PERIODIC SOLUTIONS OF RESONANT SYSTEMS WITH RAPIDLY ROTATING NONLINEARITIES

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Abstract. We obtain existence of T-periodic solutions to a second order system of ordinary differential equations of the form

$$u'' + cu' + g(u) = p$$

where $c \in \mathbb{R}$, $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic and has mean value zero, and $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is e.g. sublinear. In contrast with a well known result by Nirenberg [6], where it is assumed that the nonlinearity g has non-zero uniform radial limits at infinity, our main result allows rapid rotations in g.

1. Introduction

In [4] Lazer considered the periodic problem for the scalar differential equation

$$x'' + cx' + g(x) = p(t), \tag{1.1}$$

where c is any constant and $p(t)$ is a continuous T-periodic function with zero average. As a particular case of his main result, existence of a T-periodic solution of equation (1.1) follows when $g : \mathbb{R} \to \mathbb{R}$ is bounded, continuous, and satisfies

$$g(x) > 0 > g(-x) \tag{1.2}$$

for $x > 0$ sufficiently large.

When one interprets the equation as an oscillator, condition (1.2) means that outside a compact set the force $-g(x)$ points everywhere toward the origin. The boundedness condition is assumed in order to avoid the linear resonance occurring at $c = 0$ and $g(x) = \lambda_n x$, $n = 1, 2, \ldots$, where $\lambda_n = \left(\frac{2\pi n}{T}\right)^2$ is the n-th eigenvalue of the T-periodic problem for the linear operator $Lx = -x''$.

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The preceding result admits an immediate generalization to systems. Indeed, if we consider (1.1) as a system in $\mathbb{R}^N$, where the continuous T-periodic function $p(t)$ is vector valued with zero average and $g = (g_1, \dots, g_N)$ is a bounded continuous map of $\mathbb{R}^N$, then condition (1.2) may be replaced by

$$g_k(x_1, \dots, x_k, \dots, x_N) > 0 > g_k(x_1, \dots, -x_k, \dots, x_N) \tag{1.3}$$

for $x_k > 0$ sufficiently large and $k = 1, \dots, N$. The existence of a T-periodic solution follows from the main theorem in [5], which extends Lazer's result to $N > 1$ and applies, in particular, to the case of weakly coupled systems. Many other extensions of (1.2) were discussed in the literature around the seventies.

From a topological point of view condition (1.2) says two different things: firstly, that g does not vanish outside a compact set; secondly, that its Brouwer degree over the interval $(-R, R)$ is different from zero when R is large. Thus, one might believe that a natural extension of (1.2) to $\mathbb{R}^N$ could be to require that

$$g(x) \neq 0 \qquad \text{for } |x| \geq R \tag{1.4}$$

and

$$\deg(g, B_R(0), 0) \neq 0, \tag{1.5}$$

where deg refers to the Brouwer degree and $B_R(0)$ is the open ball of radius R centered at the origin.

This possible extension was analyzed by Ortega and Sánchez in [8], where they constructed an example showing that (1.4) and (1.5) are not sufficient to guarantee the existence of a periodic solution. The pathological g rotates very fast as x moves in some specific directions.

Motivated by this observation, the following result, which follows from the main theorem in the work of Ruiz and Ward [10], can be regarded as an extension of the preceding results.

We write $B_r(v) := \{x \in \mathbb{R}^N : |x - v| < r\}$ and $\overline{B}_r(v)$ for its closure, and $\mathrm{co}(A)$ for the convex hull of a subset A of $\mathbb{R}^N$. We denote by $C_T(\mathbb{R}, \mathbb{R}^N)$ the space of T-periodic functions $u : \mathbb{R} \to \mathbb{R}^N$ with the uniform norm $\|\cdot\|_\infty$, and the mean value of u by

$$\overline{u} := \frac{1}{T} \int_0^T u(t)\, dt.$$

Theorem 1.1. *Let $c \in \mathbb{R}$ and assume that $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded and satisfies the following condition:*
For each $r > 0$ there exists $R > r$ such that

$$0 \notin \mathrm{co}(g(\overline{B}_r(v))) \qquad \text{if } v \in \mathbb{R}^N \text{ and } |v| = R \tag{1.6}$$

and

$$\deg(g, B_R(0), 0) \neq 0.$$

Then, for each $p \in C_T(\mathbb{R}, \mathbb{R}^N)$ *with* $\overline{p} = 0$, *there exists a* T*-periodic solution of problem* (1.1).

The role of condition (1.6) is easily understood when one attempts to solve problem (1.1) using Leray-Schauder degree methods. Indeed, the key step for proving Theorem 1.1 consists in showing that, for $0 < \lambda \leq 1$, equation

$$u'' + cu' + \lambda g(u) = \lambda p(t) \tag{1.7}$$

has no T-periodic solution on $\partial\Omega$, where

$$\Omega := \{u \in C_T(\mathbb{R}, \mathbb{R}^N) : \|u - \overline{u}\|_\infty < r, \;\; |\overline{u}| < R\}$$

for some accurate r and the corresponding R given by condition (1.6).

An appropriate value of r is obtained after observing that, if u is T-periodic and satisfies (1.7), then

$$\|u'\|_\infty \leq k \left(\|p\|_{L^1(0,T)} + T\|g\|_\infty\right)$$

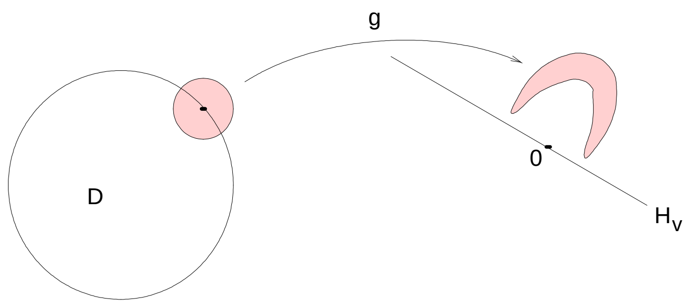
for some constant k, independent of the data. Thus, the choice of any value $r > kT(\|p\|_{L^1(0,T)} + T\|g\|_\infty)$ provides an a priori bound for $\|u - \overline{u}\|_\infty$. Then, if we assume that $|\overline{u}| = R$, a contradiction is obtained in the following way: since the convex hull of $g(\overline{B}_r(\overline{u}))$ is compact, the geometric version of the Hahn-Banach theorem asserts that there exists a hyperplane H passing through the origin such that $g(\overline{B}_r(\overline{u})) \subset \mathbb{R}^N \setminus H$. As $\|u - \overline{u}\|_\infty < r$, we conclude that $g(u(t))$ remains on the same side of H for every value of t, which contradicts the fact that $\int_0^T g(u(t))\, dt = \int_0^T p(t)\, dt = 0$.

Note that, if one fixes p and chooses r as before, the role of $B_R(0)$ may be assumed by a more general domain $D \subset \mathbb{R}^N$ if conditions (1.6) and (1.5) are replaced respectively by (1.8) and (1.9), as follows:
There exists a bounded open subset D *of* $\mathbb{R}^N$ *with the following properties: for each* $v \in \partial D$ *there exists a hyperplane* H_v *passing through the origin such that*

$$g(\overline{B}_r(v)) \subset \mathbb{R}^N \setminus H_v, \tag{1.8}$$

and

$$\deg(g, D, 0) \neq 0. \tag{1.9}$$



Moreover, as in Lazer's original result in [4], Theorem 1.1 still holds if g is unbounded but *sublinear*, that is,

$$\frac{g(u)}{|u|} \to 0 \qquad \text{as } |u| \to \infty. \tag{1.10}$$

Indeed, sublinearity implies that for any given $\varepsilon > 0$ there exists a constant M_ε such that

$$|g(u)| \leq \varepsilon\, |u| + M_\varepsilon \quad \text{for every } u \in \mathbb{R}^N.$$

Thus, if u is a T-periodic solution of (1.7) for some $\lambda \in (0,1]$, then

$$\begin{aligned}\|u'\|_\infty &\leq k(\|p\|_{L^1(0,T)} + \varepsilon\|u\|_{L^1(0,T)} + M_\varepsilon T) \\ &\leq k\left[\|p\|_{L^1(0,T)} + M_\varepsilon T + \varepsilon T(\|u - \overline{u}\|_\infty + |\overline{u}|)\right].\end{aligned}$$

Assume that $|\overline{u}| = R < \alpha KT$ for some constants $\alpha > 1$, $K > 0$. Then, if $\|u'\|_\infty \geq K$, the previous inequality yields

$$K(1 - k\varepsilon T^2(1+\alpha)) \leq k(\|p\|_{L^1(0,T)} + M_\varepsilon T).$$

Consequently, taking $\alpha > 1$,

$$0 < \varepsilon < \frac{1}{kT^2(1+\alpha)}, \qquad K > \frac{k(\|p\|_{L^1(0,T)} + M_\varepsilon T)}{1 - k\varepsilon T(T+\alpha)}, \qquad r := KT, \tag{1.11}$$

we conclude that any T-periodic solution of (1.7) for $\lambda \in (0,1]$ such that $|\overline{u}| = R < \alpha r$ must satisfy

$$\|u'\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r.$$

The aim of this paper is to prove the existence of T-periodic solutions to equation (1.1) in some situations where condition (1.8) is not satisfied. More specifically, we shall allow $g(B_r(v))$ to intersect H_v, provided that g maps an appropriate subset of $B_r(v)$ sufficiently far away from H_v.

A subset of $B_r(v)$ of the form

$$\mathcal{S}(v) = \{u \in B_r(v) : |\langle u - v, \xi_v\rangle| < \delta\},$$

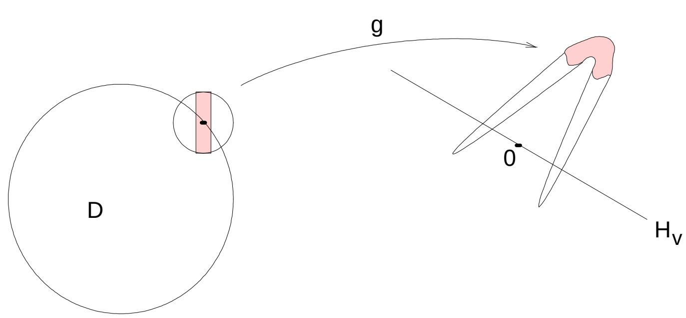
for some $\xi_v \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and $\delta > 0$, will be called a *strip of width* 2δ.

Our main theorem reads as follows.

Theorem 1.2. *Let* $c \in \mathbb{R}$ *and assume that* $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ *satisfies* (1.10) *and* $p \in C_T(\mathbb{R}, \mathbb{R}^N)$ *satisfies* $\overline{p} = 0$. *Further, assume that for some* $\alpha > 1$, *and* ε, K *and* r *satisfying* (1.11), *there exists a domain* $D \subset B_{\alpha r}(0)$ *with the following properties:*
($\mathbf{D}_1$) *For every* $v \in \partial D$ *there exist a hyperplane* H_v *passing through the origin and a strip* $\mathcal{S}(v)$ *of width* 2δ *such that* $g(\mathcal{S}(v)) \subset \mathbb{R}^N \setminus H_v$ *and*

$$\operatorname{dist}(g(\mathcal{S}(v), H_v) > \left(\frac{r}{2\delta} - 1\right) \operatorname{dist}(g(u), H_v)$$

for every $u \in B_r(v)$ *with* $g(u) \in H_v^-$, *where* H_v^- *denotes the closure of the connected component of* $\mathbb{R}^N \setminus H_v$ *not containing* $\mathcal{S}(v)$.
($\mathbf{D}_2$) $\deg(g, D, 0) \neq 0$.
Then there exists a T*-periodic solution* u *of equation* (1.1) *such that* $\overline{u} \in D$ *and* $\|u - \overline{u}\|_\infty < r$.



In particular, if (1.6) holds then $g(B_r(v)) \cap H_v^- = \emptyset$, and condition ($\mathbf{D}_1$) is trivially satisfied. Observe also that, if (1.6) does not hold and $\delta \geq \frac{r}{2}$, then condition ($\mathbf{D}_1$) simply says that $\operatorname{dist}(g(\mathcal{S}(v), H_v) > 0$.

Condition ($\mathbf{D}_1$) is motivated by some results in the scalar case involving rapidly oscilating nonlinearities. In the following section we discuss the effect of *rapidly rotating* nonlinearities and give some examples where condition ($\mathbf{D}_1$) allows to obtain existence results in situations where condition (1.6) fails. The proof of Theorem 1.2 is given in section 3. Finally, in section 4 we give further sufficient conditions on g which provide a priori bounds on the solutions for a given p, and we present an example for which the assumptions of our main theorem are satisfied.

2. The effect of rotation

We begin with a simple remark concerning the scalar case. From the discussion following Theorem 1.1 it is immediately seen that, for a given p, the condition

$$g < 0 \quad \text{in } I^- \qquad \text{and} \qquad g > 0 \quad \text{in } I^+,$$

where $I^\pm$ are large enough bounded intervals, is sufficient for the existence of a solution. Indeed, when $N = 1$ condition (1.8) for a general domain $D = (a, b)$ simply reads

$$g \neq 0 \qquad \text{in} \quad (a-r, a+r) \cup (b-r, b+r),$$

and if the signs of g over $(a-r, a+r)$ and $(b-r, b+r)$ are different the degree condition is also satisfied.

This means that, in contrast with (1.2), oscillations of g around 0 at $\pm\infty$ are, in fact, allowed, but the length of the intervals $I^\pm$ where g does not change sign is determined by g and p, and cannot be arbitrarily small. For instance, when $g(u) = \sin u$, there are well known examples of forcing terms p with zero average and $c \neq 0$ such that the problem has no solutions (see [1], [7], [9]).

There are, however, some particular situations in which g is oscillatory, but solvability for arbitrary p can still be ensured. This is the case of the so-called expansive nonlinearities, like

$$g(u) = \sin(u^{1/3}).$$

Indeed, here the gap between consecutive zeros of g becomes arbitrarily large as $|u|$ tends to infinity. Thus, for any choice of p, the existence of appropriate intervals $I^\pm$ is verified. Furthermore, since g changes sign infinitely many times, we deduce the existence of infinitely many solutions.

The preceding argument obviously fails in the case of non-expansive nonlinearities. Despite this fact, some existence results are known when g presents rapid oscillations (see e.g. [2], [3] and the references therein). For example, assume that g is bounded from below and that $g < 0$ over some large interval I^-, but no interval of positivity of g is long enough to satisfy (1.8). Then it is possible to compensate this 'smallness' by assuming that g is larger than an appropriate constant C over some subset of one of these positivity intervals. Again, the value of the constant depends on p, and also on the length of the interval: faster oscillations require larger values of g. If we expect to prove solvability for arbitrary p using this approach, then g must necessarily be unbounded. For example, we may consider a function g bounded from below with expansive nonlinearities for $u < 0$, and that behaves as

$$u^2[\sin u^2 + 1] + \sin u^2$$

for $u > 0$. It can be proved that, even if the length of the positivity intervals of g tends to 0 as $u \to +\infty$, the large factor u^2 guarantees solvability for any p.

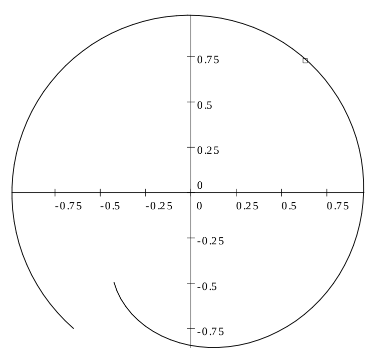
Our main theorem may be considered as an extension of this idea for rapid oscillations to the case $N > 1$. Although an extension of some of the results in [2] and [3] is rather straightforward for weakly coupled systems, there seem to be no results which extend the results in [10] for *rapidly rotating* nonlinearities.

We first observe that Theorem 1.1 provides a better understanding of the non-existence result given in [8]. Indeed, conditions (1.4) and (1.6) are equivalent for $N = 1$, but when $N = 2$ the function

$$g_\rho(z) = e^{i\frac{\mathrm{Re}(z)}{\rho}} \frac{z}{\sqrt{1+|z|^2}}, \qquad \rho > 0,$$

(in complex notation) considered in [8], satisfies (1.4) but not (1.6).

It is worth taking a closer look at this function g_ρ in order to understand why condition (1.6) is violated for some choices of r. If $r \geq \rho\pi$ and $R \gg 0$, then for $z_0 \in \partial B_R(0)$ it suffices to consider the curve $z(t) = z_0 - t$ with $t \in [-\rho\pi, \rho\pi]$. Since $R \gg 0$, the variation of $|g_\rho(z(t))|$ is small, but $g_\rho(z(t))$ rotates around the origin and contains points belonging to antipodal rays in each direction.



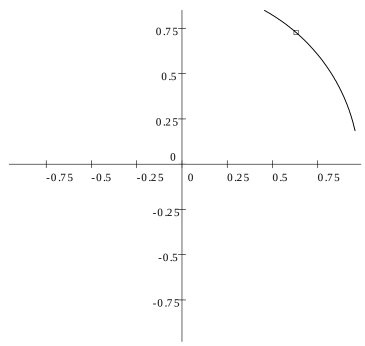
$g_1(4 + t), \;\; t \in [-\pi, \pi]$

Note that $|g_\rho(z)|$ does not depend on ρ, so the choice of the appropriate r depends only on p. For fixed p, condition (1.6) is satisfied for large values of ρ. An approximate lower bound for ρ would be $\frac{2r}{\pi}$. But (1.6) fails to hold values of ρ which are smaller than some $\rho(p)$, i.e. for those nonlinearities g_ρ which rotate too fast.

Note, however, that the effect of rotation appears only when we consider the image of the whole ball $B_r(z)$ under the function g, whereas the image of a vertical strip

$$\mathcal{S}(z) := \{u \in B_r(z) : |\mathrm{Re}(u) - \mathrm{Re}(z)| < \delta\}$$

under g remains in the same half-plane for δ small enough.



$g_1(4 + it), \;\; t \in [-\pi, \pi]$

According to our main theorem, when (1.6) fails, existence of solutions can still be proved if the distance between $g_\rho(\mathcal{S}(z))$ and some line through the origin is large enough. In this sense, our result can be regarded as a generalization of the above mentioned results for rapid oscillations in the scalar case. In particular, for a given p, existence of solutions can be proved for nonlinearities g_ρ in a range of values of ρ which is larger than the one given by condition (1.6).

3. Proof of the main theorem

Proof of Theorem 1.2. Set

$$\Omega = \{u \in C_T(\mathbb{R}, \mathbb{R}^N) : \|u - \overline{u}\|_\infty < r, \; \overline{u} \in D\}.$$

By standard continuation methods, it suffices to prove that the equation

$$u'' + cu' + \lambda g(u) = \lambda p \tag{3.1}$$

has no solutions in $\partial\Omega$ for $\lambda \in (0, 1]$.

If $u \in \overline{\Omega}$ is a solution of (3.1) for some $\lambda \in (0, 1]$ then, since we are assuming that $D \subset B_{\alpha r}(0)$, our choice of K and $r := KT$ yields

$$\|u'\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r. \tag{3.2}$$

Thus, it only remains to prove that $\overline{u} \notin \partial D$.

Note that, if we take w_v to be the unit normal vector of H_v satisfying $\langle g(v), w_v \rangle > 0$, then condition $(\mathbf{D}_1)$ is equivalent to the following one:

($\mathbf{D}_1'$) For every $v \in \partial D$ there exist a vector $w_v \in \mathbb{S}^{N-1}$ and a strip $\mathcal{S}(v)$ of width 2δ such that

$$\inf_{u\in\mathcal{S}(v)} \langle g(u), w_v\rangle + \left(\frac{r}{2\delta} - 1\right) \langle g(u), w_v\rangle > 0 \tag{3.3}$$

for every $u \in B_r(v)$ with $\langle g(u), w_v\rangle \leq 0$.

Arguing by contradiction, assume that $\overline{u} \in \partial D$ and take $w_{\overline{u}} \in \mathbb{S}^{N-1}$ and a strip $\mathcal{S}(\overline{u}) = \{u \in B_r(\overline{u}) : |\langle u - \overline{u}, \xi_{\overline{u}}\rangle| < \delta\}$ with $\xi_{\overline{u}} \in \mathbb{S}^{N-1}$ which satisfy (3.3). Since u solves (3.1), we have that

$$0 = \int_0^T \langle g(u(t)), w_{\overline{u}}\rangle \, dt = \int_0^T \langle g(u(t)) - \tilde{\Delta} w_{\overline{u}}, w_{\overline{u}}\rangle \, dt + \tilde{\Delta} T,$$

where

$$\tilde{\Delta} := \inf_{t\in[0,T]} \langle g(u(t)), w_{\overline{u}}\rangle.$$

This implies that $\tilde{\Delta} \leq 0$.

Set $\varphi(u) := \langle u, \xi_{\overline{u}}\rangle$. From the mean value theorem for vector-valued integrals we deduce that $\overline{u} \in \mathrm{co}(\mathrm{im}(u))$, where $\mathrm{im}(u)$ stands for the image of u. Hence $\varphi(\overline{u}) \in \varphi(\mathrm{im}(u))$. Thus, setting $\overline{t} \in [0, T]$ such that $\varphi(u(\overline{t})) = \varphi(\overline{u})$ and using (3.2) we obtain

$$|\varphi(u(t)) - \varphi(\overline{u})| \leq \left|u(t) - u(\overline{t})\right| \leq K \left|t - \overline{t}\right|.$$

It follows that $u(t) \in \mathcal{S}(\overline{u})$ if $\left|t - \overline{t}\right| < \frac{\delta}{K}$. Using the periodicity of u we conclude that $\mathrm{meas}(A) \geq \frac{2\delta}{K}$, where $A = \{t \in [0, T] : u(t) \in \mathcal{S}(\overline{u})\}$. Moreover, since $[0, T]$ is compact, there exists $t_0 \in [0, T]$ such that $\langle g(u(t_0)), w_{\overline{u}}\rangle = \tilde{\Delta} \leq 0$. Therefore,

$$\begin{aligned} 0 &\geq \int_A \langle g(u(t)) - \tilde{\Delta} w_{\overline{u}}, w_{\overline{u}}\rangle \, dt + T\tilde{\Delta} \\ &\geq \frac{2\delta}{K} \inf_{v\in\mathcal{S}(\overline{u})} \langle g(v), w_{\overline{u}}\rangle + \left(T - \frac{2\delta}{K}\right) \tilde{\Delta} \\ &= \frac{2\delta}{K} \left[\inf_{v\in\mathcal{S}(\overline{u})} \langle g(u), w_{\overline{u}}\rangle + \left(\frac{r}{2\delta} - 1\right) \langle g(u(t_0)), w_{\overline{u}}\rangle\right], \end{aligned}$$

contradicting (3.3). □

Note that the result is still valid if one considers a more general strip defined by

$$\mathcal{S}(v) = \{u \in B_r(v) : |\varphi(u) - \varphi(v)| < \delta\},$$

where $\delta > 0$ and $\varphi : B_r(v) \to \mathbb{R}$ is Lipschitz continuous with constant 1 and satisfies that $\varphi(v) \in \varphi(U)$ for every connected $U \subset B_r(v)$ such that $v \in \mathrm{co}(U)$.

This last condition is quite restrictive. It is an open question whether a similar result holds, for example, for a lower dimensional strip, i.e. for a δ-neighborhood of a subspace of codimension > 1 in $B_r(v)$.

4. Conditions on the nonlinearity

In this section we obtain other versions of our main theorem assuming other conditions on g instead of sublinearity.

In first place, it is easy to prove that in the scalar case no restrictions on the growth of g have to be imposed if the inequalities in (1.2) are reversed, that is to say, if $g(u)u < 0$ for $|u|$ large enough. This fact suggests to consider the assumption

$$\langle g(u), u\rangle < \kappa \qquad \text{for all } u \in \mathbb{R}^N. \tag{4.1}$$

It is readily seen that the case $\kappa < 0$ is contained in Theorem 1.1.

For $\kappa \geq 0$, we consider in fact a weaker assumption, namely, we require that

$$\langle g(u), u\rangle < \kappa + \mu|u|^\theta \qquad \text{for all } u \in \mathbb{R}^N, \tag{4.2}$$

where $\theta < 2$ and $\mu \geq 0$.

Then, if u is a T-periodic solution of the equation

$$u'' + cu' = \lambda(p - g(u)), \tag{4.3}$$

equality

$$-\int_0^T \langle u'', u - \overline{u}\rangle = \lambda \left(\int_0^T \langle g(u), u\rangle - \int_0^T \langle p, u - \overline{u}\rangle \right)$$

holds, and therefore

$$\|u'\|_{L^2}^2 \leq \|p\|_{L^2}\|u - \overline{u}\|_{L^2} + \kappa T + \mu\|u\|_{L^\theta}^\theta.$$

Now we may proceed as in the introduction in order to get bounds K, depending on some fixed $\alpha > 1$, and $r := KT$ such that any T-periodic solution of (4.3) for $\lambda \in (0, 1]$ with $|\overline{u}| < \alpha r$ satisfies

$$\|u'\|_\infty < K \quad \text{and} \quad \|u - u\|_\infty < r. \tag{4.4}$$

For example, if $g(z) = e^{i|z|}\frac{z}{\sqrt{1+|z|^2}}$, $z \in \mathbb{C}$, then

$$\langle g(z), z\rangle = \frac{|z|^2}{\sqrt{1 + |z|^2}} \cos(|z|).$$

So condition (4.1) is not satisfied. However, (4.2) holds.

Finally, let us point out that there is still another way of obtaining a priori bounds (4.4). Again, we recall the case $N = 1$, and note that condition (1.10) in Lazer's result can also be dropped if we assume instead that g is bounded from one side, i.e. that either

$$g(u) \leq M \quad \text{for all } u \in \mathbb{R}, \quad \text{or} \quad g(u) \geq M \quad \text{all } u \in \mathbb{R}.$$

This condition can be generalized to $N > 1$ by assuming that

$$g(\mathbb{R}^N) \subset \xi + \left(\mathbb{R}^N \setminus \bigcup_{j=1}^{N} H_j \right), \tag{4.5}$$

where $\xi \in \mathbb{R}^N$, and $H_j \subset \mathbb{R}^N$ are linearly independent hyperplanes through the origin. In other words, (4.5) says that the range of g is contained in an 'angular sector' of $\mathbb{R}^N$.

If (4.5) holds, a priori bounds (4.4) can be obtained as follows. Let u satisfy $u'' + cu' + \lambda g(u) = \lambda p$ for some $0 < \lambda \leq 1$, and set

$$d_j := \inf_{u \in \mathbb{R}^N} \langle g(u), w_j \rangle, \qquad v_j := d_j w_j,$$

where $\{w_1, \ldots, w_N\}$ is a basis of unit vectors of $\mathbb{R}^N$ chosen in such a way that $\langle g(u) - \xi, w_j \rangle \geq 0$ for every $u \in \mathbb{R}^N$. Then

$$\langle g(u) - v_j, w_j \rangle \geq d_j - \langle v_j, w_j \rangle = 0.$$

Thus,

$$|\langle u''(t), w_j \rangle| \leq \langle g(u) - v_j, w_j \rangle + |\langle v_j - p, w_j \rangle|$$

and, in consequence,

$$\int_0^T |\langle u''(t), w_j \rangle| \; dt \leq \int_0^T |\langle v_j - p, w_j \rangle| \; dt - T\langle \xi_j, w_j \rangle := K_j.$$

Hence, for each $t \in [0, T]$ we have

$$|\langle u'(t), w_j \rangle| \leq K_j$$

and

$$|\langle u(t) - \overline{u}, w_j \rangle| \leq K_j T.$$

Although sharper results could be obtained by taking $r_j > K_j T$ and modifying the definition of Ω accordingly, for simplicity we shall consider a value K such that $\|u'\|_\infty < K$. Then $\|u - \overline{u}\|_\infty < KT := r$. In this case R can be arbitrarily chosen.

Corollary 4.1. Theorem 1.2 *remains true if* (1.10) *is replaced by* (4.2) *or* (4.5)*, and* K, r *and* R *are defined as previously shown in this section.*

We end this paper with a simple example of a radial nonlinearity $g(u) = \gamma(|u|)u$ to which our theorem applies for arbitrary p.

Let $\gamma : [0, +\infty) \to \mathbb{R}$ satisfy

$$\gamma(s) \leq \mu s^{-\sigma}$$

for some $\mu, \sigma > 0$. Thus, condition (4.2) holds, although γ is allowed to take arbitrarily large negative values.

Let $R = \alpha r$ with $\alpha > 1$. Regarding condition $(\mathbf{D}_1)$, it proves convenient to choose $D = B_R(0)$ and, for $|v| = R$, to take $w_v = -\frac{v}{R}$ and

$$\mathcal{S}(v) = \{u \in B_r(v) : |\langle u - v, v\rangle| < \delta R\}.$$

Then, $\langle g(u), w_v\rangle = -\frac{\gamma(|u|)}{R}\langle u, v\rangle$.

Let us assume that $\gamma(R) < 0$ and that $\gamma \not< 0$ in $[R-r, R+r]$, since otherwise Theorem 1.1 applies. Then $\langle g(v), w_v\rangle > 0$ and condition (3.3) reads

$$-\sup_{u\in\mathcal{S}(v)} \gamma(|u|)\langle u, v\rangle > \left(\frac{r}{2\delta} - 1\right)\gamma(|u|)\langle u, v\rangle \quad \text{for all } u \in B_r(v),$$

or equivalently

$$-\sup_{u\in\mathcal{S}(v)} |u|\,\gamma(\,|u|\,)\cos(\beta_{u,v}) > \left(\frac{r}{2\delta} - 1\right) t\gamma(t) \quad \text{for all } t \in (R-r, R+r),$$

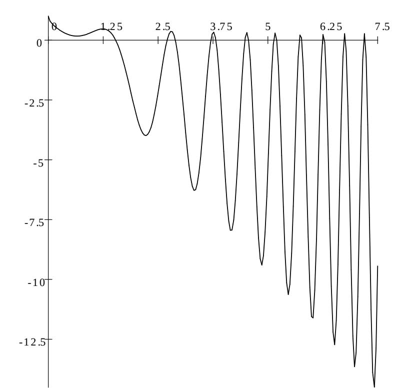
where $\beta_{u,v}$ denotes the angle between u and v. As $R = \alpha r$, a simple computation shows that $\cos(\beta_{u,v}) \geq \frac{\sqrt{\alpha^2-1}}{\alpha}$ for every $u \in B_r(v)$.

Assume that $\delta > 0$ is chosen so that $\gamma(t) < 0$ for every $t \in (R - \delta, \sqrt{R^2 + 2\delta R + r^2})$. Then $\gamma(\,|u|\,) < 0$ for every $u \in \mathcal{S}(v)$ and a sufficient condition for the above inequality to hold is

$$-\frac{\sqrt{\alpha^2-1}}{\alpha} \sup_{R-\delta<t<\sqrt{R^2+2\delta R+r^2}} t\gamma(t) > \left(\frac{r}{2\delta} - 1\right) \sup_{R-r<t<R+r} t\gamma(t). \tag{4.6}$$

For example, we may consider

$$\gamma(t) = t(\sin(t^2) - 1) + \frac{\mu}{t^\sigma + 1}.$$



Set $R_n := \sqrt{\left(2n - \frac{1}{2}\right)\pi}$ and $\delta_n := 1/2\sqrt{n}$. Since for n large enough

$$t\gamma(t) < -t^2 \qquad \text{if} \quad |t - R_n| < \delta_n$$

and

$$\sup_{R-r<t<R+r} t\gamma(t) = O(t^{1-\sigma}),$$

for any fixed p we may choose n large enough and $\delta \in (0, \delta_n)$ so that $\sqrt{R_n^2 + 2\delta R_n + r^2} < R_n + \delta_n$ and (4.6) holds.

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