Asymptotically radial solutions in expanding annular domains *

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Abstract

In this paper we consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{0.1}$$

where $p > 1$ and Ω_R is a smooth bounded domain with a hole which is diffeomorphic to an annulus and expands as $R \longrightarrow \infty$. The main goal of the paper is to prove, for large R, the existence of a positive solution to (0.1) which is close to the positive solution in the corresponding diffeomorphic annulus. The proof relies on a careful analysis of the spectrum of the linearized operator at the radial solution as well as on a delicate analysis of the nondegeneracy of suitable approximating solutions.

1 Introduction

In this paper we study the existence of positive solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$. It is well known that the answer to this problem is strictly related to the exponent p of the nonlinear term and to the geometrical and/or topological properties of the

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domain Ω. Indeed the classical Pohozaev identity ([P]) implies that, if Ω is starshaped and $p \geq \frac{N+2}{N-2}$ in dimension $N \geq 3$, then (1.1) does not admit any solution. On the other hand it has been proved in [BC] that if Ω has nontrivial topology then a positive solution exists even if p is the critical exponent $\frac{N+2}{N-2}$.

In the special case when Ω is an annulus it is easy to prove that a radial positive solution always exists, whatever p is, even supercritical (see [KW]), and this solution is unique (see [NN]). Moreover, exploiting the invariance of the annulus with respect to different symmetry groups, several authors were able to prove the existence of nonradial positive solutions for p up to a certain exponent $p_N > \frac{N+2}{N-2}$ in expanding annuli $A_R = \{x \in \mathbb{R}^N : R < |x| < R+1\}$, for R sufficiently large (see [C], [YYL], [L1], [L2], [B], [CW]). A study of the asymptotic behavior of some of these solutions, as $R \to \infty$, shows that they converge to positive solutions on an infinite strip (see [L2]). As to the positive radial solution, it is a least energy solution and of mountain pass type in the space of radial H^1_0-functions, but its Morse index in the full space becomes unbounded as $R \to \infty$.

In view of these results it is natural to expect that similar multiplicity results should also hold in expanding "annular-type" domains, i.e. in domains with an expanding hole, which are not necessarily annuli. This idea has been carried out in the papers [CW], [DY], [ACP] where, in the subcritical case, the existence of an increasing number of positive solutions is proved, as the domain expands. In particular, in [DY] the limit problem in the strip is exploited to construct positive multibump solutions using a Lyapunov-Schmidt reduction argument. The domains D_R considered in [DY] are the sets of points whose distance to a fixed convex set is larger than R and smaller than $R+1$. So they are diffeomorphic to an annulus. However the solutions that they construct (as well as those of [CW]) are very different from radial solutions in annuli, since they exhibit a finite number of bumps.

The existence of the radial positive solution in an annulus suggests that a positive solution of "radial-type" should exist also in expanding annular-type domains, regardless of the growth of the nonlinearity. It is the main goal of the present paper to prove the existence of a positive solution, in domains of this kind, which is close to the positive radial solution in an annulus. More precisely, using polar coordinates $(\rho, \theta) \in \mathbb{R}^+ \times S^{N-1}$ in $\mathbb{R}^N$ and fixing a positive C^∞-function $g : S^{N-1} \to \mathbb{R}$ on the unit sphere $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$, we consider the domains

$$\Omega_R = \left\{ (\rho, \theta) \in \mathbb{R}^+ \times S^{N-1} : R + \frac{g(\theta)}{R^s} < \rho < R + 1 + \frac{g(\theta)}{R^s} \right\} \tag{1.2}$$

for $R > 0$ with $s > \frac{N-5}{2}$ if $N \geq 5$ and $s = 0$ if $2 \leq N \leq 4$. We observe that Ω_R is diffeomorphic to the annulus

$$A_R = \{x \in \mathbb{R}^N : R < |x| < R+1\}$$

by the obvious diffeomorphism

$$T : \Omega_R \to A_R, \quad T(\rho, \theta) = (\rho - \frac{g(\theta)}{R^s}, \theta). \tag{1.3}$$

Moreover, for any $s \geq 0$, the exterior unit normal to the hypersurface

$$\left\{(R + g(\theta)R^{-s}, \theta) : \theta \in S^{N-1}\right\}$$

at the point $(R + g(\theta)R^{-s}, \theta)$ tends to the radial unit vector $(1, \theta)$ as $R \to \infty$. Hence, as it expands, Ω_R looks more and more like an annulus. Note that Ω_R becomes closer, in fact, to A_R as $R \to \infty$ if $s > 0$. Denoting by $w_R \in H_0^1(A_R)$ the positive radial solution of (1.1) with $\Omega = A_R$ we define $\widetilde{u}_R := w_R \circ T \in H_0^1(\Omega_R)$, that is,

$$\widetilde{u}_R(\rho, \theta) = w_R(\rho - \frac{g(\theta)}{R^s}, \theta). \tag{1.4}$$

Now we can state the main result of the paper.

Theorem 1.1. *There exists a sequence $R_k \to \infty$ with the property that for every $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that for any $k \geq k_\delta$ and for $R \in [R_k + \delta, R_{k+1} - \delta]$, problem* (1.1) *admits a positive solution*

$$u_R = \widetilde{u}_R + \phi_R$$

for some $\phi_R \in H_0^1(\Omega_R)$.

Moreover the distance $|R_k - R_{k+1}|$ is bounded away from zero by a constant independent of k, and $\phi_R \to 0$ in $H_0^1(\Omega_R)$ as $R \in [R_k + \delta, R_{k+1} - \delta]$.

Remark 1.2. *The proof of Theorem 1.1 yields that u_R is an isolated solution of* (1.1) *in $H_0^1(\Omega)$ and that it depends continuously on R in the H^1-norm. If $\Omega_R = A_R$ then $u_R = w_R$ exists for all $R > 0$. In this case all curves u_R, $R \in I_k$, from Theorem 1.1 are part of one solution curve. The radii R_k are bifurcation points for solutions of* (1.1) *in A_R with R as parameter. After perturbing the radial setting the bifurcations may disappear, and the solution curves u_R, $R \in I_k$, may not lie on one continuum of solutions. The global behavior of these solution branches remains to be studied.*

In the case of a nonlinearity with subcritical growth our theorem provides a new multiplicity result since, besides the positive multibump solutions constructed in [CW], [DY], [ACP], it asserts the existence of asymptotically radial solutions in a different type of expanding domains. In the critical exponent case our approach gives a direct proof of the Bahri-Coron result ([BC]) for annular shaped domains with large holes, complementing various results in domains with small holes ([Co], [R], [LYY], [CMP]). To our knowledge the only other result in this direction has been obtained in [CP] for symmetric domains. In the supercritical case Theorem 1.1 is the first existence result for a positive solution without assuming that Ω has a small hole as in [DW], [DFM].

We believe that the main interest of the present paper, however, is to give a direct construction of asymptotically radial positive solutions of (1.1) in annular shaped domains with large holes. To our knowledge this is the first result of this type. Even in the case when p is subcritical a variational characterization or minmax approach seems to be difficult because the Morse indices of the solutions which we obtain become unbounded.

The proof of Theorem 1.1 is quite long and technically difficult. It requires a delicate analysis of the asymptotic behavior, as $R \to \infty$, of the eigenvalues of the linearized operator at the radial solution w_R. We believe that this analysis has an interest of its own and could be used in other problems. This allows to prove the nondegeneracy of the asymptotically radial function $\widetilde{u}_R$, for some values of R. Then we use a fixed point argument and the contraction mapping principle in the space $H^1_0(\Omega_R)$ to find the true solution, close to $\widetilde{u}_R$. This last part is quite difficult to perform. This part of the proof is also responsible for the correction of the diffeomorphism with the term $\frac{1}{R^s}$ in dimension $N \geq 5$.

We would like to remark that it would be possible to get the result of Theorem 1.1 in dimension $N \geq 5$ also for $s = 0$ in (1.2) by correcting the approximating solutions with the addition of some extra terms. Anyway, since this would make the proof technically much more complicated we have chosen to carry out the proof for the special class of domains defined in (1.1).

The outline of the paper is as follows. In Section 2 we give some preliminary results on the radial solution w_R, while in Section 3 and 4 we perform a precise analysis of the spectrum of the linearized operator at w_R. In Section 5 we study the approximate solutions and its possible degeneracy. Finally in Section 6 we prove Theorem 1.1.

2 Preliminary results on the radial solution

As in the introduction we denote by A_R the annulus

$$A_R = \left\{x \in \mathbb{R}^N,\ R < |x| < R+1\right\}, \quad R > 1,$$

and by w_R the unique positive radial solution of (1.1) for $\Omega = A_R$. We start with analyzing the asymptotic behavior of w_R as $R \to \infty$. Obviously w_R satisfies

$$\begin{cases} -w_R'' - \frac{N-1}{r} w_R' = w_R^p & \text{in } (R, R+1), \\ w_R > 0 & \text{in } (R, R+1), \\ w_R(R) = w_R(R+1) = 0, \end{cases} \tag{2.1}$$

where we write $(r, \theta) \in \mathbb{R}^+ \times S^{N-1}$ for the polar coordinates in $\mathbb{R}^N$. Since w_R is the unique positive solution of (2.1) we have that $w_R = \beta_R^{\frac{1}{p-1}} \overline{w_R}$, where

$$\begin{aligned} \beta_R &= \inf_{\substack{u \in H^1_0(R,R+1), \\ \|u\|_{L^{p+1}}=1}} \int_R^{R+1} (u')^2 r^{N-1} dr \\ &= \inf_{\substack{u \in H^1_0(R,R+1), \\ u \not\equiv 0}} \frac{\int_R^{R+1} (u')^2 r^{N-1} dr}{\left(\int_R^{R+1} |u|^{p+1} r^{N-1} dr\right)^{\frac{2}{p+1}}} \end{aligned}$$

and $\overline{w_R}$ is a minimizer for β_R. Taking a function $\overline{\phi} \in C_0^\infty\big((0,1)\big)$, $\overline{\phi} \geq 0$ and defining $\phi \in C_0^\infty\big((R,R+1)\big)$ by $\phi(r) = \overline{\phi}(r-R)$ we have

$$\begin{aligned}\beta_R &\leq \frac{\int_R^{R+1}(\phi')^2 r^{N-1}dr}{\left(\int_R^{R+1}\phi^{p+1}r^{N-1}dr\right)^{\frac{2}{p+1}}} = \frac{\int_0^1(\overline{\phi}')^2(R+t)^{N-1}dt}{\left(\int_0^1\overline{\phi}^{p+1}(R+t)^{N-1}dt\right)^{\frac{2}{p+1}}} \\ &\leq CR^{(N-1)\left(1-\frac{2}{p+1}\right)}.\end{aligned} \tag{2.2}$$

Moreover, from equation (2.1) we deduce

$$\int_R^{R+1} w_R^{p+1} r^{N-1}dr = \beta_R^{\frac{p+1}{p-1}} \leq CR^{N-1}. \tag{2.3}$$

Now we consider the function

$$\widetilde{w}_R(t) = w_R(t+R)$$

and observe that (2.3) implies

$$\begin{aligned}\int_0^1 \widetilde{w}_R(t)^{p+1}dt &= \int_R^{R+1} w_R(r)^{p+1}dr \\ &\leq \frac{1}{R^{N-1}}\int_R^{R+1} w_R(r)^{p+1}r^{N-1}dr \leq C.\end{aligned} \tag{2.4}$$

By (2.1) we have

$$\int_R^{R+1}(w_R')^2 r^{N-1}dr = \int_R^{R+1} w_R^{p+1}r^{N-1}dr,$$

so we get

$$\begin{aligned}R^{N-1}\int_0^1\left(\widetilde{w}_R'\right)^2dt &\leq \int_0^1\left(\widetilde{w}_R'\right)^2(t+R)^{N-1}dt = \int_R^{R+1}(w_R')^2r^{N-1}dr \\ &= \int_0^1 \widetilde{w}_R^{p+1}(t+R)^{N-1}dt \leq (1+R)^{N-1}\int_0^1\widetilde{w}_R^{p+1}dt.\end{aligned}$$

As a consequence we obtain by (2.4)

$$\int_0^1\left(\widetilde{w}_R'\right)^2dt \leq C\int_0^1\widetilde{w}_R^{p+1}dt \leq C. \tag{2.5}$$

Observe that $\widetilde{w}_R$ satisfies

$$\begin{cases} -\widetilde{w}_R'' - \frac{N-1}{\rho+R}\widetilde{w}_R' = \widetilde{w}_R^p & \text{in } (0,1), \\ \widetilde{w}_R > 0 & \text{in } (0,1), \\ \widetilde{w}_R(0) = \widetilde{w}_R(1) = 0. \end{cases}$$

¿From (2.5) we deduce that $\widetilde{w}_R$ is bounded in $H_0^1(0,1)$ and hence also in $L^\infty(0,1)$ and in $C^2(0,1)$. Consequently $\widetilde{w}_R \to w_0$ uniformly, as $R \to \infty$, where

$$\begin{cases} -w_0'' = w_0^p & \text{in } (0,1), \\ w_0 \geq 0 & \text{in } (0,1), \\ w_0(0) = w_0(1) = 0. \end{cases} \tag{2.6}$$

It is not difficult to show that $w_0 \not\equiv 0$ so that it is the unique positive solution of (2.6). Indeed $\|\widetilde{w}_R\|_\infty \geq \alpha > 0$, where α is a constant independent of R, as can be seen by multiplying the equation in (1.1) by the first eigenfunction of $-\Delta$ in A_R and integrating. Now we prove that w_R is nondegenerate in the space of radial functions. It is possible that this result is already known but we are not aware of any reference.

Proposition 2.1. *The linearized problem*

$$\begin{cases} -\Delta v = p w_R^{p-1} v & \text{in } A_R, \\ v = 0 & \text{on } \partial A_R, \end{cases} \tag{2.7}$$

does not admit any nontrivial radial solution.

Proof. Arguing by contradiction let us assume that there exists a nontrivial radial solution $\overline{v}$ of (2.7). Denoting by $\widetilde{\mu}_1, \ldots, \widetilde{\mu}_k$ the radial eigenvalues of the operator

$$L_{w_R} = -\Delta - p w_R^{p-1} I$$

with zero boundary conditions in A_R, it is well known that only $\widetilde{\mu}_1$ is negative. Hence $\overline{v}$ must be the second radial eigenfunction and, so, it has only two nodal regions $A_1 = \{x \in \mathbb{R}^N : R < |x| < d\}$ and $A_2 = \{x \in \mathbb{R}^N : d < |x| < R+1\}$, and in each region the first eigenvalue of the linearized operator L_{w_R} is zero. It is easy to see that

$$z = x \cdot \nabla w_R + \frac{2}{p-1} w_R$$

solves

$$\begin{cases} -\Delta z = p w_R^{p-1} z & \text{in } A_R, \\ z(x) > 0 & \text{if } |x| = R, \\ z(x) < 0 & \text{if } |x| = R+1. \end{cases}$$

Hence z is radial and changes sign in A_R. We claim that z changes sign only once, i.e. it has only two annular nodal regions. Indeed if it had more than two nodal regions, by the boundary behavior, it would have at least four nodal regions B_i, $i = 1, \ldots, k$, $k \geq 4$. Hence there would exist B_j such that $z = 0$ on ∂B_j and $B_j \subset A_i$, for some $i = 1, 2$. This implies a contradiction for the first eigenvalues:

$$0 = \lambda_1(L_{w_R}, B_j) > \lambda_1(L_{w_R}, A_i) = 0.$$

Thus z has only two nodal regions B_1, B_2, and therefore $A_i \subset B_j$, for some $i, j = 1, 2$. Using z as a testfunction one sees that $\lambda_1(L_{w_R}, B_j) > 0$ contradicting $\lambda_1(L_{w_R}, B_j) \leq \lambda_1(L_{w_R}, A_i) = 0$. □

3 Analysis of the linearized operator

As in the previous section let w_R be the unique positive radial solution of (2.1) in the annulus A_R. We want to analyze the possible degeneracy of the linearized operator $L_{w_R} = -\Delta - p w_R^{p-1} I$. To this aim let us denote by ν a generic eigenvalue of the problem

$$\begin{cases} -\Delta v = \nu p w_R^{p-1} v & \text{in } A_R, \\ v = 0 & \text{in } \partial A_R, \end{cases} \tag{3.1}$$

and study the eigenvalues ν close to 1. We start by considering the operator

$$\widetilde{L}_R^\nu = |x|^2 \left(-\Delta - \nu p w_R^{p-1} I \right) \tag{3.2}$$

and noting that ν is an eigenvalue for (3.1) if and only if zero is an eigenvalue for $\widetilde{L}_R^\nu$. We also need the operator

$$\hat{L}_R^\nu(v) = r^2 \left(-v'' - \frac{N-1}{r} v' - \nu p w_R^{p-1} v \right) \quad \text{in } (R, R+1) \tag{3.3}$$

with zero boundary conditions. Let us denote by λ_k, $k = 0, 1, \dots$, the eigenvalues of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the $N-1$ dimensional unit sphere S^{N-1}. It is well known that $\lambda_k = k(k+N-2)$. We start with a preliminary lemma, known in the case $\nu = 1$ (see for example [Pa], [L2]). However, since in the sequel we need a more accurate analysis of the eigenvalues ν close to 1 and to recall explicit computations, we also detail the proof of this first lemma.

Lemma 3.1. *The spectra of $\widetilde{L}_R^\nu$, $\hat{L}_R^\nu$, and $-\Delta_{S^{N-1}}$ are related by*

$$\sigma(\widetilde{L}_R^\nu) = \sigma(\hat{L}_R^\nu) + \sigma(-\Delta_{S^{N-1}}). \tag{3.4}$$

Proof. Given $\mu \in \sigma(\widetilde{L}_R^\nu)$ we choose an eigenfunction ψ, i.e. ψ satisfies

$$\begin{cases} -\Delta \psi - \nu p w_R^{p-1} \psi = \mu \frac{\psi}{|x|^2} & \text{in } A_R, \\ \psi = 0 & \text{on } \partial A_R. \end{cases} \tag{3.5}$$

We choose $k \in \mathbb{N}_0$ and an eigenfunction ϕ of $-\Delta_{S^{N-1}}$ associated to λ_k. Then the function

$$w(r) := \int_{S^{N-1}} \psi(r, \theta) \phi(\theta) d\theta.$$

satisfies

$$\begin{aligned} -w'' - \frac{N-1}{r} w' &= \int_{S^{N-1}} \left(-\psi_{rr} - \frac{N-1}{r} \psi_r \right) \phi d\theta \\ &= \int_{S^{N-1}} \left(-\Delta \psi + \frac{1}{r^2} \Delta_{S^{N-1}} \psi \right) \phi d\theta \\ &= \nu p w_R^{p-1} w + \frac{\mu}{r^2} w + \frac{1}{r^2} \int_{S^{N-1}} \left(\Delta_{S^{N-1}} \psi \right) \phi d\theta. \end{aligned}$$

Integrating the last term by parts we get

$$-w'' - \frac{N-1}{r}w' - \nu p w_R^{p-1} w = \frac{\mu - \lambda_k}{r^2} w,$$

which implies that the numbers $\mu - \lambda_k$ are eigenvalues of the operator $\hat{L}_R^\nu$, hence

$$\mu = \mu - \lambda_k + \lambda_k \in \sigma(\hat{L}_R^\nu) + \sigma(-\Delta_{S^{N-1}}). \tag{3.6}$$

In order to see the converse consider $\alpha \in \sigma(\hat{L}_R^\nu)$ and $\lambda_k \in \sigma(-\Delta_{S^{N-1}})$, and choose corresponding eigenfunctions w and ϕ. Setting

$$v(x) = w(|x|)\phi\left(\frac{x}{|x|}\right),$$

there holds

$$\begin{aligned} -\Delta v &= \left(-w'' - \frac{N-1}{r}w'\right)\phi - \frac{w}{r^2}\Delta_{S^{N-1}}\phi \\ &= \left[\nu p w_R^{p-1} w + \frac{\alpha}{r^2} w\right]\phi + \frac{\lambda_k}{r^2} w\phi = \nu p w_R^{p-1} v + \frac{\alpha + \lambda_k}{r^2} v \end{aligned}$$

which implies $\alpha + \lambda_k \in \sigma(\widetilde{L}_R^\nu)$. □

As pointed out before we are interested in studying the eigenvalues of problem (3.1) close to 1 for large R. To do this we first study the asymptotic behavior of the eigenvalues of the operator $\hat{L}_R^\nu$ as $R \to \infty$. We need the eigenvalues β_n^ν of the operator $\bar{L}^\nu$ on $(0,1)$ with Dirichlet boundary conditions defined by

$$\bar{L}^\nu \psi = -\psi'' - \nu p w_0^{p-1} \psi, \tag{3.7}$$

where w_0 is from (2.6).

Lemma 3.2. *The n-th eigenvalue $\alpha_n^\nu(R)$ of the operator $\hat{L}_R^\nu$ satisfies*

$$\alpha_n^\nu(R) = \beta_n^\nu R^2 + o\left(R^2\right) \quad \textit{as } R \to \infty. \tag{3.8}$$

Proof. We consider the operator $\bar{L}_R^\nu$ on $(0,1)$ with Dirichlet boundary conditions defined by

$$\bar{L}_R^\nu \psi := \frac{(t+R)^2}{R^2}\left(-\psi'' - \frac{N-1}{t+R}\psi' - \nu p \widetilde{w}_R^{p-1}\psi\right).$$

If w is an n-th eigenfunction of $\hat{L}_R^\nu$ then $\psi_R(t) := w(t+R)$ satisfies

$$\bar{L}_R^\nu \psi_R = \frac{\alpha_n^\nu(R)}{R^2}\psi_R$$

and viceversa. Consequently there holds

$$\sigma(\hat{L}_R^\nu) = R^2 \sigma(\bar{L}_R^\nu).$$

Since the coefficients of $\bar{L}_R^\nu$ converge uniformly on $(0,1)$ towards the coefficients of $\bar{L}^\nu$ as $R \to \infty$, we obtain

$$\sigma\left(\bar{L}_R^\nu\right) = \sigma\left(\bar{L}^\nu\right) + o(1).$$

and the result follows immediately. $\square$

Remark 3.3. *Let β_1 and β_2 the first and the second eigenvalue respectively of the operator $\bar{L}^1$ from (3.7) with $\nu = 1$. It is well known that the unique positive solution w_0 of (2.6) has Morse index one and hence $\beta_1 < 0$ and $\beta_2 \geq 0$. Moreover it is easy to prove that $\beta_2 > 0$, repeating for example the proof of Proposition 2.1. Then, by the continuity of the eigenvalues there exists $\sigma > 0$ such that*

$$|\nu - 1| < \sigma \quad \Longrightarrow \quad \beta_1^\nu < 0 \text{ and } \beta_2^\nu > 0, \tag{3.9}$$

where β_1^ν and β_2^ν are the first and the second eigenvalue of $\bar{L}^\nu$. The proof of (3.9) is not difficult and can be found in [GPY].

An immediate consequence of this and Lemma 3.2 is

Corollary 3.4. *If $|\nu - 1| < \sigma$, σ as in (3.9), then*

$$\alpha_2^\nu(R) > 0 \quad \text{for } R \text{ sufficiently large,} \tag{3.10}$$

where $\alpha_2^\nu(R)$ is the second eigenvalue of the operator $\hat{L}_R^\nu$ from (3.3).

Now let us come back to the problem from which we started in this section, i.e. to understand for which values R of the radius a number ν close to 1 can be an eigenvalue of (3.1) in A_R. A consequence of the results obtained so far is

Proposition 3.5. *Assume $|\nu - 1| < \sigma$, σ as in (3.9). Then there exists $R_0 > 0$ such that ν can be an eigenvalue of (3.1) in A_R for $R > R_0$, if and only if*

$$\alpha_1^\nu(R) = -\lambda_k = -k(k + N - 2) \quad \text{for some } k \geq 1. \tag{3.11}$$

Proof. We know that ν is an eigenvalue of (3.1) if and only if

$$0 \in \sigma(\widetilde{L}_R^\nu) = \sigma(\hat{L}_R^\nu) + \sigma(-\Delta_{S^{N-1}}).$$

By the assumption on ν we can apply Lemma 3.4 so that $\alpha_1^\nu(R) < 0$ and $\alpha_2^\nu(R) > 0$ for R large. Now (3.11) follows immediately. $\square$

4 Analysis of the eigenvalues of (3.1) close to 1

We start by making a deeper analysis on the behavior of $\alpha_1^\nu(R)$ showing that, for large R, it is a strictly decreasing function of the radius R. As in Section 2 we denote by $\widetilde{w}_R(t)$ the function $w_R(t + R)$ which is the only positive solution of

$$\begin{cases} -u'' - \frac{N-1}{t+R} u' = u^p & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.1}$$

We have

Lemma 4.1. *The function $\widetilde{w}_R$ is continuously differentiable with respect to R. Moreover*

$$\lim_{R\to\infty} R^q \int_0^1 \left|\frac{\partial \widetilde{w}_R}{\partial R}\right|^q dt = 0 \quad \forall q > 1. \tag{4.2}$$

Proof. Using the nondegeneracy of the solution $\widetilde{w}_R$, which follows from Proposition 2.1, and applying the implicit function theorem to the function

$$F(\phi, R) = \phi'' + \frac{N-1}{t+R}\phi' + \phi^p$$

it easy to see that $\widetilde{w}_R$ is continuously differentiable with respect to R. The function $V(t,R) := \frac{\partial \widetilde{w}_R}{\partial R}$ satisfies

$$\begin{cases} -V'' - \frac{N-1}{t+R}V' + \frac{N-1}{(t+R)^2}\widetilde{w}_R' = p\widetilde{w}_R^{p-1}V & \text{in } (0,1), \\ V(0) = V(1) = 0. \end{cases}$$

We claim that

$$R\|V(\,.\,,R)\|_{H_0^1(0,1)} \le C \tag{4.3}$$

for some constant $C > 0$. If (4.3) does not hold then for a sequence $R_n \to \infty$ we have

$$R_n\|V(\cdot,R_n)\|_{H_0^1(0,1)} \to \infty \quad \text{as } n \to \infty.$$

Now $z_n = \frac{V(\cdot,R_n)}{\|V(\cdot,R_n)\|_{H_0^1(0,1)}}$ satisfies

$$\begin{cases} -z_n'' - \frac{N-1}{t+R_n}z_n' + \frac{(N-1)R_n\widetilde{w}_{R_n}'}{(t+R_n)^2 R_n\|V(\cdot,R_n)\|_{H_0^1}} = p\widetilde{w}_{R_n}^{p-1}z_n & \text{in } (0,1), \\ z_n(0) = z_n(1) = 0, \end{cases} \tag{4.4}$$

and $z_n \to z_0$ weakly in $H_0^1(0,1)$ and strongly in $L^q(0,1)$, for any $q > 1$. Moreover, since w_{R_n}' is bounded in $L^\infty(0,1)$ by the results of Section 2, passing to the limit in (4.4) yields

$$\begin{cases} -z_0'' = pw_0^{p-1}z_0 & \text{in } (0,1), \\ z(0) = z(1) = 0, \end{cases}$$

with w_0 from (2.6). Then $z_0 \equiv 0$ by Remark 3.3. This is a contradiction because from the equations we also get that z_n converges strongly to z_0 in $H_0^1(0,1)$ so that $\|z_0\|_{H_0^1(0,1)} = 1$. Consequently (4.3) holds. This implies that the function $RV(\cdot,R)$ converges weakly in $H_0^1(0,1)$ and strongly in $L^q(0,1)$, for any $q > 1$, to a function $\bar{V}$ which solves as before

$$\begin{cases} -\bar{V}'' = pw_0^{p-1}\bar{V} & \text{in } (0,1), \\ \bar{V}(0) = \bar{V}(1) = 0. \end{cases}$$

Again $\bar{V}$ must be identically zero so that (4.2) holds. □

Lemma 4.2. *The first eigenvalue $\alpha_1^\nu(R)$ of the operator $\hat{L}_R^\nu$ is differentiable with respect to R and*

$$\frac{\partial \alpha_1^\nu(R)}{\partial R} = 2\beta_1^\nu R + o(R) \quad \text{as } R \to \infty, \tag{4.5}$$

where β_1^ν is the first eigenvalue of $\bar{L}^\nu$ from (3.7).

Proof. We consider the first eigenfunction $v_{1,R}$ of $\hat{L}_R^\nu$ with $\|v_{1,R}\|_\infty = 1$. The function $\widetilde{v}_{1,R}(t) = v_1(t+R)$ then solves

$$\begin{cases} -\widetilde{v}_{1,R}'' - \frac{N-1}{t+R}\widetilde{v}_{1,R}' - \nu p \|w_R\|_\infty^{p-1}\widetilde{v}_{1,R} = \alpha_1^\nu(R)\frac{\widetilde{v}_{1,R}}{(t+R)^2} & \text{in } (0,1), \\ \widetilde{v}_{1,R}(0) = \widetilde{v}_{1,R}(1) = 0. \end{cases} \tag{4.6}$$

Since $\|v_{1,R}\|_\infty = 1$ we have that $\widetilde{v}_{1,R} \to \phi_1 \not\equiv 0$, as $R \to \infty$, uniformly in $(0,1)$ and $\phi_1 \geq 0$ solves

$$\begin{cases} -\phi_1'' - \nu p \|w_0\|_\infty^{p-1}\phi_1 = \beta_1^\nu \phi_1 & \text{in } (0,1), \\ \phi_1(0) = \phi_1(1) = 0. \end{cases} \tag{4.7}$$

By results of Kato (see [K], p. 380) we have that both $\widetilde{v}_{1,R}$ and $\alpha_1^\nu(R)$ depend analytically on R. Thus the function $\Phi = \Phi(t,R) = \frac{\partial \widetilde{v}_{1,R}}{\partial R}$ satisfies

$$\begin{aligned} &-\Phi'' - \frac{N-1}{t+R}\Phi' + \frac{N-1}{(t+R)^2}\widetilde{v}_{1,R}' - \nu p(p-1)\widetilde{w}_R^{p-2}\frac{\partial \widetilde{w}_R}{\partial R}\widetilde{v}_{1,R} - \nu p \widetilde{w}_R^{p-1}\Phi \\ &\quad = \frac{\partial \alpha_1^\nu(R)}{\partial R}\frac{1}{(t+R)^2}\widetilde{v}_{1,R} + \frac{\alpha_1^\nu(R)}{(t+R)^2}\Phi - 2\frac{\alpha_1^\nu(R)}{(t+R)^3}\widetilde{v}_{1,R}. \end{aligned}$$

Multiplying this equation by $\widetilde{v}_{1,R}$ and integrating we get

$$\begin{aligned} &\int_0^1 \Phi'\widetilde{v}_{1,R}'(t+R)^{N-1}dt + (N-1)\int_0^1 \widetilde{v}_{1,R}'\widetilde{v}_{1,R}(t+R)^{N-3}dt \\ &\qquad - \nu p(p-1)\int_0^1 \widetilde{w}_R^{p-2}\frac{\partial \widetilde{w}_R}{\partial R}\widetilde{v}_{1,R}^2(t+R)^{N-1}dt \\ &\qquad - \nu p \int_0^1 \widetilde{w}_R^{p-1}\widetilde{v}_{1,R}\Phi(t+R)^{N-1}dt \\ &\quad = \frac{\partial \alpha_1^\nu(R)}{\partial R}\int_0^1 \widetilde{v}_{1,R}^2(t+R)^{N-3}dt + \alpha_1^\nu(R)\int_0^1 \Phi\widetilde{v}_{1,R}(t+R)^{N-3}dt \\ &\qquad - 2\alpha_1^\nu(R)\int_0^1 \Phi\widetilde{v}_{1,R}^2(t+R)^{N-4}dt. \end{aligned}$$

Multiplying instead (4.6) by Φ and integrating we get

$$\begin{aligned} &\int_0^1 \Phi'\widetilde{v}_{1,R}'(t+R)^{N-1}dt - \nu p\int_0^1 \widetilde{w}_R^{p-1}\widetilde{v}_{1,R}\Phi(t+R)^{N-1}dt \\ &\quad = \alpha_1^\nu(R)\int_0^1 \Phi\widetilde{v}_{1,R}(t+R)^{N-3}dt. \end{aligned}$$

Subtracting the last two equations we deduce

$$(N-1)\int_0^1 \widetilde{v}_{1,R}'\widetilde{v}_{1,R}(t+R)^{N-3}dt - \nu p(p-1)\int_0^1 \widetilde{w}_R^{p-2}\frac{\partial \widetilde{w}R}{\partial R}\widetilde{v}_{1,R}^2(t+R)^{N-1}dt$$
$$= \frac{\partial \alpha_1^\nu(R)}{\partial R}\int_0^1 \widetilde{v}_{1,R}^2(t+R)^{N-3}dt - 2\alpha_1^\nu(R)\int_0^1 \widetilde{v}_{1,R}^2(t+R)^{N-4}dt.$$

Since

$$(N-1)\int_0^1 \widetilde{v}_{1,R}'\widetilde{v}_{1,R}(t+R)^{N-3}dt = -\frac{(N-1)(N-3)}{2}\int_0^1 \widetilde{v}_{1,R}^2(t+R)^{N-4}dt$$
$$= O\left(R^{N-4}\right)$$

and

$$\nu p(p-1)\int_0^1 \widetilde{w}_R^{p-2}\left(R\frac{\partial \widetilde{w}_R}{\partial R}\right)\widetilde{v}_{1,R}^2\frac{(t+R)^{N-1}}{R}dt = o\left(R^{N-2}\right)$$

by (4.2) we have

$$\frac{\partial \alpha_1^\nu(R)}{\partial R}\int_0^1 \widetilde{v}_{1,R}^2\frac{(t+R)^{N-3}}{R^{N-3}}dt = 2\alpha_1^\nu(R)\int_0^1 \widetilde{v}_{1,R}^2\frac{(t+R)^{N-4}R^2}{R^{N-3}}dt + o(R).$$

Finally using the convergence of $\widetilde{v}_{1,R}$ to ϕ_1 as in (4.7) and (3.8) we get

$$\frac{\partial \alpha_1^\nu(R)}{\partial R}\left(\int_0^1 \phi_1^2 dt + o(1)\right) = 2\beta_1^\nu R\left(\int_0^1 \phi_1^2 dt + o(1)\right) + o(R)$$

so that (4.5) holds. □

¿From (4.5) we deduce that if $|\nu - 1| < \sigma$ as in (3.9) the function $\alpha_1^\nu(R)$ is a strictly decreasing function of R, for R large. This allows to prove

Proposition 4.3. *Let $|\nu-1| < \sigma$ as in (3.9). Then there exists $\bar{R} > 0$ such that ν can be an eigenvalue of problem* (3.1) *for $R > \bar{R}$ at most for a sequence $R = R_k^\nu$ which behaves asymptotically like*

$$R_k^\nu = \sqrt{\frac{-k(k+N-2)}{\beta_1^\nu}} + o(1) \quad \text{as } k \to \infty. \tag{4.8}$$

Proof. As a consequence of Lemma 4.2 there exists $\bar{R} > 0$ such that $\alpha_1^\nu(R)$ is strictly decreasing for $R > \bar{R}$. Then for any $k \geq 1$ the equation (3.11) has at most one solution $R = R_k^\nu$. Now (3.8) yields

$$\left(\beta_1^\nu + o(1)\right)\left(R_k^\nu\right)^2 = -k(k+N-2),$$

from which (4.8) follows. □

In particular, from (4.8) we deduce that the only possible radii for which the linearized operator L_{w_R} can be degenerate, i.e. $\nu = 1$ is an eigenvalue of (3.1), are R^1_k, $k \geq 1$, and these behave asymptotically like

$$R^1_k = \sqrt{\frac{-k(k+N-2)}{\beta_1}} + o(1) \qquad \text{as } k \to \infty. \tag{4.9}$$

This implies that for any $R > \bar{R}$ with $R \neq R^1_k$ for all $k \geq 1$ the linearized operator L_{w_R} is nondegenerate. Observe that

$$\tau := \lim_{k\to\infty} \left(R^1_{k+1} - R^1_k\right) = \frac{1}{\sqrt{|\beta_1|}}. \tag{4.10}$$

We conclude this section by making a finer analysis of the nondegeneracy of L_{w_R}. More precisely we prove that when R is at certain distance from the "bad radii" R^1_k then the eigenvalues ν_R of (3.1) are bounded away from 1 by a constant which depends on the distance between R and R^1_k, but is independent from k.

Proposition 4.4. *For $\delta > 0$ there exists $\gamma(\delta) > 0$ and $k(\delta) \in \mathbb{N}$ such that for $k \geq k(\delta)$ and $R \in \left(R^1_k, R^1_{k+1}\right)$ with* $\min\left\{R - R^1_k, |R - R^1_{k+1}|\right\} \geq \delta$

$$|\nu_R - 1| \geq \gamma(\delta) \tag{4.11}$$

for any eigenvalue ν_R of problem (3.1).

Proof. Arguing by contradiction we assume there exists a sequence $k_n \to \infty$, a sequence of radii $R_n \in \left(R^1_{k_n}, R^1_{k_n+1}\right)$ with $\min\left\{R_n - R^1_{k_n}, |R_n - R^1_{k_n+1}|\right\} \geq \delta$, and a sequence of eigenvalues ν_n of problem (3.1) such that

$$\lim_{n\to\infty} \nu_n = 1$$

Then, obviously, $|\nu_n - 1| < \sigma$, σ as in (3.9), for n sufficiently large and hence by (4.8)

$$R_n = \frac{\sqrt{h_n(h_n+N-2)}}{\sqrt{-\beta_1^{\nu_n}}} + o(1) \tag{4.12}$$

for a sequence of positive integers $h_n \to \infty$. This implies $R_n - R^1_{k_n} \to 0$ because $\beta_1^\nu \to \beta_1$ as $\nu \to 1$. □

5 Study of the approximate solutions

In this section we go back to the domain Ω_R defined in (1.2) which is diffeomorphic to the annulus A_R by the diffeomorphism T defined in (1.3). As in the previous sections we denote by w_R the unique positive radial solution of (1.1) for $\Omega = A_R$ and by $\widetilde{u}_R$ the function defined in Ω_R as in (1.4), i.e. $\widetilde{u}_R(\rho,\theta) = w_R\left(T(\rho,\theta)\right)$. We will prove that $\widetilde{u}_R$ is an "approximate" solution of (1.1) in Ω_R, for large R, and we derive some useful estimates.

Lemma 5.1.
$$-\Delta \widetilde{u}_R = \widetilde{u}_R^p + O\left(\frac{1}{R^{2+s}}\right), \tag{5.1}$$
where $s = 0$ *if* $N \leq 4$, $s > \frac{N-5}{2}$ *if* $N \geq 5$, *is from* (1.2).

Proof. Using polar coordinates and the inverse transformation

$$T^{-1}(r,\theta) = (\rho,\theta) = \left(r + \frac{g(\theta)}{R^s}, \theta\right)$$

we get for $\widetilde{u}_R = w_R \circ T$

$$\begin{aligned}
-\Delta \widetilde{u}_R &= -\frac{\partial^2 \widetilde{u}_R}{\partial \rho^2} - \frac{N-1}{\rho}\frac{\partial \widetilde{u}_R}{\partial \rho} - \frac{1}{\rho^2}\Delta_{S^{N-1}} \widetilde{u}_R \\
&= -\frac{\partial^2 w_R}{\partial r^2} - \frac{N-1}{r}\frac{\partial w_R}{\partial r} + \frac{(N-1)\frac{g(\theta)}{R^s}}{r\left(r + \frac{g(\theta)}{R^s}\right)}\frac{\partial w_R}{\partial r} - \frac{1}{\left(r + \frac{g(\theta)}{R^s}\right)^2}\Delta_{S^{N-1}} \widetilde{u}_R \\
&= w_R^p + O\left(\frac{1}{R^{2+s}}\right),
\end{aligned}$$

having used the equation satisfied by w_R and that $|\Delta_{S^{N-1}} \widetilde{u}_R| = O(1/R^s)$. The last fact holds because w_R is radial so that

$$\frac{\partial \widetilde{u}_R}{\partial \theta} = \frac{\partial w_R}{\partial r} \cdot \left(-\frac{\partial g}{\partial \theta}\frac{1}{R^s}\right),$$

and similarly for the second derivatives. □

Now we consider the functional

$$I_R(u) = \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_R} |u|^{p+1}$$

and observe that for $p > 1$, and $p + 1 \leq 2^* = \frac{2N}{N-2}$ if $N \geq 3$, I_R is well defined and of class C^2 in $H_0^1(\Omega_R)$. It is well known that in this case the solutions of (1.1) correspond to the critical points of $I_R(u)$ and that for any $u \in H_0^1(\Omega_R)$ the derivative $I_R'(u)$ can be represented by the element $\text{grad} I_R(u) \in H_0^1(\Omega_R)$ given by
$$\text{grad} I_R(u) = u - \left(-\Delta^{-1}\right)\left(|u|^{p-1}u\right) \tag{5.2}$$
We have

Lemma 5.2. *If* $p > 1$, *and* $p \leq 2^* - 1 = \frac{N+2}{N-2}$ *if* $N \geq 3$, *then*

$$\|\text{grad} I_R(\widetilde{u}_R)\|_{H_0^1(\Omega_R)} \leq \frac{C_1}{R^\alpha} \tag{5.3}$$

with $\alpha = \frac{5-N+2s}{2} > 0$, s *as in (1.2) and* C_1 *independent of* R.

Proof. Setting

$$z_R = \mathrm{grad} I_R(\widetilde{u}_R) = \widetilde{u}_R - \left(-\Delta^{-1}\right)(\widetilde{u}_R^p)$$

we have

$$-\Delta \widetilde{u}_R - \widetilde{u}_R^p = -\Delta z_R$$

and hence, using (5.1),

$$\int_{\Omega_R} |\nabla z_R|^2 = \int_{\Omega_R} (-\Delta \widetilde{u}_R - \widetilde{u}_R^p) z_R \le \left(\int_{\Omega_R} |-\Delta \widetilde{u}_R - \widetilde{u}_R^p|^2\right)^{\frac{1}{2}} \cdot \left(\int_{\Omega_R} |z_R|^2\right)^{\frac{1}{2}}$$
$$\le C \left[\int_{\Omega_R} \left(\frac{1}{R^{2+s}}\right)^2\right]^{\frac{1}{2}} P_0 \left(\int_{\Omega_R} |\nabla z_R|^2\right)^{\frac{1}{2}},$$

where P_0 is the constant of the Poincaré inequality which is independent of R (see [A, Lemma 5.14]). Consequently

$$\|z_R\| \le C_1 \frac{1}{R^{2+s}} R^{\frac{N-1}{2}} = \frac{C_1}{R^\alpha}$$

because meas $\Omega_R = O\left(R^{N-1}\right)$. □

The previous estimates imply that $\widetilde{u}_R$ are approximate solutions to (1.1). In order to prove that near $\widetilde{u}_R$ there is a true solution of (1.1) we need to prove that the linear operator $I_R''(\widetilde{u}_R)$ is invertible. To do this we need some preliminary results.

Lemma 5.3. *Let v be any function in $H_0^1(A_R)$ and set $\widetilde{v} := v \circ T : \Omega_R \to \mathbb{R}$ with T from* (1.3)*. Then $\widetilde{v} \in H_0^1(\Omega_R)$ and*

$$\int_{\Omega_R} |\nabla \widetilde{v}|^2 dx = \int_{A_R} |\nabla v|^2 dy + O\left(\frac{1}{R^{s+1}} \int_{A_R} |\nabla v|^2 dy\right). \tag{5.4}$$

Proof. Using spherical coordinates $(\rho, \theta_1, \ldots, \theta_{N-1})$ in $\mathbb{R}^N$, $N \ge 2$, we have

$$|\nabla \widetilde{v}|^2 = \left(\frac{\partial \widetilde{v}}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left(\frac{\partial \widetilde{v}}{\partial \theta_i}\right)^2, \tag{5.5}$$

where $(\theta_1, \ldots, \theta_{N-1})$ and $a_i(\theta)$ are bounded functions independent of ρ. Moreover, by the definition of the diffeomorphism T given in (1.3),

$$\begin{cases} \dfrac{\partial \widetilde{v}}{\partial \rho} = \dfrac{\partial v}{\partial r}, \qquad r = \rho - \dfrac{g(\theta)}{R^s}, \\ \dfrac{\partial \widetilde{v}}{\partial \theta_i} = \dfrac{\partial v}{\partial \theta_i} - \dfrac{1}{R^s} \dfrac{\partial v}{\partial r} \dfrac{\partial g}{\partial \theta_i}. \end{cases} \tag{5.6}$$

Hence (5.5) yields

$$
\begin{aligned}
\int_{\Omega_R} |\nabla \widetilde{v}|^2 dx &= \int_{A_R} \left[\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{\left(r + \frac{g(\theta)}{R^s}\right)^2} \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left(\frac{1}{R^{2s}} \left(\frac{\partial v}{\partial r}\right)^2 \left(\frac{\partial g}{\partial \theta_i}\right)^2 \right.\right. \\
&\qquad \left.\left. + \left(\frac{\partial v}{\partial \theta_i}\right)^2 - \frac{2}{R^s} \frac{\partial v}{\partial r} \frac{\partial v}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \right) \right] dy \\
&= \int_{A_R} \left[\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2} \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left(\frac{\partial v}{\partial \theta_i}\right)^2 \right] dy \\
&\qquad + \int_{A_R} \frac{1}{\left(r + \frac{g(\theta)}{R^s}\right)^2} \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left[\frac{1}{R^{2s}} \left(\frac{\partial v}{\partial r}\right)^2 \left(\frac{\partial g}{\partial \theta_i}\right)^2 - \frac{2}{R^s} \frac{\partial v}{\partial r} \frac{\partial v}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \right] dy \\
&\qquad + \int_{A_R} \left[\frac{1}{\left(r + \frac{g(\theta)}{R^s}\right)^2} - \frac{1}{r^2} \right] \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left(\frac{\partial v}{\partial \theta_i}\right)^2 dy \\
&= \int_{A_R} |\nabla v|^2 \, dy + I_1 + I_2,
\end{aligned}
$$

having denoted by I_1, I_2 the last two integrals. Now (5.4) follows from

$$
\begin{aligned}
|I_1| &\le \frac{C}{R^{2s+2}} \int_{A_R} \left(\frac{\partial v}{\partial r}\right)^2 dy + \int_{A_R} \frac{C}{r^2 R^s} \sum_{i=1}^{N-1} \left[r \left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r} \left(\frac{\partial v}{\partial \theta_i}\right)^2 \right] dy \\
&\le \frac{C}{R^{2s+2}} \int_{A_R} \left(\frac{\partial v}{\partial r}\right)^2 dy + \frac{C}{R^{s+1}} \int_{A_R} \left(\frac{\partial v}{\partial r}\right)^2 dy \\
&\qquad + \frac{C}{R^{s+1}} \int_{A_R} \sum_{i=1}^{N-1} \frac{1}{r^2} \left(\frac{\partial v}{\partial \theta_i}\right)^2 dy \\
&\le \frac{C}{R^{s+1}} \int_{A_R} |\nabla v|^2 \, dy
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\le \int_{A_R} \frac{\left| 2r \frac{g(\theta)}{R^s} + \frac{g^2(\theta)}{R^{2s}} \right|}{\left(r + \frac{g(\theta)}{R^s}\right)^2 r^2} \sum_{i=1}^{N-1} |a_i(\theta)|^2 \left(\frac{\partial v}{\partial \theta_i}\right)^2 dy \\
&\le \frac{C}{R^{s+1}} \int_{A_R} \frac{1}{r^2} \sum_{i=1}^{N-1} \left(\frac{\partial v}{\partial \theta_i}\right)^2 dy \le \frac{C}{R^{s+1}} \int_{A_R} |\nabla v|^2 \, dy.
\end{aligned}
$$

□

Let us define the functionals

$$
Q_R(v) = \frac{\int_{A_R} |\nabla v|^2 \, dy}{\int_{A_R} p w_R^{p-1} v^2 dy}, \qquad v \in H_0^1(A_R), \ v \not\equiv 0,
$$

and

$$\widetilde{Q}_R(u) = \frac{\int_{\Omega_R} |\nabla u|^2 \, dx}{\int_{\Omega_R} p\widetilde{u}_R^{p-1} u^2 dx}, \qquad u \in H_0^1(\Omega_R),\ u \not\equiv 0.$$

Then we consider the eigenvalue problem

$$\begin{cases} -\Delta v = \widetilde{\nu} p \widetilde{u}_R^{p-1} v & \text{in } \Omega_R \\ v = 0 & \text{on } \partial\Omega_R \end{cases} \tag{5.7}$$

and denote by $\phi_1^R, \dots, \phi_k^R$ and $\widetilde{\psi}_1^R, \dots, \widetilde{\psi}_k^R$ the eigenfunctions of (3.1) and (5.7) respectively, with $\|\phi_i^R\|_{L^2(A_R, pw_R^{p-1})} = \|\widetilde{\psi}_i^R\|_{L^2(A_R, pw_R^{p-1})} = 1$, and by $\nu_1^R, \dots, \nu_k^R$, $\widetilde{\nu}_1^R, \dots, \widetilde{\nu}_k^R$ the corresponding eigenvalues.

Lemma 5.4. *Let V_k^R be the subspace of $H_0^1(A_R)$ spanned by $\phi_1^R, \dots, \phi_k^R$, then*

$$\widetilde{Q}_R(\widetilde{v}) = \nu_k^R + O\left(R^{-1}\right) \nu_k^R \qquad \text{as } R \to \infty \tag{5.8}$$

for any $\widetilde{v} \in \widetilde{V}_k^R$ where $\widetilde{V}_k^R$ is the space spanned by $\widetilde{\phi}_1^R = \phi_1^R \circ T, \dots, \widetilde{\phi}_k^R = \phi_k^R \circ T$. Conversely if $\widetilde{W}_k^R$ is the subspace of $H_0^1(\Omega_R)$ spanned by $\widetilde{\psi}_1^R, \dots, \widetilde{\psi}_k^R$ we have

$$\widetilde{Q}_R(v) = \widetilde{\nu}_k^R + O\left(R^{-1}\right) \widetilde{\nu}_k^R \qquad \text{as } R \to \infty \tag{5.9}$$

for any $v \in W_k^R$ which is the space spanned by $\psi_1^R = \widetilde{\psi}_1^R \circ T, \dots, \psi_k^R = \widetilde{\psi}_k^R \circ T$.

Proof. We start by observing that the functions $\phi_1^R, \dots, \phi_k^R$ are linearly independent and the same is true for $\widetilde{\phi}_1^R, \dots, \widetilde{\phi}_k^R$. Writing $\widetilde{v}$ as $\sum_{i=1}^k \alpha_i \widetilde{\phi}_i^R$ we have

$$\widetilde{Q}_R(\widetilde{v}) = \frac{\sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega_R} \nabla \widetilde{\phi}_i^R \nabla \widetilde{\phi}_j^R dx}{\sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega_R} p\widetilde{u}_R^{p-1} \widetilde{\phi}_i \widetilde{\phi}_j dx}.$$

As in (5.5), using spherical coordinates we have

$$\nabla \widetilde{\phi}_i^R \nabla \widetilde{\phi}_j^R = \frac{\partial \widetilde{\phi}_i^R}{\partial \rho} \frac{\partial \widetilde{\phi}_j^R}{\partial \rho} + \frac{1}{\rho^2} \sum_{l=1}^k a_l^2(\theta) \frac{\partial \widetilde{\phi}_i^R}{\partial \theta_l} \frac{\partial \widetilde{\phi}_j^R}{\partial \theta_l},$$

and analogously

$$\nabla \phi_i^R \nabla \phi_j^R = \frac{\partial \phi_i^R}{\partial r} \frac{\partial \phi_j^R}{\partial r} + \frac{1}{r^2} \sum_{l=1}^k a_l^2(\theta) \frac{\partial \phi_i^R}{\partial \theta_l} \frac{\partial \phi_j^R}{\partial \theta_l}.$$

Moreover, by the definition of the diffeomorphism T, formulas analogous to (5.6) hold. Therefore we have

$$
\begin{aligned}
\nabla\widetilde{\phi}_i^R\nabla\widetilde{\phi}_j^R &= \frac{\partial\phi_i^R}{\partial r}\frac{\partial\phi_j^R}{\partial r}+\frac{1}{\left(r+\frac{g(\theta)}{R^s}\right)^2}\sum_{l=1}^k a_l^2(\theta)\left(-\frac{1}{R^s}\frac{\partial\phi_i^R}{\partial r}\frac{\partial g}{\partial\theta_l}+\frac{\partial\phi_i^R}{\partial\theta_l}\right)\cdot\\
&\quad\cdot\left(-\frac{1}{R^s}\frac{\partial\phi_j^R}{\partial r}\frac{\partial g}{\partial\theta_l}+\frac{\partial\phi_j^R}{\partial\theta_l}\right)\\
&=\frac{\partial\phi_i^R}{\partial r}\frac{\partial\phi_j^R}{\partial r}+\left(\frac{1}{r^2}+O\left(R^{-3-s}\right)\right)\sum_{l=1}^k a_l^2(\theta)\left[\frac{\partial\phi_i^R}{\partial r}\frac{\partial\phi_j^R}{\partial r}\left(\frac{\partial g}{\partial\theta_l}\right)^2+\right.\\
&\quad\left.+\frac{\partial\phi_i^R}{\partial\theta_l}\frac{\partial\phi_j^R}{\partial\theta_l}-\frac{\partial\phi_i^R}{\partial r}\frac{\partial\phi_j^R}{\partial\theta_l}\frac{\partial g}{\partial\theta_l}-\frac{\partial\phi_i^R}{\partial\theta_l}\frac{\partial\phi_j^R}{\partial r}\frac{\partial g}{\partial r}\right]\\
&\le\nabla\phi_i^R\nabla\phi_j^R+O\left(R^{-1}\right)\nabla\phi_i^R\nabla\phi_j^R+C\left(\frac{1}{r^2}+O\left(R^{-3-s}\right)\right)\cdot\\
&\quad\cdot\sum_{l=1}^k\left[r\left(\frac{\partial\phi_i^R}{\partial r}\right)^2+r\left(\frac{\partial\phi_j^R}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial\phi_i^R}{\partial\theta_l}\right)^2+\frac{1}{r}\left(\frac{\partial\phi_j^R}{\partial\theta_l}\right)^2\right]\\
&\le\left(1+O\left(R^{-1}\right)\right)\nabla\phi_i^R\nabla\phi_j^R+C\left(\frac{1}{r^2}+O\left(R^{-3-s}\right)\right)r\left(\left|\nabla\phi_i^R\right|^2+\left|\nabla\phi_j^R\right|^2\right)\\
&=\left(1+O\left(R^{-1}\right)\right)\nabla\phi_i^R\nabla\phi_j^R+O\left(R^{-1}\right)\left(\left|\nabla\phi_i^R\right|^2+\left|\nabla\phi_j^R\right|^2\right).
\end{aligned}
$$

On the other hand we have that

$$
\begin{aligned}
p\int_{\Omega_R}\widetilde{u}_R^{p-1}\widetilde{\phi}_i^R\widetilde{\phi}_j^R dx &= p\int_{S^{N-1}}\int_{R+g(\theta)}^{R+1+g(\theta)}\widetilde{u}_R^{p-1}\widetilde{\phi}_i^R\widetilde{\phi}_j^R\rho^{N-1}b(\phi)d\rho d\theta\\
&=p\int_{S^{N-1}}\int_R^{R+1}w_R^{p-1}\phi_i^R\phi_j^R\left(r+\frac{g(\theta)}{R^s}\right)^{N-1}b(\theta)drd\theta\\
&=p\int_{S^{N-1}}\int_R^{R+1}w_R^{p-1}\phi_i^R\phi_j^R r^{N-1}b(\theta)drd\theta\\
&\quad+\int_R^{R+1}\int_{S^{N-1}}w_R^{p-1}\phi_i^R\phi_j^R\left[\left(r+\frac{g(\theta)}{R^s}\right)^{N-1}-r^{N-1}\right]b(\theta)drd\theta\\
&=p\int_{A_R}w_R^{p-1}\phi_i^R\phi_j^R dy+O\left(R^{-1-s}\right)\int_{A_R}w_R^{p-1}|\phi_i^R||\phi_j^R|dy
\end{aligned}
$$

with $O\left(R^{-1-s}\right)>0$. Hence, using also (5.4), the Rayleigh quotient becomes

$$
\widetilde{Q}_R(\widetilde{v})=\frac{\sum\limits_{i,j=1}^k\alpha_i\alpha_j\int_{A_R}\nabla\phi_i^R\nabla\phi_j^R dy+O\left(R^{-1}\right)\sum\limits_{i,j=1}^k\alpha_i\alpha_j\int_{A_R}\left[|\nabla\phi_i^R|^2+|\nabla\phi_j^R|^2\right]dy}{\sum\limits_{i,j=1}^k\alpha_i\alpha_j p\int_{A_R}w_R^{p-1}\phi_i^R\phi_j^R dy+O\left(R^{-1-s}\right)\sum\limits_{i,j=1}^k\alpha_i\alpha_j p\int_{A_R}w_R^{p-1}|\phi_i^R||\phi_j^R|dy}.
$$

Recalling that $\phi_1^R, \dots, \phi_k^R$ are orthogonal in $H_0^1(A_R)$ and in $L^2(A_R, w_R^{p-1})$, and that

$$\sum_{i,j=1}^{k} \alpha_i\alpha_j p \int_{A_R} w_R^{p-1}|\phi_i^R||\phi_j^R|dy = p\int_{A_R} w_R^{p-1}\left(\sum_{i=1}^{k}\alpha_i|\phi_i^R|\right)^2 dy \geq 0$$

we have

$$\widetilde{Q}_R(\widetilde{v}) \leq \frac{\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R}|\nabla\phi_i^R|^2dy + O\left(R^{-1}\right)\left(\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R}|\nabla\phi_i^R|^2dy + \sum\limits_{j=1}^{k}\alpha_j^2\int_{A_R}|\nabla\phi_j^R|^2dy\right)}{p\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R} w_R^{p-1}\left(\phi_i^R\right)^2dy},$$

and using $p\int_{A_R} w_R^{p-1}|\phi_i^R|^2dy = 1$ we get

$$\begin{aligned}
\widetilde{Q}_R(\widetilde{v}) &\leq \frac{\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R}|\nabla\phi_i^R|^2dy + O\left(R^{-1}\right)\left(\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R}|\nabla\phi_i^R|^2dy + \sum\limits_{j=1}^{k}\alpha_j^2\int_{A_R}|\nabla\phi_j^R|^2dy\right)}{p\sum\limits_{i=1}^{k}\alpha_i^2} \\
&\leq \nu_k^R + O\left(R^{-1}\right)\frac{\sum\limits_{i=1}^{k}\alpha_i^2\int_{A_R}|\nabla\phi_i^R|^2 + \sum\limits_{j=1}^{k}\alpha_j^2\int_{A_R}|\nabla\phi_j^R|^2}{\sum\limits_{i=1}^{k}\alpha_i^2} \\
&\leq \nu_k^R + O\left(R^{-1}\right)\nu_k^R
\end{aligned}$$

which gives (5.8). The inequality (5.9) is obtained in the same way. □

We conclude this section showing an analogue of Proposition 4.4 for the eigenvalues of problem (5.7). Recall the values R_k^1 from Proposition 4.3.

Proposition 5.5. *For $\delta > 0$ let $\gamma(\delta) > 0$ and $k(\delta) \in \mathbb{N}$ be as in Proposition 4.4. Then there exists $\bar{k}(\delta) \geq k(\delta)$ such that for any $k \geq \bar{k}(\delta)$ and any $R \in \left[R_k^1 + \delta, R_{k+1}^1 - \delta\right]$ we have*

$$|\widetilde{\nu}_R - 1| \geq \frac{\gamma(\delta)}{2} \tag{5.10}$$

for any eigenvalue $\widetilde{\nu}_R$ of (5.7).

Proof. Consider $R \in \left[R_k^1 + \delta, R_{k+1}^1 - \delta\right]$ with $k \geq k(\delta)$. Then (4.11) holds and we denote by $\nu_1^R, \dots, \nu_{n(R)}^R$ the eigenvalues of (3.1) smaller than 1 so that $\nu_{n(R)+1}^R$ is the first eigenvalue of (3.1) larger than 1 and

$$\nu_{n(R)}^R < 1 - \gamma, \quad \nu_{n(R)+1}^R > 1 + \gamma. \tag{5.11}$$

Consider the eigenfunctions $\phi_1^R, \dots, \phi_{n(R)}^R$ of (3.1) corresponding to $\nu_1, \dots, \nu_{n(R)}$ and the transformed functions $\widetilde{\phi}_i^R = \phi_i^R \circ T$, $i = 1, \dots, n(R)$, in Ω_R. As observed in the proof of Lemma 5.4 the functions $\widetilde{\phi}_i^R$ are linearly independent and hence the space $\widetilde{V}_{n(R)}$ spanned by them is $n(R)$-dimensional. Computing the quotient $\widetilde{Q}_R$ for functions in $\widetilde{V}_{n(R)}$ and using (5.8) and (5.11) we have that

$$\widetilde{Q}_R(\psi) < 1 - \frac{\gamma}{2} \qquad \text{for all } \psi \in \widetilde{V}_{n(R)} \tag{5.12}$$

for R sufficiently large, i.e. for k sufficiently large. By the variational characterization of the eigenvalues we have that the eigenvalue $\widetilde{\nu}_{n(R)}$ of (5.7) is smaller than $1 - \frac{\gamma}{2}$. Hence, arguing by contradiction, if (5.7) has an eigenvalue $\widetilde{\nu}_{m(R)}$ which contradicts (5.10) we must have $m(R) > n(R)$. Now we consider the eigenspace $\widetilde{W}_{m(R)}$ corresponding to the eigenvalue $\widetilde{\nu}_{m(R)}$ for which

$$|\widetilde{\nu}_{m(R)} - 1| < \frac{\gamma}{2}. \tag{5.13}$$

Computing the quotient Q_R on the space $W_{m(R)}$ obtained from $\widetilde{W}_{m(R)}$ by applying the inverse transformation T^{-1} and using (5.9) we see that the eigenvalue $\nu_{m(R)}$ of (3.1) satisfies

$$\nu_{m(R)} \le \widetilde{\nu}_{m(R)} + \frac{\gamma}{2} \tag{5.14}$$

for R, and hence k, sufficiently large. Since $\nu_{m(R)} > 1$ we have $\nu_{m(R)} - 1 < \gamma$ because of (5.13), contradicting (4.11). $\square$

6 Proof of Theorem 1.1

Let $R_k := R_k^1$ be the radii from Proposition 4.3 and recall from (4.10) that $R_{k+1} - R_k \to \tau = |\beta_1|^{-1/2}$ as $R \to \infty$. We start by proving Theorem 1.1 in the case the exponent p in (1.1) is subcritical:

$$1 < p \le \frac{N+2}{N-2} = 2^* - 1 \quad \text{if } N \ge 3, \quad \text{any } p > 1 \text{ if } N = 2. \tag{6.1}$$

As in Section 5 we consider the functional

$$I_R(u) = \frac{1}{2} \int_{\Omega_R} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega_R} |u|^{p+1}$$

which is of class C^2 in $H_0^1(\Omega_R)$ and whose derivative $I_R'(u)$ is represented by the element $\text{grad} I_R(u) \in H_0^1(\Omega_R)$ given by (5.2). Analogously the second derivative $I_R''(u)$ can be identified with a linear continuous operator from $H_0^1(\Omega_R)$ to $H_0^1(\Omega_R)$ as follows:

$$\langle I_R''(u), v \rangle = v - \left(-\Delta^{-1}\right) \left(p|u|^{p-1} v\right) \qquad \forall v \in H_0^1(\Omega_R). \tag{6.2}$$

Let $\widetilde{u}_R = w_R \circ T$ be the "approximate solution" of (1.1) defined in Section 5. The following lemma shows that for a certain range of R and that the norm of the inverse operator in the space

$$\mathcal{L}_R = \left\{T : H_0^1(\Omega_R) \to H_0^1(\Omega_R) \mid \; T \text{ linear and continuous}\right\} \tag{6.3}$$

is bounded.

Lemma 6.1. *Fix $\delta > 0$, and let $\bar{k}(\delta) \in \mathbb{N}$ be given by Proposition 5.5. Then for $k \geq \bar{k}(\delta)$ and $R \in [R_k + \delta, R_{k+1} - \delta]$ the operator $I_R''(\widetilde{u}_R)$ is invertible and*

$$\left\| [I_R''(\widetilde{u}_R)]^{-1} \right\|_{\mathcal{L}_R} \leq \gamma_1(\delta). \tag{6.4}$$

where $\gamma_1(\delta) > 0$ is independent of k.

Proof. The linear operator $I_R''(\widetilde{u}_R)$ is invertible if it does not admit zero as eigenvalue i.e., using (6.2), if there does not exist a nontrivial solution of the equation

$$\begin{cases} -\Delta v - p\widetilde{u}_R^{p-1} v = 0 & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R. \end{cases}$$

In other words $I_R''(\widetilde{u}_R)$ is invertible if 1 is not an eigenvalue of (5.7). By (5.10) this is indeed the case for $k \geq \bar{k}(\delta)$, $R \in (R_k + \delta, R_{k+1} - \delta)$. Then, denoting by μ_h^R the eigenvalues of $[I_R''(\widetilde{u}_R)]^{-1}$ we have

$$\left\| [I_R''(\widetilde{u}_R)]^{-1} \right\| = \sup\left\{ \left|\mu_h^R\right|, h \geq 1 \right\}.$$

It is easy to see that $\mu_h^R = \frac{\widetilde{\nu}_h^R}{\widetilde{\nu}_h^R - 1}$, where $\widetilde{\nu}_h^R$ is an eigenvalue of the problem (5.7). Hence $\left\| [I_R''(\widetilde{u}_R)]^{-1} \right\|_{\mathcal{L}_R}$ is bounded if and only if the eigenvalues $\widetilde{\nu}_h^R$ are bounded away from 1 which is precisely the result of Proposition 5.5. □

We also need the following estimates.

Lemma 6.2. *The map $G_R : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ defined by*

$$G_R(\phi) = \text{grad} I_R(\widetilde{u}_R + \phi) - \text{grad} I_R(\widetilde{u}_R) - \langle I_R''(\widetilde{u}_R), \phi \rangle \tag{6.5}$$

satisfies for $\|\phi\|_{H_0^1}^2 \leq 1$

$$\|G_R(\phi)\|_{H_0^1} \leq \begin{cases} C_3 \|\phi\|_{H_0^1}^p & \text{for } 1 < p \leq 2, \\ C_3 \|\phi\|_{H_0^1}^2 & \text{for } p > 2, \end{cases} \tag{6.6}$$

with C_3 a constant independent of R.

Proof. By (5.2) and (6.2) we have

$$
\begin{aligned}
G_R(\phi) = \widetilde{u}_R + \phi - \left(-\Delta^{-1}\right)\left(|\widetilde{u}_R + \phi|^{p-1}(\widetilde{u}_R + \phi)\right) - \widetilde{u}_R \\
+ \left(-\Delta^{-1}\right)\left(|\widetilde{u}_R|^{p-1}\widetilde{u}_R\right) - \phi + \left(-\Delta^{-1}\right)\left(p\widetilde{u}_R^{p-1}\phi\right) \\
= -\left(-\Delta^{-1}\right)\left(|\widetilde{u}_R + \phi|^{p-1}\left(\widetilde{u}_R + \phi\right) - \widetilde{u}_R^p - p\widetilde{u}_R^{p-1}\phi\right).
\end{aligned}
$$

We set

$$
z_R = |\widetilde{u}_R + \phi|^{p-1}\left(\widetilde{u}_R + \phi\right) - \widetilde{u}_R^p - p\widetilde{u}_R^{p-1}\phi
$$

and

$$
\zeta_R = \left(-\Delta^{-1}\right)(z_R)
$$

so that

$$
\begin{cases} -\Delta\zeta_R = -z_R & \text{in } \Omega_R, \\ \zeta_R = 0 & \text{on } \partial\Omega_R. \end{cases} \tag{6.7}
$$

Since the function $\widetilde{u}_R$ is uniformly bounded we have

$$
|z_R| \leq \begin{cases} C|\phi|^p & \text{if } 1 < p \leq 2, \\ C\left(|\phi|^p + |\phi|^2\right) & \text{for } p > 2. \end{cases} \tag{6.8}
$$

Now we distinguish two cases:
Case 1. $N = 2$ and $p > 1$, or $N \geq 3$ and $1 \leq p \leq \frac{N}{N-2}$.
Case 2. $N \geq 3$ and $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$.
In Case 1 (6.7) implies

$$
\begin{aligned}
\int_{\Omega_R} |\nabla\zeta_R|^2\,dx = -\int_{\Omega_R} z_R\zeta_R dx \leq \left(\int_{\Omega_R} |\zeta_R|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega_R} |z_R|^2\,dx\right)^{\frac{1}{2}} \\
\leq P_0\|z_R\|_{L^2}\|\zeta_R\|_{H_0^1},
\end{aligned}
$$

where P_0 is the Poincaré constant which is independent of R. Hence by (6.8), if $p \leq 2$,

$$
\|G_R(\phi)\|_{H_0^1} = \|\zeta_R\|_{H_0^1} \leq C_2\left(\int_{\Omega_R} |\phi|^{2p}\right)^{\frac{1}{2}} \leq C_3\|\phi\|_{H_0^1}^p,
$$

having used the Sobolev inequality with a constant independent of R, because $2p > 2$. If $p > 2$, from (6.8) we get in the same way $\|G_R(\phi)\|_{H_0^1} \leq C_3\|\phi\|_{H_0^1}^2$, because $\|\phi\|_{H_0^1} \leq 1$. In Case 2 by (6.7) we have

$$
\begin{aligned}
\int_{\Omega_R} |\nabla\zeta_R|^2 \leq \left(\int_{\Omega_R} |\zeta_R|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}\left(\int_{\Omega_R} |z_R|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \\
\leq \frac{1}{S}\|\zeta_R\|_{H_0^1}\|z_R\|_{L^{\frac{2N}{N+2}}},
\end{aligned}
$$

where S is the best Sobolev constant for the embedding of $H_0^1(\Omega_R)$ in $L^{\frac{2N}{N-2}}(\Omega_R)$ which is independent of R. Hence, by (6.8), if $p \leq 2$,

$$\|G_R(\phi)\|_{H_0^1} \leq C_2 \left(\int_{\Omega_R} |\phi|^{p\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \leq C_3 \|\phi\|_{H_0^1}^p,$$

having used, as before, the Sobolev inequality with a constant independent of R, because $p\frac{2N}{N+2} > 2$. If $p > 2$, from (6.8) and $\|\phi\|_{H_0^1} \leq 1$ we obtain in the same way

$$\|G_R(\phi)\|_{H_0^1} \leq C_3 \|\phi\|_{H_0^1}^2.$$

□

Lemma 6.3. *Let G_R be as in Lemma 6.2. Then for $\|\phi\|_{H_0^1}^1, \|\phi\|_{H_0^1}^2 \leq 1$ there holds:*

$$\|G_R(\phi_1) - G_R(\phi_2)\|_{H_0^1} \leq \begin{cases} C_4 \left(\|\phi_1\|_{H_0^1}^{p-1} + \|\phi_2\|_{H_0^1}^{p-1} \right) \|\phi_1 - \phi_2\|_{H_0^1} & \text{if } 1 < p \leq 2, \\ C_4 \left(\|\phi_1\|_{H_0^1} + \|\phi_2\|_{H_0^1} \right) \|\phi_1 - \phi_2\|_{H_0^1} & \text{if } p > 2. \end{cases}$$

Proof. The proof is similar to that of Lemma 6.2 using

$$\left| |\widetilde{u}_R + \phi_1|^{p-1} (\widetilde{u}_R + \phi_1) - |\widetilde{u}_R + \phi_2|^{p-1} (\widetilde{u}_R + \phi_2) - p\widetilde{u}_R^{p-1}(\phi_1 - \phi_2) \right| \leq \begin{cases} C \left(|\phi_1|^{p-1} + |\phi_2|^{p-1} \right) |\phi_1 - \phi_2| & \text{if } 1 < p \leq 2, \\ C \left(|\phi_1|^{p-1} + |\phi_2|^{p-1} + |\phi_1| + |\phi_2| \right) |\phi_1 - \phi_2| & \text{if } p > 2, \end{cases}$$

with a constant C independent of R, because $\widetilde{u}_R$ is uniformly bounded. □

Now we reformulate (1.1) as fixed point problem. We look for a solution u of (1.1) in Ω_R in the form

$$u = \widetilde{u}_R + \phi_R. \tag{6.9}$$

Hence we need to find $\phi_R \in H_0^1(\Omega_R)$ such that

$$\text{grad} I_R(\widetilde{u}_R + \phi_R) = 0$$

which is equivalent to

$$\phi_R = - \left[I_R''(\widetilde{u}_R) \right]^{-1} \left[\text{grad} I_R(\widetilde{u}_R) + G_R(\phi_R) \right] \tag{6.10}$$

with G_R defined as in (6.5).

Proof of Theorem 1.1 for p as in (6.1). We consider the map $F_R : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ defined by

$$F_R(\phi) = \left[I_R''(\widetilde{u}_R) \right]^{-1} \left[\text{grad} I_R(\widetilde{u}_R) + G_R(\phi) \right], \tag{6.11}$$

so that (6.10) becomes the fixed point equation

$$\phi = F_R(\phi). \tag{6.12}$$

We fix $\delta > 0$, and take $R \in [R_k + \delta, R_{k+1} - \delta]$ with k sufficiently large (to be determined below) and prove that, for such R, the operator F_R maps the set

$$B_{\delta,R} := \left\{ \phi \in H_0^1(\Omega_R) : \|\phi\|_{H_0^1} \le \frac{A(\delta)}{R^\alpha} \right\}$$

to itself; here α is as in Lemma 5.2 and $A(\delta) = 2\gamma_1(\delta)\bar{C}$, with $\gamma_1(\delta)$ as in Lemma 6.1, $\bar{C} = \max\{C_1, C_3, C_4\}$, the constants from Lemma 5.2), Lemma 6.2 and Lemma 6.3. For $k \ge \bar{k}(\delta)$ as in Lemma 6.1 we have, by (6.11), (6.4), (5.3), and (6.6)

$$\begin{aligned} \|F_R(\phi)\|_{H_0^1} &\le \gamma_1(\delta) \left[\|\mathrm{grad} I_R(\widetilde{u}_R)\|_{H_0^1} + \|G_R(\phi)\|_{H_0^1} \right] \\ &\le \gamma_1(\delta) \left[\frac{C_1}{R^\alpha} + C_3 \|\phi\|_{H_0^1}^q \right] \end{aligned}$$

where $q := \min\{p, 2\} > 1$. Then we have

$$\|F_R(\phi)\|_{H_0^1} \le \frac{\gamma_1(\delta)\bar{C}}{R^\alpha} + \frac{\gamma_1(\delta)\bar{C}A(\delta)^q}{R^{\alpha q}} < \frac{A(\delta)}{R^\alpha}$$

if R is sufficiently large. Moreover from Lemmas 6.1 and 6.3 we deduce, for R sufficiently large,

$$\begin{aligned} \|F_R(\phi_1) - F_R(\phi_2)\|_{H_0^1} &\le \gamma_1(\delta) \left[\|G_R(\phi_1) - G_R(\phi_2)\|_{H_0^1} \right] \\ &\le 2\gamma_1(\delta)\bar{C} \left(\frac{A(\delta)}{R^\alpha} \right)^d \|\phi_1 - \phi_2\|_{H_0^1} < \frac{1}{2} \|\phi_1 - \phi_2\|_{H_0^1} \end{aligned}$$

where d is either $p-1$ or 1. Thus, for $R \in [R_k + \delta, R_{k+1} - \delta]$ and $k \in \mathbb{N}$ large, the map F_R is a contraction in the set $B_{\delta,R}$ and so the fixed point equation (6.12) has a solution $\phi_R \in B_{\delta,R}$ with

$$\|\phi\|_{H_0^1} \le \frac{A(\delta)}{R^\alpha} \tag{6.13}$$

with α as in (5.3). Hence $u_R = \widetilde{u}_R + \phi_R$ is a solution of (1.1) in Ω_R. The existence of the intervals and that the solution is positive will be shown below for general $p > 1$. □

Proof of Theorem 1.1 in the supercritical case. We consider the map

$$F_R : H_0^1 \cap L^\infty(\Omega_R) \to H_0^1 \cap L^\infty(\Omega_R)$$

defined in 6.11. Let $z_R = |\widetilde{u}_R + \phi|^{p-1}(\widetilde{u}_R + \phi) - \widetilde{u}_R^p - p\widetilde{u}_R^{p-1}\phi$ be as in the proof of Lemma 6.2. We start with the following

Lemma 6.4. *There exists $C > 0$ independent of R such that for R large*

$$\|F_R(\phi)\|_{L^\infty(\Omega_R)} \le C\left(\|F_R(\phi)\|_{L^2(\Omega_R)} + \|z_R\|_{L^\infty(\Omega_R)} + \frac{1}{R^2}\right). \tag{6.14}$$

Proof. The function

$$\omega_R := [I''_R(\widetilde{u}_R)]^{-1}[\text{grad} I_R(\widetilde{u}_R) + G_R(\phi)]$$

solves the problem

$$\begin{cases} -\Delta\omega_R - p\widetilde{u}_R^{p-1}\omega_R = \Delta\widetilde{u}_R + \widetilde{u}_R^p + z_R(\phi) & \text{in } \Omega_R, \\ \omega_R \in H_0^1(\Omega_R). \end{cases} \tag{6.15}$$

By Lemma 5.1 we have for R large

$$\|\Delta\widetilde{u}_R + \widetilde{u}_R^p\|_{L^\infty(\Omega_R)} \le \frac{C}{R^2}. \tag{6.16}$$

Now we choose a point $x_R \in \Omega_R$ with

$$\|\omega_R\|_{L^\infty(\Omega_R)} = \omega_R(x_R),$$

and set $B_R^\delta = B(x_R, \delta) \cap \Omega_R$ where $B(x_R, \delta)$ is the ball centered at x_R and with radius $\delta > 0$. Applying [GT, Theorem 8.17] we obtain

$$\sup_{B_R^1} |\omega_R| \le C\left[\|\omega_R\|_{L^2(B_R^2)} + \|\Delta\widetilde{u}_R + \widetilde{u}_R^p\|_{L^\infty(B_R^2)} + \|z_R\|_{L^\infty(B_R^2)}\right]. \tag{6.17}$$

The claim follows from $\|\omega_R\|_{L^\infty(\Omega_R)} = \|\omega_R\|_{L^\infty(B_R^1)}$. □

As for the subcritical case we fix $\delta > 0$ and take $R \in [R_k + \delta, R_{k+1} - \delta]$.
We now set for $0 < \beta < \alpha$,

$$C_{\delta,R} = \left\{\phi \in H_0^1 \cap L^\infty(\Omega_R) : \|\phi\|_{H_0^1(\Omega_R)} \le \frac{A(\delta)}{R^\alpha},\ \|\phi\|_{L^\infty(\Omega_R)} \le \frac{1}{R^\beta}\right\}$$

where $A(\delta)$ and α are the same as in the proof of Theorem 1.1 with p as in (6.1). For $M > 0$ we choose $w_M \in C^2(\mathbb{R})$ with

$$w_M(s) = \begin{cases} |s|^{p+1} & \text{if } |s| \le M, \\ M+1 & \text{if } |s| \ge M+1, \end{cases}$$

and consider the functional $I_{R,M} : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ defined by

$$I_{R,M}(u) = \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_R} w_M(u).$$

For $M \ge M_0 := 2\,||\widetilde{u}_R||_{L^\infty(\Omega_R)}$ the linear operators $I'_{R,M}(\widetilde{u}_R)$ and $I''_{R,M}(\widetilde{u}_R)$ coincide with $I'_R(\widetilde{u}_R)$ and $I''_R(\widetilde{u}_R)$, respectively. We denote by $F_{R,M}$ the counterpart of the operator F_R. In the following we always assume that $M \ge M_0$.

As in the proof of Theorem 1.1 in the case $p \leq \frac{N+2}{N-2}$ we will show that $F_{R,M}$ is a contraction mapping from $C_{\delta,R}$ to itself for R large enough. Indeed, repeating the same proof we obtain

$$\|\phi\|_{H_0^1(\Omega_R)} \leq \frac{A(\delta)}{R^\alpha} \quad \Longrightarrow \quad \|F_{R,M}(\phi)\|_{H_0^1(\Omega_R)} \leq \frac{A(\delta)}{R^\alpha}. \tag{6.18}$$

Also we have by (6.8), for $\|z_R\|_{L^\infty(\Omega_R)} \leq \frac{1}{R^\beta}$,

$$\|z_R\|_{L^\infty(\Omega_R)} \leq C\left(\|\phi_R\|^2_{L^\infty(\Omega_R)} + \|\phi_R\|^p_{L^\infty(\Omega_R)}\right) \leq C\left(\frac{1}{R^{2\beta}} + \frac{1}{R^{p\beta}}\right).$$

Hence Lemma 6.4 implies

$$\|F_{R,M}(\phi)\|_{L^\infty(\Omega_R)} \leq C\left(\frac{1}{R^\alpha} + \frac{1}{R^{2\beta}} + \frac{1}{R^{p\beta}} + \frac{1}{R^2}\right) \leq \frac{1}{R^\beta}$$

for R large, because $0 < \beta < \alpha \leq 2$. This proves that $F_{R,M}$ maps $C_{\delta,R}$ into itself. In the same way it is possible to show that for $R \in [R_k + \delta, R_{k+1} - \delta]$ and $k \in \mathbb{N}$ large,

$$\|F_{R,M}(\phi_1) - F_{R,M}(\phi_2)\|_{H_0^1} \leq \gamma \|\phi_1 - \phi_2\|_{H_0^1}$$

with $\gamma < 1$, and similarly with the L^∞-norm. Hence the contraction mapping theorem applies and yields a solution $u_R = \widetilde{u}_R + \phi_R$ of

$$\begin{cases} -\Delta u_R = |u_R|^{p-1} u_R & \text{in } \Omega_R, \\ u_R = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{6.19}$$

with $\widetilde{u}_R$ as in (1.4) and $\phi_R \in C_{\delta,R}$.

We complete the proof of Theorem 1.1 by showing that u_R is positive in Ω_R. Since $\widetilde{u}_R > 0$ in Ω_R and $\phi_R \to 0$ in $H_0^1(\Omega_R)$ it is easy to show that if D_R is any regular set such that $u_R \leq 0$ in D_R then $\text{meas}(D_R) \to 0$ as $R \to \infty$. Multiplying (6.19) by u_R^- and integrating on D_R we get

$$\int_{\Omega_R} |\nabla u_R^-|^2 dx = \int_{\Omega_R} |u_R|^{p-1} \left(u_R^-\right)^2 dx,$$

from which, using the Poincaré inequality, we deduce

$$\lambda_1(D_R) \int_{\Omega_R} \left(u_R^-\right)^2 dx \leq \|u_R\|_\infty^{p-1} \int_{\Omega_R} \left(u_R^-\right)^2 dx \tag{6.20}$$

where $\lambda_1(D_R)$ is the first eigenvalue of $-\Delta$ in D_Rd with homogenous Dirichlet boundary conditions. From (6.20) we get

$$\lambda_1(D_R) \leq \|u_R\|_\infty^{p-1} \leq C$$

which is impossible because $\lambda_1(D_R)$ should tend to ∞ as $\text{meas}(D_R) \to 0$. □

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