partial\Gamma_R)\ge CR$ for some $C>0$ and all R and i. □

4 Proof of the Main Results

4.1 The Finite Dimensional Reduction

Recall the definition of φ_R from Section 3.

Lemma 4.1. *The map φ_R is a C^2-immersion of $\mathcal{U}_{2,R}$ in $H_0^1(\Omega_R)$.*

Proof. By (H2) and (H4) it is sufficient to show that for fixed R the map $h\colon \Gamma_R\to L^2(\mathbb{R}^N)$, $x\mapsto \widetilde{U}^{\pm}_{x,R}$ is in C^2. Since $U^{\pm}\in H^1(\mathbb{R}^N)$, the C^1-differentiability of h is obvious. For C^2-differentiability one has to keep in mind that only directions tangential to Γ_R have to be considered.

That φ_R is an immersion is immediate. □

Denote $\Sigma_R := \varphi_R(\mathcal{U}_{2,R})$. If $\partial\Gamma\neq\varnothing$ or $n\le 2$ then φ_R is injective, and hence Σ_R is a submanifold of $H_0^1(\Omega_R)$. If $\partial\Gamma=\varnothing$ and $n\ge 4$ then φ_R is not injective because it is invariant under the nontrivial permutation subgroup generated by the coordinate permutation $x_i\mapsto x_{i+2}$, acting on $(\Gamma_R)^n$. Nevertheless, since this subgroup acts freely, also in this case Σ_R is a submanifold of $H_0^1(\Omega_R)$.

For $u\in\Sigma_R$ denote $S_{u,R} := (T_u\Sigma_R)^{\perp}$ and let $P_{S_{u,R}}$ denote the orthogonal projection onto $S_{u,R}$. Also denote $L_{u,R} := P_{S_{u,R}}\mathrm{D}^2 J_{\Omega_R}(u)|_{S_{u,R}}$. Here, for any $u\in H_0^1(\Omega_R)$ we consider $\mathrm{D}^2 J_{\Omega_R}(u)$ as an element of $\mathcal{L}(H_0^1(\Omega_R))$, as the differential of $\nabla J_{\Omega_R}(u)$.

Lemma 4.2. *If R is large enough and $u\in\Sigma_R$, then $L_{u,R}$ is invertible in $\mathcal{L}(S_{u,R})$ and*

$$\limsup_{R\to\infty}\sup_{u\in\Sigma_R}\left\|L_{u,R}^{-1}\right\|_{\mathcal{L}(S_{u,R})}<\infty.$$

Proof. The proof of this fact is standard, see [7, 13, 16] or the proofs of (3.19) and Lemma 3.8(v) in [1]. □

Lemma 4.3. *There are $r_0 > 0$ and $R_1 \geq 1$ such that for $R \geq R_1$ and for every $u \in \Sigma_R$ there is a unique $v_u \in u + S_{u,R}$ that satisfies $\|u - v_u\|_{H_0^1(\Omega_R)} < r_0$ and $P_{S_{u,R}} \nabla J_{\Omega_R}(v_u) = 0$. It holds that*

$$\|u - v_u\|_{H_0^1(\Omega_R)} = O(\|\nabla J_{\Omega_R}(u)\|_{H_0^1(\Omega_R)}) \tag{4.1}$$

and

$$|J_{\Omega_R}(u) - J_{\Omega_R}(v_u)| = O(\|\nabla J_{\Omega_R}(u)\|^2_{H_0^1(\Omega_R)}) \tag{4.2}$$

as $R \to \infty$, independently of $u \in \Sigma_R$. Moreover, the operator

$$P_{S_{u,R}} \mathrm{D}^2 J_{\Omega_R}(v_u)|_{S_{u,R}}$$

is invertible in $\mathcal{L}(S_{u,R})$.

Proof. In this proof we denote by $B_r Z$ the open ball of radius r and with center 0 in a normed space Z, and by $\overline{B}_r Z$ its closure.

Using Lemma 4.2 pick some $M \geq 1$ satisfying

$$M > \limsup_{R \to \infty} \sup_{u \in \Sigma_R} \left\| L_{u,R}^{-1} \right\|_{\mathcal{L}(S_{u,R})}. \tag{4.3}$$

It is clear that

$$C_0 := \limsup_{R \to \infty} \sup_{u \in \Sigma_R} \|u\|_{H_0^1(\Omega_R)} < \infty.$$

By Condition (H4)

$$\limsup_{R \to \infty} \|J_{\Omega_R}\|_{C^{2,\bar{\alpha}}(B_{2C_0} H_0^1(\Omega_R))} < \infty \tag{4.4}$$

for some $\bar{\alpha} \in (0, 1]$.

By Lemma 4.2 and (4.4) there is $r_0 > 0$ such that R large enough, $u \in \Sigma_R$, $v \in H_0^1(\Omega_R)$, and $\|u - v\| \leq r_0$ imply that

$$\|\mathrm{D}^2 J_{\Omega_R}(u) - \mathrm{D}^2 J_{\Omega_R}(v)\|_{\mathcal{L}(H_0^1(\Omega_R))} \leq \frac{1}{2M} \tag{4.5}$$

and $P_{S_{u,R}} \mathrm{D}^2 J_{\Omega_R}(v)|_{S_{u,R}}$ is invertible in $\mathcal{L}(S_{u,R})$.

Suppose now that R is large enough such that

$$\sup_{u \in \Sigma_R} \|\nabla J_{\Omega_R}(u)\|_{H_0^1(\Omega_R)} \leq \frac{r_0}{2M}. \tag{4.6}$$

This is possible because of Propositions 3.4 and 3.11. Fix $u \in \Sigma_R$ and define the auxiliary map $g\colon S_{u,R} \to S_{u,R}$ by $g(w) := w - L_{u,R}^{-1} P_{S_{u,R}} \nabla J_{\Omega_R}(u + w)$. If $w \in \overline{B}_{r_0} S_{u,R}$ it follows from (4.5) and (4.6) that

$$\begin{aligned} \|g(w)\| &\leq M \|\mathrm{D}^2 J_{\Omega_R}(u) w - \nabla J_{\Omega_R}(u + w)\| \\ &= M \left\| -\nabla J_{\Omega_R}(u) - \int_0^1 (\mathrm{D}^2 J_{\Omega_R}(u + tw) - \mathrm{D}^2 J_{\Omega_R}(u)) w \, \mathrm{d}t \right\| \\ &\leq M \left(\|\nabla J_{\Omega_R}(u)\| + \frac{\|w\|}{2M} \right) \\ &\leq r_0. \end{aligned} \tag{4.7}$$

Hence g maps $\overline{B}_{r_0} S_{u,R}$ into itself. Moreover, by (4.3) and (4.5) we have

$$\|g'(w)\| \le \|L_{u,R}^{-1}\| \|\mathrm{D}^2 J_{\Omega_R}(u) - \mathrm{D}^2 J_{\Omega_R}(u+w)\| \le \frac{1}{2}.$$

Hence g is a strict contraction on $\overline{B}_{r_0} S_{u,R}$ and has a unique fixed point w_u, by Banach's fixed point theorem. Clearly $v_u := u + w_u$ is then the only zero v of $P_{S_{u,R}} \nabla J_{\Omega_R}$ such that $\|u - v\|_{H_0^1(\Omega_R)} \le r_0$.

Inserting $g(w_u) = w_u$ in (4.7) yields $\|w_u\| \le 2M \|\nabla J_{\Omega_R}(u)\|$ and hence (4.1). Moreover, since $\mathrm{D}J_{\Omega_R}(v_u)[w_u] = 0$

$$\begin{aligned} |J_{\Omega_R}(u) - J_{\Omega_R}(v_u)| &\le |\mathrm{D}J_{\Omega_R}(v_u)[w_u]| \\ &\quad + \int_0^t (1-t) |\mathrm{D}^2 J_{\Omega_R}(u + (1-t) w_u)[w_u, w_u]| \, \mathrm{d}t \le C \|w_u\|^2 \end{aligned} \tag{4.8}$$

with some constant C that is independent of R and u. Now (4.1) and (4.8) imply (4.2). Finally, if R is large enough (4.1) implies the stronger inequality $\|u - v_u\|_{H_0^1(\Omega_R)} < r_0$, as stated in the lemma. □

We now fix r_0 and R_1 as in Lemma 4.3. If $R \ge R_1$, and $X \in \mathcal{U}_{2,R}$ then set $u := \varphi_R(X)$ and define $\psi_R(X) := v_u$, where v_u is given by Lemma 4.3. Moreover, define $G_R \colon \mathcal{U}_{2,R} \to \mathbb{R}$ by

$$G_R := J_{\Omega_R} \circ \psi_R.$$

Lemma 4.4. *For $R \ge R_1$ the map G_R is in $C^1(\mathcal{U}_{2,R}, \mathbb{R})$. If $X \in \mathcal{U}_{2,R}$ is a critical point of G_R then $\psi_R(X)$ is a critical point of J_{Ω_R}.*

Proof. Denote by $\Pi \colon \mathcal{U}_{2,R} \times H_0^1(\Omega_R) \to \mathcal{U}_{2,R}$ the projection of the trivial Hilbert bundle over $\mathcal{U}_{2,R}$ with fiber $H_0^1(\Omega_R)$. For every $X \in \mathcal{U}_{2,R}$ and $u := \varphi_R(X)$ consider the subspace $T_u \Sigma_R$ of $H_0^1(\Omega_R)$. Since φ_R is a C^2-immersion, the collection of these subspaces forms a C^1-vector subbundle of Π with total space $\mathcal{T}$ and projection $\Pi_{\mathrm{t}} \colon \mathcal{T} \to \mathcal{U}_{2,R}$. Define the C^1-subbundle $\mathcal{N} := \mathcal{T}^\perp$ of Π with projection Π_{n}. Then Π_{t} is the pullback of the tangent bundle of Σ_R and $\mathcal{N}$ is the pullback of the normal bundle of Σ_R, both with respect to φ_R.

Set $U := \mathcal{N} \cap (\mathcal{U}_{2,R} \times B_{r_0} H_0^1(\Omega_R))$, an open neighborhood of the zero section of $\mathcal{N}$, and consider the C^1-map

$$\begin{aligned} g \colon \qquad U &\to \mathcal{U}_{2,R} \times H_0^1(\Omega_R), \\ (X, v) &\mapsto \nabla J_{\Omega_R}(\varphi_R(X) + v). \end{aligned}$$

If $(X, v) \in U$ and $u := \varphi_R(X)$ then $(\mathrm{D}g(X,v)(S_{u,R})) \oplus T_u \Sigma_R = H_0^1(\Omega_R)$ by Lemma 4.3. Hence g is transverse to the submanifold $\mathcal{T}$ of $\mathcal{U}_{2,R} \times H_0^1(\Omega_R)$, $g^{-1}(\mathcal{T})$ is a C^1-submanifold of $\mathcal{N}$, and it is the image of a C^1-cross section $h \colon \mathcal{U}_{2,R} \to \mathcal{N}$. If $\Phi \colon \mathcal{N} \to H_0^1(\Omega_R)$ is the C^1-map $\Phi(X, v) := \varphi_R(X) + v$, then $\psi_R = \Phi \circ h$ and hence G_R is C^1.

Now let $X \in \mathcal{U}_{2,R}$ be a critical point of G_R and set $u := \varphi_R(X)$. The linear map $L := P_{T_u\Sigma_R}\mathrm{D}\psi_R(X) \in \mathcal{L}(T_X\mathcal{U}_{2,R}, T_u\Sigma_R)$ is bijective, and $\nabla J_{\Omega_R}(\psi_R(X)) \in T_u\Sigma_R$. Let $v := L^{-1}\nabla J_{\Omega_R}(\psi_R(X))$. Then

$$\|\nabla J_{\Omega_R}(\psi_R(X))\|^2_{H^1_0(\Omega_R)} = \langle \nabla J_{\Omega_R}(\psi_R(X)), Lv\rangle_{H^1_0(\Omega_R)} = \mathrm{D}G_R(X)v = 0,$$

so $\psi_R(X)$ is a critical point of J_{Ω_R}. □

4.2 The proof of Theorems 1.1 and 1.2

Recall Propositions 3.4, 3.5, and 3.6 that apply to the closed tube setting $\gamma(0) = \gamma(1)$, and Propositions 3.11, 3.12, and 3.13 that apply to the open ended setting $\gamma(0) \neq \gamma(1)$. In both cases we obtain from Lemma 4.3

$$\min G_R(\partial\mathcal{U}_{1,R}) \geq E_n + \beta\mathrm{e}^{-\mu g_1(R)} + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) + O(\mathrm{e}^{-2\alpha'\mu g_2(R)})$$

and

$$\min G_R(\overline{\mathcal{U}_{1,R}}) \leq E_n + o(\mathrm{e}^{-\mu g_1(R)}) + O(R^{-2/3}) + O(\mathrm{e}^{-2\alpha'\mu g_2(R)}).$$

With the choices

$$g_1(R) := \frac{1}{2\mu}\log R \qquad \text{and} \qquad g_2(R) := \left(\frac{1}{2} + \frac{1}{4\alpha'}\right) g_1(R)$$

we satisfy (3.32), (3.33), and (3.34), since $\alpha' > 1/2$. Moreover, it holds that

$$R^{-2/3} = o(\mathrm{e}^{-\mu g_1(R)}) \qquad \text{and} \qquad \mathrm{e}^{-2\alpha'\mu g_2(R)} = o(\mathrm{e}^{-\mu g_1(R)}).$$

Therefore

$$\min G_R(\overline{\mathcal{U}_{1,R}}) < \min G_R(\partial\mathcal{U}_{1,R})$$

if R is large enough. Hence G_R has a critical point X_R in $\mathcal{U}_{1,R}$, and $\psi_R(X_R)$ is a critical point of J_{Ω_R} by Lemma 4.4. By (4.1) we have that $\psi_R(X_R) = \varphi_R(X_R) + o(1)$ in $H^1_0(\Omega_R)$ as $R \to \infty$. Together with (3.29) this proves (1.6), and together with (3.65) and (3.45) this proves (1.7).

Finally, $X_R \in \mathcal{U}_{1,R}$ and (3.33) imply that $|x_{R,i} - x_{R,j}| \to \infty$ if $i \neq j$ and $\mathrm{dist}(x_{R,i}, \partial\Gamma_R) \to \infty$ for all i, as $R \to \infty$. The proofs of Theorems 1.1 and 1.2 are complete.

References

[1] Ackermann, N.: A nonlinear superposition principle and multibump solutions of periodic Schrödinger equations. J. Funct. Anal. **234**, 277–320 (2006)

[2] Ackermann, N., Clapp, M., and Pacella, F.: Positive multibump solutions of nonlinear elliptic equations in expanding shells. In preparation

[3] Bartsch, T. and Weth, T.: The effect of the domain's configuration space on the number of nodal solutions of singularly perturbed elliptic equations. Topol. Methods Nonlinear Anal. **26**, 109–133 (2005)

[4] Bartsch, T. and Weth, T.: Three nodal solutions of singularly perturbed elliptic equations on domains without topology. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 259–281 (2005)

[5] Berestycki, H., Caffarelli, L.A., and Nirenberg, L.: Inequalities for second-order elliptic equations with applications to unbounded domains. I. Duke Math. J. **81**, 467–494 (1996), A celebration of John F. Nash, Jr.

[6] Berestycki, H. and Nirenberg, L.: Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains. In: Analysis, et cetera, pp. 115–164, Boston, MA: Academic Press (1990)

[7] Cerami, G. and Wei, J.: Multiplicity of multiple interior peak solutions for some singularly perturbed Neumann problems. Internat. Math. Res. Notices pp. 601–626 (1998)

[8] Dancer, E.N.: The effect of domain shape on the number of positive solutions of certain nonlinear equations. J. Differential Equations **74**, 120–156 (1988)

[9] Dancer, E.N.: The effect of domain shape on the number of positive solutions of certain nonlinear equations. II. J. Differential Equations **87**, 316–339 (1990)

[10] Dancer, E.N.: Real analyticity and non-degeneracy. Math. Ann. **325**, 369–392 (2003)

[11] Dancer, E.N. and Yan, S.: Multibump solutions for an elliptic problem in expanding domains. Comm. Partial Differential Equations **27**, 23–55 (2002)

[12] D'Aprile, T. and Pistoia, A.: Nodal solutions for some singularly perturbed dirichlet problems. To appear in Trans. Amer. Math. Soc.

[13] Floer, A. and Weinstein, A.: Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J. Funct. Anal. **69**, 397–408 (1986)

[14] Fromm, S.J.: Potential space estimates for Green potentials in convex domains. Proc. Amer. Math. Soc. **119**, 225–233 (1993)

[15] Gilbarg, D. and Trudinger, N.S.: Elliptic partial differential equations of second order, vol. 224 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, second edn. 1983

[16] Gui, C. and Wei, J.: Multiple interior peak solutions for some singularly perturbed Neumann problems. J. Differential Equations **158**, 1–27 (1999)

[17] Simon, B.: Schrödinger semigroups. Bull. Amer. Math. Soc. (N.S.) **7**, 447–526 (1982)

[18] Spain, P.G.: The Fermat Point of a Triangle. Math. Mag. **69**, 131–133 (1996)

[19] Walter, W.: A theorem on elliptic differential inequalities with an application to gradient bounds. Math. Z. **200**, 293–299 (1989)

[20] Ziemer, W.P.: Weakly differentiable functions, vol. 120 of Graduate Texts in Mathematics. New York: Springer-Verlag 1989, Sobolev spaces and functions of bounded variation