# Configuration spaces, transfer, and $2$-nodal solutions of a semiclassical nonlinear Schrödinger equation

Thomas Bartsch$^{1,*}$ Mónica Clapp$^{2,\dagger}$ Tobias Weth$^{1,\ddagger}$

$^1$ Mathematisches Institut, Universität Giessen, Arndtstr. 2, 35392 Giessen, Germany
$^2$ Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior,
Ciudad Universitaria, 04510 México, D.F., México

**Albrecht Dold und Dieter Puppe gewidmet**[1]
**In memoriam Dieter Puppe**[2]

## 1 Introduction

We consider the stationary nonlinear Schrödinger equation

$$\begin{cases} -\varepsilon^2\Delta u + a(x)u = |u|^{p-2}\,u \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $p > 2$ and $p < 2^* := 2N/(N-2)$ if $N \geq 3$, $a : \mathbb{R}^N \to \mathbb{R}$ is bounded and uniformly continuous, and $a_0 := \inf_{x\in\mathbb{R}^N} a(x) > 0$. We are especially interested in the existence of bound states in the semiclassical case $\varepsilon \to 0$.

Equations of this kind have been the subject of extensive research in the last two decades. They are motivated by various problems from mathematical physics, and present mathematically interesting and challenging problems.

There are many papers which investigate existence, multiplicity and shape of positive small energy solutions. We mention, in particular, the work of Floer and Weinstein [20] which started this research, and the more recent results of X. Wang [28], del Pino and Felmer [16, 17], Ambrosetti, Badiale and Cingolani [4], and Y. Li [22]. They establish the existence of one-peak or multipeak solutions with each peak concentrating at a different bounded connected component of the critical set of $a$. Cingolani and Lazzo [12] (see also Ambrosetti et al. [5]) showed that

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there might be more than one solution concentrating at the same component. In fact, they showed that, if the set of global minima of $a$ is bounded, then there is an effect of its topology on the number of positive one-peak solutions concentrating at this set. A similar result for problems with symmetries was proved in [13].

Whereas there are many results on positive solutions, little is known about sign changing solutions. An exception is the case where $a$ is radial and one looks for radially symmetric solutions with a prescribed number of nodal domains. This case is substantially simpler and it allows methods, like the shooting method or Nehari's method, which do not work in the nonradial setting considered here.

We propose a new approach to the existence of 2-nodal solutions of $(\mathcal{P}_\varepsilon)$, that is, solutions with precisely two nodal domains. There are a number of recent papers about this type of solutions on bounded domains, see [6, 7, 14]. On $\mathbb{R}^N$ we are only aware of the paper [3] by Alves and Soares who obtained one such solution $u_\varepsilon$ for $\varepsilon > 0$ small, whose positive part $u_\varepsilon^+ = \max\{u_\varepsilon, 0\}$ and negative part $u_\varepsilon^- = \min\{u_\varepsilon, 0\}$ concentrate near a given bounded component of the set

$$M^0 := \{y \in \mathbb{R}^N : a(y) = a_0\}$$

of minima of $a$.

Our approach is based on a dynamical systems point of view applied to the negative gradient flow on $H^1(\mathbb{R}^N)$ of the energy functional associated to $(\mathcal{P}_\varepsilon)$. The main ingredients are as follows: Given a compact and isolated subset $C$ of $M^0$ and a neighborhood $V$ of $C$ in $\mathbb{R}^N$ we construct, for $\varepsilon > 0$ small,

1) a symmetric subset $\mathcal{Z}_\varepsilon(V)$ of $H^1(\mathbb{R}^N)$ which is positively invariant for the negative gradient flow, such that the energy functional satisfies the Palais-Smale condition on $\mathcal{Z}_\varepsilon(V)$, and the critical points in $\mathcal{Z}_\varepsilon(V)$ are 2-nodal solutions of $(\mathcal{P}_\varepsilon)$ whose positive and negative parts are concentrated at different points in $V$,

2) a map $\tau_\varepsilon$ in cohomology from the set

$$C_\varepsilon = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \text{dist}(x, C), \text{dist}(y, C) \le \sqrt{\varepsilon},\ |x - y| \ge 2\sqrt{\varepsilon}\}$$

to $\mathcal{Z}_\varepsilon(V)$, and a map $\theta_\varepsilon : \mathcal{Z}_\varepsilon(V) \to F(V) = \{(x, y) \in V \times V : x \neq y\}$ such that the composition $\tau_\varepsilon \circ \theta_\varepsilon^*$ is the homomorphism induced by the inclusion $C_\varepsilon \hookrightarrow F(V)$ in cohomology.

In fact, the whole construction is symmetric with respect to the action of the group $G = \mathbb{Z}/2$ on $\mathcal{Z}_\varepsilon(V)$ given by $u \mapsto -u$ and the actions on $C_\varepsilon$ and $F(V)$ given by $(x, y) \mapsto (y, x)$. The cohomology mentioned in 2) is $H_G^*(X, Y) := H^*(X/G, Y/G)$ where $H^*$ stands for Alexander-Spanier cohomology with coefficients in the field of two elements and $X/G$ is the $G$-orbit space of $X$. Thus we have homomorphisms

$$H_G^*(F(V)) \xrightarrow{\theta_\varepsilon^*} H_G^*(\mathcal{Z}_\varepsilon(V)) \xrightarrow{\tau_\varepsilon} H_G^*(C_\varepsilon)$$

such that $\tau_\varepsilon \circ \theta_\varepsilon^* = i^*$ where $i : C_\varepsilon \hookrightarrow F(V)$. The homomorphism $\tau_\varepsilon$ is not induced by an actual map $C_\varepsilon \to \mathcal{Z}_\varepsilon(V)$. Rather, it is constructed only on the cohomology level using a version of Dold's fixed point transfer.

As an immediate consequence we obtain a lower bound for the number of solutions of $(\mathcal{P}_\varepsilon)$ lying in $\mathcal{Z}_\varepsilon(V)$ in terms of the $H^*_G$-cuplength of $C_\varepsilon$ in $F(V)$. We define this concept in Section 2 below and provide some computations of it. In particular, this number is at least $N$, even if $V$ is a contractible neighborhood of an isolated minimum of $a$. Thus we always obtain at least $N$ pairs $\pm u_j$, $j = 1, \dots, N$, of solutions of $(\mathcal{P}_\varepsilon)$ whose positive and negative parts concentrate near any compact isolated subset of the set $M^0$ of minima of $a$. Via a similar construction we also obtain solutions whose positive and negative parts concentrate near different components of $M^0$.

It is an interesting feature of our construction that we can localize the concentration points of the solutions near prescribed compact isolated subsets of $M^0$ without using finite-dimensional reductions, penalization techniques, or other modifications of the functional as in the references mentioned above. Instead, this localization is done by means of a generalized barycenter map as defined in [6, 9]. As a result, we can also treat unbounded sets $M^0$ of minima of $a$, and in some cases we obtain *infinitely many* solutions for *fixed* small $\varepsilon$. Results of this type can not be obtained via a priori modifications or artificial constraints. In fact, they require a study of compactness questions for the original energy functional, and in the past this has been found difficult to pursue for bounded potentials $a$ without a prescribed behaviour as $|x| \to \infty$. We provide a rather simple and unified approach to these questions under the (apparently crucial) assumption that $a$ is uniformly continuous. Another interesting feature is that we obtain information on the cohomology of the set $\mathcal{Z}_\varepsilon(V)$ itself, not just on its Conley index.

The approach we present here can be extended to a variety of other situations. First, one may consider other classes of potentials and more general nonlinearities $f : \mathbb{R} \to \mathbb{R}$, not just the homogeneous nonlinearity $f(t) = |t|^{p-2}t$, and one can use it for problems on bounded domains. It is not essential for our approach that the limit equation $-\Delta u + a_0 u = f(u)$ has a unique, nondegenerate positive radial solution. It is essential however, that $f$ is odd. In order to keep technicalities to a minimum and to present the main ideas in rather simple setting we only consider the homogeneous nonlinearity. Second, one may consider flows other than the negative gradient flow, most importantly the parabolic semiflow associated to $u_t - \varepsilon^2 \Delta u + a(x)u = |u|^{p-2}u$. In additon to the existence of a certain type of stationary 2-nodal solutions we can analyze the dynamics in $\mathcal{Z}_\varepsilon(V)$ to some extent and obtain heteroclinic orbits connecting the 2-nodal solutions in $\mathcal{Z}_\varepsilon(V)$. Third, one may use the flow invariant sets $\mathcal{Z}_\varepsilon(V)$ to construct a Morse decomposition associated to low energy solutions of $(\mathcal{P}_\varepsilon)$. This is important for analyzing the dynamics further and for finding connecting orbits between the flow invariant sets and the set of positive or negative solutions, for instance. It is essential here that we do not need to make any modifications of the nonlinearity. Again the dynamics is especially interesting for the parabolic semiflow. Working with the parabolic semiflow requires however different techniques for proving appropriate compactness properties and it is therefore not treated here. The interested reader is referred to the papers [1, 2] for results on the parabolic semiflow with superlinear nonlinearity.

The paper is organized as follows. In Section 2 we introduce some notation and present our main results. In Section 3 we construct a flow invariant set of sign changing functions in $H^1(\mathbb{R}^N)$, and we state some crucial technical results concerning energy bounds, compactness and localization of functions in this set, see Propositions 3.3–3.5. These propositions are proved in Section 4, where we also generalize the usual representation lemma for Palais-Smale sequences to families of functionals. In Section 5 we complete the proof of our main theorems. Section 6 contains new lower bounds on the equivariant cuplength of certain configuration spaces. These bounds are of independent interest beyond our application. In the appendix we discuss a version of Dold's fixed point transfer for vector bundles which is a main ingredient in our construction.

## 2 The main results

Let $a : \mathbb{R}^N \to \mathbb{R}$ be bounded and uniformly continuous with $a_0 = \inf_{x\in\mathbb{R}^N} a(x) > 0$. For $\varepsilon > 0$ let

$$J_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}, \qquad J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + a(x)u^2) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p,$$

be the energy functional associated to $(\mathcal{P}_\varepsilon)$. It is well known that $J_\varepsilon$ is of class $C^2$ and that the critical points of $J_\varepsilon$ are the solutions of $(\mathcal{P}_\varepsilon)$. We consider the scalar product

$$\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + a(x)uv)$$

on $H^1(\mathbb{R}^N)$, and write

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + a(x)u^2)\right)^{1/2}$$

for the corresponding norm. Clearly, $\|\cdot\|_\varepsilon$ is equivalent to the standard norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\right)^{1/2}.$$

The gradient of $J_\varepsilon$ with respect to $\langle \cdot, \cdot\rangle_\varepsilon$ is denoted by $\nabla_\varepsilon J_\varepsilon$ and is given by

$$\nabla_\varepsilon J_\varepsilon(u) = u - (-\varepsilon^2\Delta + a)^{-1}(|u|^{p-2}u).$$

Our main theorems deal with the flow $\varphi_\varepsilon$ associated to the negative gradient vector field $-\nabla_\varepsilon J_\varepsilon$, defined by

$$\begin{cases} \dfrac{d}{dt}\varphi_\varepsilon(t,u) = -\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t,u)) \\ \varphi_\varepsilon(0,u) = u \end{cases}$$

In order to state our results we need some notation. For a subset $A$ of $\mathbb{R}^N$ and $\varepsilon > 0$ set

$$\begin{aligned} B_\varepsilon(A) &:= \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) \le \varepsilon\}, \\ A_\varepsilon &:= \{(x,y) : x, y \in B_{\sqrt{\varepsilon}}(A),\ |x-y| \ge 2\sqrt{\varepsilon}\}, \\ F(A) &:= \{(x,y) \in A \times A : x \ne y\}. \end{aligned}$$

$F(A)$ is the configuration space of ordered pairs in $A$. The group $G = \mathbb{Z}/2$ acts on $F(A)$ via permutation of the coordinates $(x,y) \mapsto (y,x)$. The orbit space

$$F(A)/G = \{\{x,y\} \subset A : x \ne y\}$$

is the configuration space of unordered pairs in $A$. For example, if $A = \{x_0\}$ is a point, then $B_\varepsilon(A) = B_\varepsilon(x_0)$ is the closed ball of radius $\varepsilon$ around $x_0$, $A_\varepsilon = \{(x_0 + x, x_0 - x) : |x| = \sqrt{\varepsilon}\}$

is equivariantly homeomorphic to the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ with the antipodal action $x \mapsto -x$, and the inclusion $A_\varepsilon \hookrightarrow F(B_\delta(x_0))$, $\delta \geq \sqrt{\varepsilon}$, is an equivariant homotopy equivalence whose inverse is given by

$$(x,y) \mapsto \left( x_0 + \sqrt{\varepsilon}\frac{x-y}{|x-y|}, x_0 - \sqrt{\varepsilon}\frac{x-y}{|x-y|} \right).$$

Let $H^*$ denote Alexander-Spanier cohomology [24, 27] or Čech cohomology [18] with coefficients in the field of two elements. Both cohomology theories are the same on locally closed subsets of an $ANR$. They coincide with singular cohomology on ANR's, in particular on Banach manifolds. The definition and the properties of the cup product in $H^*$ can be found in [24]. For a space $X$ with a free action of $G$ we set

$$H^*_G(X) := H^*(X/G).$$

We shall apply this cohomology, in particular, to configuration spaces $X = F(A)$ and to symmetric subsets $\mathcal{A} = -\mathcal{A}$ of $H^1(\mathbb{R}^N) \smallsetminus \{0\}$ where $G$ acts via the antipodal map $u \mapsto -u$. For the example above,

$$H^*_G(F(B_\delta(x_0))) \cong H^*(\mathbb{S}^{N-1}/G) = H^*(\mathbb{R}P^{N-1}) \tag{2.1}$$

is the cohomology of the real projective space $\mathbb{R}P^{N-1}$ which is a polynomial ring with $\mathbb{Z}/2$-coefficients on one generator $\omega$ subject to the relation $\omega^N = 0$.

Finally we need a generalized barycenter map. By this we mean a continuous map $\beta : L^2(\mathbb{R}^N) \smallsetminus \{0\} \to \mathbb{R}^N$ which is equivariant with respect to the action of the group of euclidian motions in $\mathbb{R}^N$, that is, for every $x \in \mathbb{R}^N$, every orthogonal $N \times N$-matrix $\Theta$ and every $u \in L^2(\mathbb{R}^N) \smallsetminus \{0\}$, one has

$$\beta(x * u) = x + \beta(u) \qquad \text{and} \qquad \beta(u \circ \Theta^{-1}) = \Theta(\beta(u)), \tag{2.2}$$

where $(x*u)(\xi) = u(\xi - x)$. Such a map has been constructed in [6,9]. For matters of convenience we also assume equivariance with respect to scaling, that is

$$\beta(u \circ \varepsilon) = \varepsilon^{-1}\beta(u) \tag{2.3}$$

for $u \in L^2(\mathbb{R}^N) \smallsetminus \{0\}$, $\varepsilon > 0$, where $(u \circ \varepsilon)(x) = u(\varepsilon x)$. This property is easily built into the construction. Indeed, if $\beta_0$ satisfies (2.2), then $\beta$ defined by $\beta(u) = |u|_2^{2/N}\beta_0(u \circ |u|_2^{2/N})$ satisfies (2.2) and (2.3) where $|\cdot|_2$ denotes the $L^2$-norm. Note that the map

$$u \mapsto \left( \int_{\mathbb{R}^N} u^2 \right)^{-1} \int_{\mathbb{R}^N} xu^2$$

has the invariance properties (2.2) and (2.3), but it is neither well defined on $L^2(\mathbb{R}^N) \smallsetminus \{0\}$ nor on $H^1(\mathbb{R}^N) \smallsetminus \{0\}$.

As before we set

$$a_0 = \inf_{x \in \mathbb{R}^N} a(x), \qquad M^0 = \{x \in \mathbb{R}^N : a(x) = a_0\}.$$

Moreover, if $\mathcal{Z} \subset H^1(\mathbb{R}^N)$, we say that the functional $J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{Z}$ if every sequence $(u_n)$ in $\mathcal{Z}$ for which $J_\varepsilon(u_n)$ is bounded and $\nabla_\varepsilon J_\varepsilon(u_n) \to 0$ is relatively compact.

**Theorem 2.1.** *Let $C$ be a nonempty compact isolated subset of $M^0$. Let $V$ be a compact neighborhood of $C$ in $\mathbb{R}^N$ such that $V \cap M^0 = C$, and let $\beta : L^2(\mathbb{R}^N) \smallsetminus \{0\} \to \mathbb{R}^N$ be a generalized barycenter map. Then there exists $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, a subset $\mathcal{Z}_\varepsilon(V)$ of $H^1(\mathbb{R}^N)$, with the following properties:*

(i) *$\mathcal{Z}_\varepsilon(V) = -\mathcal{Z}_\varepsilon(V)$ is positively invariant under the negative gradient flow $\varphi_\varepsilon$.*

(ii) *The functional $J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{Z}_\varepsilon(V)$.*

(iii) *Every $u \in \mathcal{Z}_\varepsilon(V)$ changes sign and, if $u$ is a critical point of $J_\varepsilon$, then it has precisely two nodal domains.*

(iv) *For $u \in \mathcal{Z}_\varepsilon(V)$ the barycenters $\beta(u^+)$ and $\beta(u^-)$ are different and lie in $V$, so there is a continuous map*

$$\theta_\varepsilon : \mathcal{Z}_\varepsilon(V) \to F(V), \qquad \theta_\varepsilon(u) = (\beta(u^+), \beta(u^-)).$$

(v) *There exists a homomorphism*

$$\tau_\varepsilon : H_G^*(\mathcal{Z}_\varepsilon(V)) \to H_G^*(C_\varepsilon)$$

*such that the composition*

$$\tau_\varepsilon \circ \theta_\varepsilon^* : H_G^*(F(V)) \to H_G^*(C_\varepsilon)$$

*is the homomorphism induced by the inclusion $C_\varepsilon \hookrightarrow F(V)$.*

(vi) *$\varepsilon_0$ only depends on the difference $\min\limits_{\partial V} a - a_0 > 0$.*

In order to obtain explicit lower bounds on the number of solutions of $(\mathcal{P}_\varepsilon)$ in $\mathcal{Z}_\varepsilon(V)$ we introduce the $H_G^*$-cuplength. For a topological space $B$ with a free action of $G$ and a $G$-invariant subspace $A \neq \emptyset$ of $B$, $H_G^*$-$\mathrm{cupl}_B(A)$ is the smallest integer $k \geq 1$ such that for any $k$ cohomology classes $\zeta_1, \ldots, \zeta_k \in \widetilde{H}_G^*(B)$ the cup-product $(\zeta_1 \smile \cdots \smile \zeta_k)|_A = 0 \in \widetilde{H}_G^*(A)$. Here $\widetilde{H}_G^*(X) = \widetilde{H}^*(X/G)$ is the reduced Alexander-Spanier cohomology of $X/G$, and $\zeta|_A \in H_G^*(A)$ is the image of $\zeta \in H_G^*(B)$ under the homomorphism $i^* : H_G^*(B) \to H_G^*(A)$ induced by the inclusion $i : A \hookrightarrow B$. If $A/G$ or $B/G$ is contractible then $\widetilde{H}_G^*(A) = 0$ or $\widetilde{H}_G^*(B) = 0$, respectively, and $H_G^*$-$\mathrm{cupl}_B(A) = 1$. We will also write

$$H_G^*\text{-cupl}(i) \text{ for } H_G^*\text{-cupl}_B(A), \quad \text{and} \quad H_G^*\text{-cupl}(A) \text{ for } H_G^*\text{-cupl}_A(A).$$

Compared with the cuplength $CL(X)$ of a topological space $X$ as defined in [11, p. 9] we have $H_G^*$-$\mathrm{cupl}(A) = CL(A/G) + 1$. We then get a lower bound for the number of solutions contained in $\mathcal{Z}_\varepsilon(V)$.

**Corollary 2.2.** *Let $C$ be a nonempty compact isolated set of minima of $a$, and let $V$ be a neighborhood of $C$ in $\mathbb{R}^N$. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exist at least $H_G^*$-$\mathrm{cupl}_{F(V)}(C_\varepsilon)$ pairs of 2-nodal solutions $\pm u$ to problem $(\mathcal{P}_\varepsilon)$ such that $\beta(u^+)$ and $\beta(u^-)$ are different and lie in $V$, where $\varepsilon_0$ and $\beta$ are as in Theorem 2.1.*

Going back to our example, if $C = \{x_0\}$ and $V = B_\delta(x_0)$, $\delta \geq \sqrt{\varepsilon}$, then the inclusion $C_\varepsilon \hookrightarrow F(V)$ is an equivariant homotopy equivalence and (2.1) yields

$$H_G^*\text{-cupl}_{F(V)}(C_\varepsilon) = H_G^*\text{-cupl}(\mathbb{R}P^{N-1}) = N.$$

We shall prove the following estimate.

**Proposition 2.3.** *Let $C \subset \mathbb{R}^N$ be a nonempty compact set, and let $V$ be a neighborhood of $C$ in $\mathbb{R}^N$. Then $H_G^*\text{-cupl}_{F(V)}(C_\varepsilon) \geq N$ for $0 < \varepsilon < \operatorname{dist}(C, \mathbb{R}^N \setminus V)$.*

This estimate can be improved if $C$ has some topology. Configuration spaces play an important role in algebraic topology and the additive structure of $H_G^*(F(V)) = H^*(F(V)/G)$ has been computed for a number of topological spaces $V$. Surprisingly, not much is known on the multiplicative structure. In Section 6 we provide some computations for the cuplength.

We now turn to 2-nodal solutions of $(\mathcal{P}_\varepsilon)$, such that $\beta(u^+)$ and $\beta(u^-)$ lie near disjoint compact subsets $C$ and $C'$ of $M^0$.

**Theorem 2.4.** *Let $C, C'$ be disjoint nonempty compact subsets of $M^0$ which are isolated in $M^0$. Let $V, V'$ be disjoint compact neighborhoods of $C$ and $C'$ in $\mathbb{R}^N$ with $V \cap M^0 = C$ and $V' \cap M^0 = C'$, and let $\beta : L^2(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be a generalized barycenter map. Then there exists $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, a subset $\mathcal{Z}_\varepsilon(V, V')$ of $H^1(\mathbb{R}^N)$ with the following properties:*

(i) *$\mathcal{Z}_\varepsilon(V, V') = -\mathcal{Z}_\varepsilon(V, V')$ is positively invariant under the negative gradient flow $\varphi_\varepsilon$.*

(ii) *The functional $J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{Z}_\varepsilon(V, V')$.*

(iii) *Every $u \in \mathcal{Z}_\varepsilon(V, V')$ changes sign and, if $u$ is a critical point of $J_\varepsilon$, then it has precisely two nodal domains.*

(iv) *The barycenter map $\beta$ induces a map*

$$\theta_\varepsilon : \mathcal{Z}_\varepsilon(V, V') \to [V \times V'] \cup [V' \times V], \qquad \theta_\varepsilon(u) = (\beta(u^+), \beta(u^-)).$$

*which is well defined and continuous.*

(v) *There exists a homomorphism*

$$\tau_\varepsilon : H_G^*(\mathcal{Z}_\varepsilon(V, V')) \to H_G^*\big([B_{\sqrt{\varepsilon}}(C) \times B_{\sqrt{\varepsilon}}(C')] \cup [B_{\sqrt{\varepsilon}}(C') \times B_{\sqrt{\varepsilon}}(C)]\big)$$

*such that the composition*

$$\tau_\varepsilon \circ \theta_\varepsilon^* : H_G^*\big([V \times V'] \cup [V' \times V]\big) \to H_G^*\big([B_{\sqrt{\varepsilon}}(C) \times B_{\sqrt{\varepsilon}}(C')] \cup [B_{\sqrt{\varepsilon}}(C') \times B_{\sqrt{\varepsilon}}(C)]\big)$$

*is the homomorphism $i_\varepsilon^*$ induced by the inclusion*

$$i_\varepsilon : [B_{\sqrt{\varepsilon}}(C) \times B_{\sqrt{\varepsilon}}(C')] \cup [B_{\sqrt{\varepsilon}}(C') \times B_{\sqrt{\varepsilon}}(C)] \hookrightarrow [V \times V'] \cup [V' \times V].$$

(vi) *$\varepsilon_0$ only depends on the difference $\min\limits_{\partial V \cup \partial V'} a - a_0 > 0$.*

Here $\varepsilon_0 > 0$ is so small that $B_{\sqrt{\varepsilon_0}}(C) \subset V$ and $B_{\sqrt{\varepsilon_0}}(C') \subset V'$. Since $V \cap V' = \emptyset$ we must have $\mathrm{dist}(C, C') > 2\sqrt{\varepsilon_0}$. If $C$ and $C'$ are not too irregular, we may take $V$ and $V'$ to be such that the inclusions $C \hookrightarrow V$ and $C' \hookrightarrow V'$ are homotopy equivalences. In this case the cohomology

$$H^*_G\big([V \times V'] \cup [V' \times V]\big) \cong H^*_G\big([C \times C'] \cup [C' \times C]\big)$$

is a direct summand of $H^*_G(\mathcal{Z}_\varepsilon(V, V'))$.

**Corollary 2.5.** *Fix $C, C', V, V'$ and $\beta$ as in Theorem 2.4. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exist at least $H^*_G$-cupl$(i_\varepsilon)$ pairs of 2-nodal solutions $\pm u$ to problem $(\mathcal{P}_\varepsilon)$ such that $\beta(u^+) \in V$ and $\beta(u^-) \in V'$, where $\varepsilon_0$ and $i_\varepsilon$ are as in Theorem 2.4.*

Observe that we always have $H^*_G$-cupl$(i_\varepsilon) \geq 1$, so that we always have at least one pair $\pm u$ of 2-nodal solutions with $\beta(u^+) \in V$ and $\beta(u^-) \in V'$. If $C$ or $C'$ have nontrivial topology we obtain more solutions. The following result is helpful for computations.

**Proposition 2.6.** $H^*_G$-cupl$(i_\varepsilon) = H^*$-$\mathrm{cupl}_V(B_{\sqrt{\varepsilon}}(C)) + H^*$-$\mathrm{cupl}_{V'}(B_{\sqrt{\varepsilon}}(C')) - 1$.

Thus, if the inclusions $C \hookrightarrow V$ and $C' \hookrightarrow V'$ are homotopy equivalences, then we have at least $H^*_G$-cupl$(C) + H^*_G$-cupl$(C') - 1$ pairs of solutions $\pm u$ to problem $(\mathcal{P}_\varepsilon)$ with precisely two nodal domains. These solutions satisfy $\beta(u^+) \in V$ and $\beta(u^-) \in V'$.

We close this section with a result yielding infinitely many solutions for fixed small $\varepsilon > 0$.

**Corollary 2.7.** *Suppose that there exist an infinite collection of nonempty compact subsets $C_i \subset M^0$, $i \in \mathbb{N}$, and compact neighborhoods $V_i$ of $C_i$ such that $V_i \cap M^0 = C_i$ and $V_i \cap V_j = \varnothing$ for $i \neq j$. Moreover, suppose that*

$$\inf a\Big(\bigcup_{i \in \mathbb{N}} \partial V_i\Big) > a_0. \tag{2.4}$$

*Then, for fixed small $\varepsilon > 0$, problem $(\mathcal{P}_\varepsilon)$ has infinitely many pairs of solutions $\pm u$. More precisely, given a generalized barycenter map $\beta$, there exist $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, solutions $\pm u_{ik}, i \in \mathbb{N}, k = 1, \ldots, N$ and $\pm v_{ij}, i, j \in \mathbb{N}, i < j$, such that*

$$\beta(u^\pm_{ik}) \in V_i, \qquad \beta(v^+_{ij}) \in V_i, \qquad \beta(v^-_{ij}) \in V_j, \tag{2.5}$$

*for $i, j \in \mathbb{N}$, $i < j$ and $k = 1, \ldots, N$.*

This follows easily from our results above. Indeed, the differences $\inf_{\partial V_i} a - a_0$ and $\inf_{\partial V_i \cup \partial V_j} a - a_0$ are bounded away from zero independently of $i, j$ by (2.4). Hence, by Theorem 2.1(vi) and Theorem 2.4(vi) there is $\varepsilon_0 > 0$ such that the assertions of Corollary 2.2 resp. Corollary 2.5 hold for each $\varepsilon \in (0, \varepsilon_0)$ and each $C_i, V_i$, $i \in \mathbb{N}$ resp. $C_i, C_j, V_i, V_j$, $i, j \in \mathbb{N}$, $i < j$. Consequently, Proposition 2.3 and Proposition 2.6 guarantee the existence of solutions $u_{ik}$ and $v_{ij}$ which safisfy (2.5).

## 3 A flow invariant subset of sign changing functions

For $\varepsilon > 0$ we take

$$\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + a(x)uv), \quad \|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + a(x)u^2)\right)^{1/2},$$

as the scalar product and the corresponding norm in $H^1(\mathbb{R}^N)$. We write

$$\operatorname{dist}_\varepsilon(u, \mathcal{D}) := \inf\{\|u - v\|_\varepsilon : v \in \mathcal{D}\}$$

for the distance to a subset $\mathcal{D}$ of $H^1(\mathbb{R}^N)$ with respect to this norm, and $|\cdot|_p$ for the norm in $L^p(\mathbb{R}^N)$. Set

$$S_\varepsilon := \inf\left\{\frac{\|u\|_\varepsilon^2}{|u|_p^2} : u \in H^1(\mathbb{R}^N) \smallsetminus \{0\}\right\}$$

so that $S_\varepsilon^{-1/2}$ is the norm of the embedding $(H^1(\mathbb{R}^N), \|\cdot\|_\varepsilon) \hookrightarrow L^p(\mathbb{R}^N)$.

The solutions of $(\mathcal{P}_\varepsilon)$ are the critical points of the $C^2$-functional

$$J_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}, \qquad J_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{p}|u|_p^p.$$

Nontrivial solutions lie on the Nehari manifold

$$\begin{aligned} \mathcal{N}_\varepsilon &= \{u \in H^1(\mathbb{R}^N) : u \neq 0,\ J_\varepsilon'(u)u = 0\} \\ &= \{u \in H^1(\mathbb{R}^N) : u \neq 0,\ \|u\|_\varepsilon^2 = |u|_p^p\}, \end{aligned}$$

which is a $C^2$-manifold, radially diffeomorphic to the unit sphere in $H^1(\mathbb{R}^N)$. The radial projection onto $\mathcal{N}_\varepsilon$ is given by

$$\rho_\varepsilon : H^1(\mathbb{R}^N) \smallsetminus \{0\} \to \mathcal{N}_\varepsilon, \qquad \rho_\varepsilon(u) = \left(\frac{\|u\|_\varepsilon^2}{|u|_p^p}\right)^{1/(p-2)} u,$$

and an easy computation shows

$$\inf_{\mathcal{N}_\varepsilon} J_\varepsilon = \frac{p-2}{2p} S_\varepsilon^{p/(p-2)}.$$

In the following $\nabla_\varepsilon J_\varepsilon$ denotes the gradient of $J_\varepsilon$ with respect to the scalar product $\langle\cdot,\cdot\rangle_\varepsilon$.

We consider the negative gradient flow $\varphi_\varepsilon : \mathcal{G}_\varepsilon \to H^1(\mathbb{R}^N)$ of $J_\varepsilon$ defined by

$$\begin{cases} \dfrac{d}{dt}\varphi_\varepsilon(t, u) = -\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t, u)) \\ \varphi_\varepsilon(0, u) = u \end{cases}$$

Here $\mathcal{G}_\varepsilon = \{(t, u) : u \in H^1(\mathbb{R}^N),\ 0 \leq t < T_\varepsilon(u)\}$, where $T_\varepsilon(u) \in (0, \infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi_\varepsilon(t, u)$. A subset $\mathcal{D}$ of $H^1(\mathbb{R}^N)$ is called strictly positively invariant for the flow $\varphi_\varepsilon$ if

$$\varphi_\varepsilon(t, u) \in \operatorname{int}(\mathcal{D}) \quad \text{for every } u \in \mathcal{D} \text{ and every } t \in (0, T_\varepsilon(u)),$$

where $\mathrm{int}(\mathcal{D})$ denotes the interior of $\mathcal{D}$ in $H^1(\mathbb{R}^N)$. If $\mathcal{D}$ is strictly positively invariant, then the set

$$\mathcal{A}_\varepsilon(\mathcal{D}) := \{u \in H^1(\mathbb{R}^N) : \varphi_\varepsilon(t,u) \in \mathcal{D} \text{ for some } t \in (0, T_\varepsilon(u))\},$$

is an open subset of $H^1(\mathbb{R}^N)$. Let $\mathcal{P} := \{u \in H^1(\mathbb{R}^N) : u \geq 0\}$ be the convex cone of nonnegative functions in $H^1(\mathbb{R}^N)$, and let

$$B(\varepsilon, \pm\mathcal{P}) := \left\{ u \in H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon(u, \pm\mathcal{P}) \leq \frac{1}{2} S_\varepsilon^{p/2(p-2)} \right\}.$$

Consider the set

$$\mathcal{E}_\varepsilon = \{u \in H^1(\mathbb{R}^N) : u^+, u^- \in \mathcal{N}_\varepsilon\},$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. The following holds.

**Proposition 3.1.** a) $[B(\varepsilon,\mathcal{P}) \cup B(\varepsilon,-\mathcal{P})] \cap \mathcal{E}_\varepsilon = \emptyset$.
b) $B(\varepsilon,\mathcal{P})$ and $B(\varepsilon,-\mathcal{P})$ are strictly positively invariant for $\varphi_\varepsilon$.

*Proof.* Note that

$$|u^-|_p = \min_{v \in \mathcal{P}} |u - v|_p \leq S_\varepsilon^{-1/2} \min_{v \in \mathcal{P}} \|u - v\|_\varepsilon = S_\varepsilon^{-1/2} \mathrm{dist}_\varepsilon(u, \mathcal{P}). \tag{3.1}$$

In order to prove a) let us assume that there exists $u \in B(\varepsilon,\mathcal{P}) \cap \mathcal{E}_\varepsilon$. Then

$$0 < S_\varepsilon^{p/(p-2)} \leq \|u^-\|_\varepsilon^2 = |u^-|_p^p \leq S_\varepsilon^{-p/2} \mathrm{dist}_\varepsilon(u,\mathcal{P})^p \leq \frac{1}{2^p} S_\varepsilon^{p/(p-2)},$$

which is a contradiction. Hence, $B(\varepsilon,\mathcal{P}) \cap \mathcal{E}_\varepsilon = \emptyset$. Similarly, $B(\varepsilon,-\mathcal{P}) \cap \mathcal{E}_\varepsilon = \emptyset$.

We now prove assertion b) for $B(\varepsilon,\mathcal{P})$. The proof for $B(\varepsilon,-\mathcal{P})$ proceeds analogously. The gradient $\nabla_\varepsilon J_\varepsilon : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is given by $\nabla_\varepsilon J_\varepsilon = Id - K_\varepsilon$ where $K_\varepsilon(u) \in H^1(\mathbb{R}^N)$ is uniquely determined by the relation

$$\langle K_\varepsilon(u), v\rangle_\varepsilon = \int_{\mathbb{R}^N} |u|^{p-2}uv \qquad \text{for all } v \in H^1(\mathbb{R}^N).$$

Using (3.1) we obtain

$$\begin{aligned} \mathrm{dist}_\varepsilon(K_\varepsilon(u),\mathcal{P}) \left\|K_\varepsilon(u)^-\right\|_\varepsilon &\leq \left\|K_\varepsilon(u)^-\right\|_\varepsilon^2 = \left\langle K_\varepsilon(u), K_\varepsilon(u)^-\right\rangle \varepsilon = \int_{\mathbb{R}^N} |u|^{p-2}uK_\varepsilon(u)^- \\ &\leq \int_{\mathbb{R}^N} |u^-|^{p-2}u^- K_\varepsilon(u)^- \leq |u^-|_p^{p-1} \left|K_\varepsilon(u)^-\right|_p \leq S_\varepsilon^{-p/2} \mathrm{dist}_\varepsilon(u,\mathcal{P})^{p-1} \left\|K_\varepsilon(u)^-\right\|_\varepsilon. \end{aligned}$$

Thus $\mathrm{dist}_\varepsilon(K_\varepsilon(u),\mathcal{P}) \leq S_\varepsilon^{-p/2} \mathrm{dist}_\varepsilon(u,\mathcal{P})^{p-1}$ for all $u \in H^1(\mathbb{R}^N)$, and therefore, if $u \in B(\varepsilon,\mathcal{P})$, then $\mathrm{dist}_\varepsilon(K_\varepsilon(u),\mathcal{P}) \leq \frac{1}{2^{p-1}} S_\varepsilon^{p/2(p-2)}$ hence, $K_\varepsilon(u) \in \mathrm{int}B(\varepsilon,\mathcal{P})$. Since $B(\varepsilon,\mathcal{P})$ is convex, this implies that

$$u + \lambda(-\nabla_\varepsilon J_\varepsilon(u)) = (1-\lambda)u + \lambda K_\varepsilon(u) \in B(\varepsilon, P)$$

for every $u \in B(\varepsilon, P)$, $0 \leq \lambda \leq 1$. It follows from [15, Theorem 5.2] that

$$\varphi_\varepsilon(t,u) \in B(\varepsilon,\mathcal{P}) \qquad \text{for every } u \in B(\varepsilon,\mathcal{P}),\ 0 \leq t < T_\varepsilon(u). \tag{3.2}$$

Now assume, by contradiction, that there are $u \in B(\varepsilon, \mathcal{P})$ and $0 < t < T_\varepsilon(u)$ such that $\varphi_\varepsilon(t,u) \in \partial B(\varepsilon, \mathcal{P})$. By Mazur's separation theorem, there exists a continuous linear functional $\ell \in H^1(\mathbb{R}^N)^*$ and an $\alpha > 0$ such that $\ell(\varphi_\varepsilon(t,u)) = \alpha$ and $\ell(v) > \alpha$ for $v \in \mathrm{int} B(\varepsilon, \mathcal{P})$. It follows that

$$\frac{\partial}{\partial s}\Big|_{s=t} \ell(\varphi_\varepsilon(s,u)) = \ell(-\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t,u))) = \ell(K_\varepsilon(\varphi_\varepsilon(t,u))) - \alpha > 0.$$

Hence there exists $t_1 < t$ such that $\ell(\varphi_\varepsilon(s,u)) < \alpha$ for $t_1 < s < t$. This implies that $\varphi_\varepsilon(s,u) \notin B(\varepsilon, \mathcal{P})$ for $t_1 < s < t$, contradicting (3.2). Therefore, $\varphi_\varepsilon(t,u) \in \mathrm{int} B(\varepsilon, \mathcal{P})$ for every $u \in B(\varepsilon, \mathcal{P})$, $0 < t < T_\varepsilon(u)$, as claimed. $\square$

For $\varepsilon > 0$ define

$$\mathcal{D}_\varepsilon := B(\varepsilon, \mathcal{P}) \cup B(\varepsilon, -\mathcal{P}) \cup J_\varepsilon^0. \tag{3.3}$$

Here we write, as usual, $J_\varepsilon^c := \{u \in H^1(\mathbb{R}^N) : J_\varepsilon(u) \leq c\}$ for $c \in \mathbb{R}$. Since $J_\varepsilon(u) > 0$ for every nontrivial critical point of $J_\varepsilon$, the set $J_\varepsilon^0 \smallsetminus \{0\}$ is strictly positively invariant for $\varphi_\varepsilon$. Thus, by Proposition 3.1, $\mathcal{D}_\varepsilon$ is strictly positively invariant for $\varphi_\varepsilon$ and

$$\mathcal{D}_\varepsilon \cap \mathcal{E}_\varepsilon = \emptyset.$$

We define

$$\mathcal{Z}_\varepsilon := H^1(\mathbb{R}^N) \smallsetminus \mathcal{A}_\varepsilon(\mathcal{D}_\varepsilon).$$

Clearly $\mathcal{Z}_\varepsilon$ is a closed symmetric set ($\mathcal{Z}_\varepsilon = -\mathcal{Z}_\varepsilon$) containing only sign-changing functions, and it is positively invariant for the flow $\varphi_\varepsilon$, that is,

$$\varphi_\varepsilon(t,u) \in \mathcal{Z}_\varepsilon \quad \text{for every } u \in \mathcal{Z}_\varepsilon \text{ and every } t \in (0, T_\varepsilon(u)).$$

We shall see that $\mathcal{Z}_\varepsilon$ is an appropriate constraint for finding sign changing solutions to problem $(\mathcal{P}_\varepsilon)$. Note that, by definition of $B(\varepsilon, \mathcal{P})$,

$$\|u^\pm\|_\varepsilon = \|u - u^\mp\|_\varepsilon > \frac{1}{2} S_\varepsilon^{p/2(p-2)} \quad \text{for every } u \in \mathcal{Z}_\varepsilon. \tag{3.4}$$

The following version of Ekeland's variational principle holds on $\mathcal{Z}_\varepsilon$.

**Lemma 3.2.** *Let $\varepsilon > 0$, $c > 0$ and $u \in \mathcal{Z}_\varepsilon$ be such that*

$$J_\varepsilon(u) \leq \inf_{\mathcal{Z}_\varepsilon} J_\varepsilon + c.$$

*Then there exists $v \in \mathcal{Z}_\varepsilon$ such that $J_\varepsilon(v) \leq J_\varepsilon(u)$, $\|u - v\|_\varepsilon \leq \sqrt{c}$, and $\|\nabla_\varepsilon J_\varepsilon(v)\|_\varepsilon \leq \sqrt{c}$.*

*Proof.* Let

$$t_0 := \inf\{t > 0 : \|\varphi_\varepsilon(t,u) - u\|_\varepsilon \geq \sqrt{c}\} \in (0, \infty]$$

and assume, by contradiction, that

$$\|\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t,u))\|_\varepsilon > \sqrt{c} \quad \text{for all } t \in (0, t_0).$$

Then, since $J_\varepsilon$ is bounded below on $\mathcal{Z}_\varepsilon$, it follows immediately from the definition of $\varphi_\varepsilon$ that $t_0 < \infty$. Moreover,

$$\begin{aligned}\sqrt{c} = \|\varphi_\varepsilon(t_0, u) - u\|_\varepsilon &\le \int_0^{t_0} \big\|\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t, u))\big\|_\varepsilon dt \\ &< \frac{1}{\sqrt{c}} \int_0^{t_0} \|\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t, u))\|_\varepsilon^2 dt = \frac{1}{\sqrt{c}}\big[J_\varepsilon(u) - J_\varepsilon(\varphi_\varepsilon(t_0, u))\big] \le \sqrt{c}.\end{aligned}$$

This is a contradiction. □

Associated to the problems $(\mathcal{P}_\varepsilon)$, $\varepsilon > 0$, is the limiting problem

$$-\Delta u + a_0 u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N),$$

where $a_0 = \inf_{\mathbb{R}^N} a(x)$. The solutions to this problem are the critical points of the $C^2$-functional

$$I_{a_0} : H^1(\mathbb{R}^N) \to \mathbb{R}, \qquad I_{a_0}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a_0 u^2) - \frac{1}{p}|u|_p^p.$$

All nontrivial critical points of $I_{a_0}$ lie on the Nehari manifold

$$\mathcal{N}(a_0) = \left\{ u \in H^1(\mathbb{R}^N) : u \ne 0, \int_{\mathbb{R}^N} (|\nabla u|^2 + a_0 u^2) = |u|_p^p \right\},$$

which is radially diffeomorphic to the unit sphere in $H^1(\mathbb{R}^N)$. We write $\rho_{a_0} : H^1(\mathbb{R}^N) \smallsetminus \{0\} \to \mathcal{N}(a_0)$ for the radial projection. It is well known that the infimum

$$c(a_0) := \inf_{\mathcal{N}(a_0)} I_{a_0}$$

is achieved at a unique positive radial function $W_0 = W_0(a_0)$. We fix a radial cut-off function $\chi \in C^\infty(\mathbb{R}^N)$ with $0 \le \chi \le 1$, $\chi(x) = 1$ if $|x| \le \frac{1}{2}$, and $\chi(x) = 0$ if $|x| \ge 1$. For $s > 0$, we define

$$W_s(x) := \chi(sx)W_0(x).$$

Then $W_s \to W_0$ as $s \to 0$ and therefore

$$I_{a_0}(\rho_{a_0}(W_s)) \to c(a_0) \quad \text{as } s \to 0.$$

Let

$$M_\varepsilon^0 := \left\{(x, y) : x, y \in B_{\sqrt{\varepsilon}}(M^0),\ |x - y| \ge 2\sqrt{\varepsilon}\right\},$$

where $M^0$ is the set of minima of $a$ and $B_{\sqrt{\varepsilon}}(M^0) = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M^0) \le \sqrt{\varepsilon}\}$. For $x \in \mathbb{R}^N$, $\gamma \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$ we define $x * u : \mathbb{R}^N \to \mathbb{R}$ and $u \circ \gamma : \mathbb{R}^N \to \mathbb{R}$ by

$$(x * u)(\xi) := u(\xi - x) \quad \text{and} \quad (u \circ \gamma)(\xi) := u(\gamma\xi),$$

and we consider the radial functions

$$w_\varepsilon := W_{\sqrt{\varepsilon}} \circ \varepsilon^{-1} \in H^1(\mathbb{R}^N), \qquad \varepsilon > 0.$$

Note that the support of $w_\varepsilon$ is contained in the ball $B_{\sqrt{\varepsilon}}(0)$ of radius $\sqrt{\varepsilon}$ centered at 0. Hence the map

$$g_\varepsilon : M_\varepsilon^0 \to \mathcal{E}_\varepsilon, \qquad g_\varepsilon(x,y) = \rho_\varepsilon(x * w_\varepsilon) - \rho_\varepsilon(y * w_\varepsilon) \tag{3.5}$$

is well defined and satisfies $g_\varepsilon(y,x) = -g_\varepsilon(x,y)$.

The following technical results are fundamental for the proof of the main theorems. They will be proved in the following section. We recall that $\beta : L^2(\mathbb{R}^N) \smallsetminus \{0\} \to \mathbb{R}^N$ is a generalized barycenter map if it satisfies the properties (2.2) and (2.3) stated in Section 2.

**Proposition 3.3.** a) $\inf_{\mathcal{Z}_\varepsilon} J_\varepsilon \geq 2c(a_0)\varepsilon^N$ *for every* $\varepsilon > 0$.

b) *If* $u \in \mathcal{Z}_\varepsilon$ *is a critical point of* $J_\varepsilon$ *such that* $J_\varepsilon(u) < 3c(a_0)\varepsilon^N$ *then* $u$ *has precisely two nodal domains.*

**Proposition 3.4.** *Let* $\beta$ *be a generalized barycenter map. Let* $\varepsilon > 0$ *and let* $(u_n)$ *be a sequence in* $\mathcal{Z}_\varepsilon$ *such that*

(i) $\lim_{n\to\infty} J_\varepsilon(u_n) < 3c(a_0)\varepsilon^N$,

(ii) $\lim_{n\to\infty} \|\nabla_\varepsilon J_\varepsilon(u_n)\|_\varepsilon = 0$,

(iii) *the sequences* $(\beta(u_n^\pm))$ *are bounded in* $\mathbb{R}^N$.

*Then* $(u_n)$ *is relatively compact in* $H^1(\mathbb{R}^N)$.

**Proposition 3.5.** *Given a generalized barycenter map* $\beta$ *and* $\delta > 0$, *there exist* $\varepsilon_0 = \varepsilon_0(\beta,\delta) > 0$ *and* $d \in (2c(a_0), \frac{5}{2}c(a_0))$ *such that, for every* $0 < \varepsilon < \varepsilon_0$, *we have:*

a) $\beta(u^+) \neq \beta(u^-)$ *and* $a(\beta(u^\pm)) \leq a_0 + \delta$ *for all* $u \in \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$.

b) $g_\varepsilon(M_\varepsilon^0) \subset \mathcal{E}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$.

Note that Proposition 3.5 does not assert that $\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$ is nonempty. This, however, will follow from the results in Section 5 below.

# 4 Compactness for sequences of functionals

Throughout this section, $\|\cdot\|$ stands for the standard norm on $H^1(\mathbb{R}^N)$, i. e. $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)$. Moreover, if $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is a $C^1$-functional, then $\nabla I$ denotes the gradient with respect to the corresponding scalar product. The change of variable $(u \circ \varepsilon)(x) = u(\varepsilon x)$, transforms problem $(\mathcal{P}_\varepsilon)$ into the equivalent problem

$$-\Delta(u\circ\varepsilon) + (a\circ\varepsilon)(u\circ\varepsilon) = |u\circ\varepsilon|^{p-2}(u\circ\varepsilon), \quad u\circ\varepsilon \in H^1(\mathbb{R}^N),$$

which turns out to be more suitable for studying compactness. So we turn our attention to problems of this type.

For $\alpha \in L^\infty(\mathbb{R}^N)$ with $\inf\limits_{x\in\mathbb{R}^N} \alpha(x) > 0$ we consider the $C^2$-functional

$$I_\alpha : H^1(\mathbb{R}^N) \to \mathbb{R}, \qquad I_\alpha(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)u^2) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p.$$

Critical points $u \in H^1(\mathbb{R}^N)$ of $I_\alpha$ are weak solutions of the equation

$$-\Delta u + \alpha(x)u = |u|^{p-2}u. \tag{4.1}$$

All nontrivial critical points of $I_\alpha$ lie on the Nehari manifold

$$\mathcal{N}(\alpha) = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I'_\alpha(u)u = 0\},$$

which is a $C^1$-manifold radially diffeomorphic to the unit sphere in $H^1(\mathbb{R}^N)$. The radial projection $\rho_\alpha : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathcal{N}(\alpha)$ is given by

$$\rho_\alpha(u) = \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \alpha(x)u^2}{|u|_p^p}\right)^{1/(p-2)} u. \tag{4.2}$$

$\mathcal{N}(\alpha)$ is a natural constraint in the sense that $u \in \mathcal{N}(\alpha)$ is a critical point of $I_\alpha$ if and only if it is a critical point of the restricted functional $I_\alpha|_{\mathcal{N}(\alpha)}$. We set

$$c(\alpha) := \inf_{u\in\mathcal{N}(\alpha)} I_\alpha(u). \tag{4.3}$$

**Lemma 4.1.** *Let $\alpha \in L^\infty(\mathbb{R}^N)$ be such that $\inf_{x\in\mathbb{R}^N} \alpha(x) > 0$. If $u$ is a sign changing critical point of $I_\alpha$, then $I_\alpha(u) > 2c(\alpha)$. If, moreover, $I_\alpha(u) < 3c(\alpha)$, then $u$ has precisely two nodal domains.*

*Proof.* If $u$ is a sign changing critical point of $I_\alpha$, then $I'_\alpha(u^\pm)u^\pm = I'_\alpha(u)u^\pm = 0$. Hence, $u^\pm \in \mathcal{N}(\alpha)$ and

$$I_\alpha(u) = I_\alpha(u^+) + I_\alpha(u^-) \geq 2c(\alpha).$$

Assume, by contradiction, that $I_\alpha(u) = 2c(\alpha)$. Then $I_\alpha(u^\pm) = c(\alpha)$. In particular, $u^+$ is a minimizer of $I_\alpha|_{\mathcal{N}(\alpha)}$ and, hence, a solution of (4.1). By the maximum principle, $u^+ > 0$ on $\mathbb{R}^N$. This implies that $u^- = 0$, contradicting the fact that $u$ is sign changing. Therefore, $I_\alpha(u) > 2c(\alpha)$. Now assume that $u$ has at least three nodal domains $\Omega_1, \Omega_2, \Omega_3$ and let $\chi_{\Omega_i}$ be the corresponding characteristic functions. Then $u\chi_{\Omega_i} \in H^1(\mathbb{R}^N)$ and

$$I'_\alpha(u\chi_{\Omega_i})u\chi_{\Omega_i} = I'_\alpha(u)u\chi_{\Omega_i} = 0.$$

Hence, $u\chi_{\Omega_i} \in \mathcal{N}(\alpha)$ and, therefore, $I_\alpha(u) \geq I_\alpha(u\chi_{\Omega_1}) + I_\alpha(u\chi_{\Omega_2}) + I_\alpha(u\chi_{\Omega_3}) \geq 3c(\alpha)$. □

**Lemma 4.2.** *Let $\alpha_1, \alpha_2 \in L^\infty(\mathbb{R}^N)$ be such that $\inf_{x\in\mathbb{R}^N} \alpha_1(x) > 0$ and $\alpha_2 \geq \alpha_1$ almost everywhere on $\mathbb{R}^N$. Then $c(\alpha_2) \geq c(\alpha_1)$ and, if $I_{\alpha_2}(u) = c(\alpha_1)$ for some $u \in \mathcal{N}(\alpha_2)$, then $\alpha_1 = \alpha_2$ almost everywhere on $\mathbb{R}^N$.*

*Proof.* An easy calculation shows that

$$c(\alpha) = \inf_{0\neq u\in H^1(\mathbb{R}^N)} \frac{p-2}{2p}\left(\frac{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha(x)u^2}{|u|_p^2}\right)^{p/(p-2)}.$$

Therefore, $c(\alpha_2)\geq c(\alpha_1)$. Let $u\in\mathcal{N}(\alpha_2)$ satisfy $I_{\alpha_2}(u)=c(\alpha_1)$. By the maximum principle we may assume that $u>0$ on $\mathbb{R}^N$. Setting $v=\rho_{\alpha_1}(u)\in\mathcal{N}(\alpha_1)$ we obtain

$$\begin{aligned} c(\alpha_1)\leq I_{\alpha_1}(v) = \frac{p-2}{2p}|v|_p^p &= \frac{p-2}{2p}\cdot\left(\frac{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha_1(x)u^2}{|u|_p^p}\right)^{p/(p-2)}|u|_p^p \\ &= \left(\frac{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha_1(x)u^2}{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha_2(x)u^2}\right)^{p/(p-2)} I_{\alpha_2}(u) = \left(\frac{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha_1(x)u^2}{\int_{\mathbb{R}^N}|\nabla u|^2+\alpha_2(x)u^2}\right)^{p/(p-2)} c(\alpha_1). \end{aligned}$$

We conclude that $\alpha_1=\alpha_2$ almost everywhere on $\mathbb{R}^N$. □

We now prove a representation lemma for Palais-Smale sequences of a sequence of functionals $I_{\alpha_n}$. Fix constants $0<a_0\leq a_\infty<\infty$, and set

$$C(\mathbb{R}^N,[a_0,a_\infty]) := \{\alpha\in C(\mathbb{R}^N): a_0\leq\alpha(x)\leq a_\infty\}.$$

A subset $\mathcal{S}\subset C(\mathbb{R}^N)$ is called uniformly equicontinuous if, for any $\gamma>0$ there exists $\delta>0$ such that $|f(y)-f(x)|<\gamma$ for every $x,y\in\mathbb{R}^N$ with $|y-x|<\delta$ and every $f\in\mathcal{S}$. Recall that for $x\in\mathbb{R}^N$ and $u:\mathbb{R}^N\to\mathbb{R}$ we defined $(x*u)(\xi):=u(\xi-x)$.

**Lemma 4.3.** *Let $(\alpha_n)$ be a uniformly equicontinuous sequence of functions in $C(\mathbb{R}^N,[a_0,a_\infty])$, and let $u_n\in H^1(\mathbb{R}^N)$ be such that*

$$I_{\alpha_n}(u_n)\to b,\qquad \nabla I_{\alpha_n}(u_n)\to 0.$$

*Then $b\geq 0$, and there exists a subsequence, which we continue to denote by $(u_n)$, and, for some $0\leq k\leq\frac{b}{c(a_0)}$, there exist functions $\alpha^1,\dots,\alpha^k\in C(\mathbb{R}^N,[a_0,a_\infty])$, nontrivial weak solutions $v^1,\dots,v^k\in H^1(\mathbb{R}^N)$ of the equations*

$$-\Delta v^k+\alpha^k(x)v^k = |v^k|^{p-2}v^k,$$

*and sequences $(x_n^i)$, $i=1,\dots,k$, such that, as $n\to\infty$,*

a) $|x_n^i-x_n^j|\to\infty$ *for* $i\neq j$,

b) $x_n^i*\alpha_n\to\alpha^i$ *uniformly on bounded subsets of* $\mathbb{R}^N$,

c) $\|u_n-\sum_{i=1}^k(-x_n^i)*v^i\|\to 0$ *as* $n\to\infty$,

d) $b=\sum_{i=1}^k I_{\alpha^i}(v^i)$.

*Proof.* We have as $n \to \infty$

$$
\begin{aligned}
b + o(1) = I_{\alpha_n}(u_n) &= \frac{p-2}{2p} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + \alpha_n(x) u_n^2\right) + \frac{1}{p} I'_{\alpha_n}(u_n) u_n \\
&\geq \min\{a_0, 1\} \frac{p-2}{2p} \|u_n\|^2 + o(1)\|u_n\|.
\end{aligned}
$$

Hence, $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$ and $b \geq 0$. Moreover, $b = 0$ if and only if $u_n \to 0$ in $H^1(\mathbb{R}^N)$. We now assume that $u_n \not\to 0$. Then, by a standard argument (cf. [23, Lemma I.1]), there exist a subsequence which we continue to denote by $(u_n)$, a sequence $(x_n^1)$ in $\mathbb{R}^N$ and a $v^1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$
\begin{aligned}
&x_n^1 * u_n \rightharpoonup v^1 \quad \text{weakly in } H^1(\mathbb{R}^N), \\
&x_n^1 * u_n \to v^1 \quad \text{a.e. on } \mathbb{R}^N.
\end{aligned}
$$

Since the sequence $(x_n^1 * \alpha_n)$ is equicontinuous, the Arzelà-Ascoli theorem allows us to assume that $x_n^1 * \alpha_n \to \alpha^1 \in C(\mathbb{R}^N, [a_0, a_\infty])$ uniformly on bounded subsets of $\mathbb{R}^N$. Then, for every $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} (x_n^1 * \alpha_n)(x_n^1 * u_n)\psi \to \int_{\mathbb{R}^N} \alpha^1 v^1 \psi \qquad \text{as } n \to \infty.
$$

Consequently,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(\nabla v^1 \nabla \psi + \alpha^1 v^1 \psi\right) - \int_{\mathbb{R}^N} \left|v^1\right|^{p-2} v^1 \psi \\
&\quad = \lim_{n\to\infty} \left( \int_{\mathbb{R}^N} \left(\nabla(x_n^1 * u_n)\nabla\psi + (x_n^1 * \alpha_n)(x_n^1 * u_n)\psi\right) - \int_{\mathbb{R}^N} |(x_n^1 * u_n)|^{p-2}(x_n^1 * u_n)\psi \right) \\
&\quad = \lim_{n\to\infty} \left( \int_{\mathbb{R}^N} (\nabla u_n \nabla((-x_n^1) * \psi) + \alpha_n u_n((-x_n^1) * \psi) - \int_{\mathbb{R}^N} |u_n|^{p-2} u_n((-x_n^1) * \psi) \right) \\
&\quad = \lim_{n\to\infty} I_{\alpha_n}(u_n)((-x_n^1) * \psi) = 0.
\end{aligned}
$$

Thus, $v^1$ is a weak solution of the equation

$$
-\Delta v^1 + \alpha^1(x) v^1 = |v^1|^{p-2} v^1.
$$

We claim that, as $n \to \infty$,

$$
\begin{aligned}
I_{x_n^1 * \alpha_n}(x_n^1 * u_n - v^1) &\to b - I_{\alpha^1}(v^1), && (4.4) \\
\nabla I_{x_n^1 * \alpha_n}(x_n^1 * u_n - v^1) &\to 0 \qquad \text{in } H^1(\mathbb{R}^N). && (4.5)
\end{aligned}
$$

In order to see (4.4) note that, since $x_n^1 * u_n \rightharpoonup v^1$ weakly in $H^1(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} \left(|\nabla(x_n^1 * u_n - v^1)|^2 - |\nabla(x_n^1 * u_n)|^2 + |\nabla v^1|^2\right) \to 0. \tag{4.6}
$$

Moreover, since $x_n^1 * \alpha_n \to \alpha^1$ uniformly on bounded subsets and $(x_n^1 * u_n)v^1 \to (v^1)^2$ in $L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left((x_n^1 * \alpha_n)(x_n^1 * u_n - v^1)^2 - (x_n^1 * \alpha_n)(x_n^1 * u_n)^2 + \alpha^1(v^1)^2\right) \to 0. \tag{4.7}$$

Finally, by the Brezis-Lieb lemma [8],

$$\int_{\mathbb{R}^N} \left(|x_n^1 * u_n - v^1|^p - |x_n^1 * u_n|^p + |v^1|^p\right) \to 0. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) we obtain

$$\lim_{n\to\infty} \left(I_{x_n^1 * \alpha_n}(x_n^1 * u_n - v^1) - I_{x_n^1 * \alpha_n}(x_n^1 * u_n) + I_{\alpha^1}(v^1)\right) = 0.$$

Since $I_{x_n^1 * \alpha_n}(x_n^1 * u_n) = I_{\alpha_n}(u_n) \to b$, this yields (4.4). To establish (4.5) it suffices to show that, for $\psi \in H^1(\mathbb{R}^N)$,

$$\left(I'_{x_n^1 * \alpha_n}(x_n^1 * u_n - v^1) - I'_{x_n^1 * \alpha_n}(x_n^1 * u_n) + I'_{\alpha^1}(v^1)\right)\psi = o(1)\|\psi\|. \tag{4.9}$$

We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \left((x_n^1 * \alpha_n)(x_n^1 * u_n - v^1) - (x_n^1 * \alpha_n)(x_n^1 * u_n) + \alpha^1 v^1\right)\psi\right|^2 \\
&\qquad = \left|\int_{\mathbb{R}^N} \left(\alpha^1 - (x_n^1 * \alpha_n)\right)v^1\psi\right|^2 \\
&\qquad \le \left(\int_{\mathbb{R}^N} |\alpha^1 - (x_n^1 * \alpha_n)| \cdot |v^1|^2\right)\left(\int_{\mathbb{R}^N} \alpha^1 - (x_n^1 * \alpha_n)| \cdot |\psi|^2\right) \\
&\qquad \le a_\infty \|\psi\|^2 \int_{\mathbb{R}^N} |\alpha^1 - (x_n^1 * \alpha_n)| \cdot |v^1|^2 \\
&\qquad = o(1)\|\psi\|^2.
\end{aligned}$$

Moreover, a standard argument yields

$$\left|\int_{\mathbb{R}^N} \left(|x_n^1 * u_n - v^1|^{p-2}(x_n^1 * u_n - v^1) - |x_n^1 * u_n|^{p-2}(x_n^1 * u_n) + |v^1|^{p-2}v^1\right)\psi\right| = o(1)\|\psi\|$$

(see e.g. [10, p. 84]). We conclude that (4.9), hence (4.5) holds.

By translation, (4.4) and (4.5) are equivalent to

$$\begin{aligned}
I_{\alpha_n}(u_n - (-x_n^1) * v^1) &\to b - I_{\alpha^1}(v^1), \\
\nabla I_{\alpha_n}(u_n - (-x_n^1) * v^1) &\to 0 \quad \text{in } H^1(\mathbb{R}^N).
\end{aligned}$$

Now, if $u_n - (-x_n^1) * v^1 \to 0$ in $H^1(\mathbb{R}^N)$, then the assertion of the lemma is true with $k = 1$. On the other hand, if $u_n - (-x_n^1) * v^1 \not\to 0$, we repeat the argument given above replacing $u_n$ by $u_n - (-x_n^1) * v^1$ and $b$ by $b - I_{\alpha^1}(v^1)$. In at most $\left[\frac{b}{c(a_0)}\right]$ steps we obtain the assertion. □

The following compactness lemma for a sequence of functionals $I_{\alpha_n}$ is the key step in the proofs of Propositions 3.3, 3.4 and 3.5.

**Lemma 4.4.** *Let $\beta$ be a generalized barycenter map, let $(\alpha_n)$ be a uniformly equicontinuous sequence in $C(\mathbb{R}^N, [a_0, a_\infty])$, and let $(u_n)$ and $(v_n)$ be two sequences in $H^1(\mathbb{R}^N)$ such that, as $n \to \infty$,*

(i) $\|u_n - v_n\| \to 0$,

(ii) $I_{\alpha_n}(v_n) \to b < 3c(a_0)$,

(iii) $\nabla I_{\alpha_n}(v_n) \to 0$,

(iv) $\liminf \|v_n^\pm\| > 0$.

*Then $b \geq 2c(a_0)$, and the sequences $((-\beta(u_n^+)) * u_n^+)$ and $((-\beta(u_n^-)) * u_n^-)$ are relatively compact in $H^1(\mathbb{R}^N)$. Moreover, if $b = 2c(a_0)$, then $|\beta(u_n^+) - \beta(u_n^-)| \to \infty$ and $\beta(u_n^\pm) * \alpha_n \to a_0$ uniformly on bounded subsets of $\mathbb{R}^N$.*

*Proof.* We apply Lemma 4.3 to the sequence $(v_n)$. The number $k$ given by Lemma 4.3 is then either 1 or 2. We first consider the case $k = 1$. Then there exists a sequence $(x_n)$ in $\mathbb{R}^N$ such that, passing to a subsequence, $x_n * v_n \to v$ in $H^1(\mathbb{R}^N)$ and $x_n * \alpha_n \to \alpha^1 \in C(\mathbb{R}^N, [a_0, a_\infty])$ uniformly on bounded subsets, where $v$ is a critical point of $I_{\alpha^1}$ and $I_{\alpha^1}(v) = b$. Since $x_n * v_n^\pm \to v^\pm$, it follows from assumption (iv) that $v$ changes sign, so Lemmas 4.2 and 4.1 yield

$$b > 2c(\alpha^1) \geq 2c(a_0). \tag{4.10}$$

Moreover, since $\beta$ is a generalized barycenter map, $x_n + \beta(v_n^\pm) = \beta(x_n * v_n^\pm) \to \beta(v^\pm)$. Therefore,

$$\big(-\beta(u_n^\pm)\big) * u_n^\pm \to \big(-\beta(v^\pm)\big) * v^\pm \qquad \text{in } H^1(\mathbb{R}^N).$$

Thus, the sequences $\big(\big(-\beta(u_n^\pm)\big) * u_n^\pm\big)$ are relatively compact.

In the case $k = 2$ there are sequences $(x_n^1)$, $(x_n^2)$ in $\mathbb{R}^N$, functions $\alpha^1, \alpha^2 \in C(\mathbb{R}^N, [a_0, a_\infty])$, and critical points $v^1$ and $v^2$ of the functionals $I_{\alpha^1}$ and $I_{\alpha^2}$ respectively, such that, as $n \to \infty$,

a) $|x_n^1 - x_n^2| \to \infty$,

b) $x_n^i * \alpha_n \to \alpha^i$ uniformly on bounded subsets of $\mathbb{R}^N$,

c) $\|u_n - (-x_n^1) * v^1 - (-x_n^2) * v^2\| \to 0$,

d) $b = I_{\alpha^1}(v^1) + I_{\alpha^2}(v^2)$.

>From assumption (ii) we obtain that $I_{\alpha^1}(v^1), I_{\alpha^2}(v^2) < 2c(a_0)$. So, by Lemma 4.1, $v^1$ and $v^2$ do not change sign. By c) and assumption (iv), we may assume that $v^1$ is positive and $v^2$ is negative. We claim that

$$|x_n^1 * u_n^+ - v^1|_2 \to 0 \qquad \text{and} \qquad |x_n^2 * u_n^- - v^2|_2 \to 0. \tag{4.11}$$

Indeed, using a) and c) we obtain

$$\begin{aligned}
&|u_n^+ - (-x_n^1) * v^1|_2 \\
&\quad\le \left|u_n^+ - \big((-x_n^1) * v^1 + (-x_n^2) * v^2\big)^+\right|_2 + \left|\big((-x_n^1) * v^1 + (-x_n^2) * v^2\big)^+ - (-x_n^1) * v^1\right|_2 \\
&\quad\le \left|u_n - \big((-x_n^1) * v^1 + (-x_n^2) * v^2\big)\right|_2 + o(1) \\
&\quad\le \left\|u_n - \big((-x_n^1) * v^1 + (-x_n^2) * v^2\big)\right\| + o(1) = o(1),
\end{aligned}$$

and similarly for the minus sign case. It follows from (4.11) that

$$\begin{aligned}
\beta(v^1) &= \lim_{n\to\infty} \beta(x_n^1 * u_n^+) = \lim_{n\to\infty} (x_n^1 + \beta(u_n^+)), \\
\beta(v^2) &= \lim_{n\to\infty} \beta(x_n^2 * u_n^-) = \lim_{n\to\infty} (x_n^2 + \beta(u_n^-)).
\end{aligned}$$

Hence, $(-\beta(u_n^+)) * u_n^+ \to (-\beta(v^1)) * v^1$ and $(-\beta(u_n^-)) * u_n^- \to (-\beta(v^2)) * v^2$. Moreover, a) yields $|\beta(u_n^+) - \beta(u_n^-)| \to \infty$.

It remains to consider the case $b = 2c(a_0)$ which, by (4.10), can only occur for $k = 2$. In this case we deduce from d) that $I_{\alpha^1}(v^1) = c(a_0) = I_{\alpha^2}(v^2)$. Hence, Lemma 4.2 forces $\alpha^1 \equiv a_0 \equiv \alpha^2$ on $\mathbb{R}^N$. We conclude, by b), that $x_n^1 * \alpha_n \to a_0$ and $x_n^2 * \alpha_n \to a_0$ uniformly on bounded subsets and, hence, that $\beta(u_n^\pm) * \alpha_n \to a_0$ uniformly on bounded subsets, as claimed. $\square$

We come back to the variational setting introduced in section 3. The rescaling operation $u \mapsto u \circ \varepsilon$ has the following properties, for every $u, v \in H^1(\mathbb{R}^N)$, every $\varepsilon > 0$:

$$\int_{\mathbb{R}^N} \big(\nabla(u\circ\varepsilon)\nabla v + (a\circ\varepsilon)(u\circ\varepsilon)v\big) = \varepsilon^{-N}\langle u, v\circ\varepsilon^{-1}\rangle_\varepsilon,$$

$$\int_{\mathbb{R}^N} |u\circ\varepsilon|^{p-2}(u\circ\varepsilon)v = \varepsilon^{-N}\int_{\mathbb{R}^N} |u|^{p-2}u(v\circ\varepsilon^{-1}).$$

We deduce that $I_{a\circ\varepsilon}(u\circ\varepsilon) = \varepsilon^{-N}J_\varepsilon(u)$, where $J_\varepsilon$ is the functional defined in Section 3. Moreover, $u$ is a critical point of $J_\varepsilon$ if and only if $u\circ\varepsilon$ is a critical point of $I_{a\circ\varepsilon}$. Setting

$$a_0 = \inf_{x\in\mathbb{R}^N} a(x), \text{and} \qquad a_\infty := \sup_{x\in\mathbb{R}^N} a(x).$$

we also deduce that

$$\varepsilon^{-N(p-2)/p} S_\varepsilon \ge \inf_{u\in H^1(\mathbb{R}^N)\smallsetminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + a_0u^2)}{|u|_p^2} =: S(a_0). \tag{4.12}$$

Recall that $\nabla_\varepsilon$ denotes the gradient of a functional on $H^1(\mathbb{R}^N)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_\varepsilon$ introduced in Section 3 for $\varepsilon > 0$, whereas $\nabla$ stands for the gradient with respect to the standard $H^1(\mathbb{R}^N)$-scalar product.

**Lemma 4.5.** *Let* $(\varepsilon_n)$ *be a sequence of positive numbers and let* $(u_n)$ *be a sequence in* $H^1(\mathbb{R}^N)$ *such that* $\varepsilon_n^{-N/2}\|\nabla_{\varepsilon_n}J_{\varepsilon_n}(u_n)\|_{\varepsilon_n} \to 0$. *Then* $\|\nabla I_{a\circ\varepsilon_n}(u_n\circ\varepsilon_n)\| \to 0$.

*Proof.* Put $v_n := u_n \circ \varepsilon_n$, $\alpha_n := a \circ \varepsilon_n$, and let $v \in H^1(\mathbb{R}^N)$. Then we have

$$\begin{aligned}|I'_{\alpha_n}(v_n)v| &= \varepsilon_n^{-N}|J'_{\varepsilon_n}(u_n)(v \circ \varepsilon_n^{-1})| \le \varepsilon_n^{-N}\|\nabla_{\varepsilon_n}J_{\varepsilon_n}(u_n)\|_{\varepsilon_n} \cdot \|v \circ \varepsilon_n^{-1}\|_{\varepsilon_n}\\ &= \varepsilon_n^{-N/2}\|\nabla_{\varepsilon_n}J_{\varepsilon_n}(u_n)\|_{\varepsilon_n}\left(\int_{\mathbb{R}^N}(|\nabla v|^2 + a(\varepsilon_n x)v^2)\right)^{1/2}\\ &\le \varepsilon_n^{-N/2}\|\nabla_{\varepsilon_n}J_{\varepsilon_n}(u_n)\|_{\varepsilon_n}\sqrt{\max\{1, a_\infty\}}\|v\| = o(1)\|v\|.\end{aligned}$$

We conclude that $\|\nabla I_{\alpha_n}(v_n)\| \to 0$ as claimed. □

*Proof of Proposition* 3.3. Let $(u_n)$ be a sequence in $\mathcal{Z}_\varepsilon$ be such that $J_\varepsilon(u_n) \to \inf_{\mathcal{Z}_\varepsilon} J_\varepsilon =: c_\varepsilon$. By Lemma 3.2 we may assume that $\|\nabla_\varepsilon J_\varepsilon(u_n)\|_\varepsilon \to 0$. As pointed out in (3.4), since $u_n \in \mathcal{Z}_\varepsilon$ we have $\|u_n^\pm\|_\varepsilon > \frac{1}{2}S_\varepsilon^{p/2(p-2)}$. Put $v_n := u_n \circ \varepsilon$ and $\alpha_n := a \circ \varepsilon$. If $c_\varepsilon < 3c(a_0)$, using Lemma 4.5, we conclude that the sequences $(v_n)$ and $(\alpha_n)$ satisfy assumptions (ii), (iii) and (iv) of Lemma 4.4, with $b = \varepsilon^{-N}c_\varepsilon$. Lemma 4.4 yields $b \ge 2c(a_0)$, which proves claim a). Claim b) follows immediately from Lemma 4.1. □

*Proof of Proposition* 3.4. Let $\varepsilon > 0$ and let $(u_n)$ be a sequence in $\mathcal{Z}_\varepsilon$ such that

$$\lim_{n\to\infty} \varepsilon^{-N}J_\varepsilon(u_n) < 3c(a_0), \quad \lim_{n\to\infty}\|\nabla_\varepsilon J_\varepsilon(u_n)\|_\varepsilon = 0, \quad \text{and} \quad \big(\beta(u_n^\pm)\big) \text{ are bounded.}$$

Since $u_n \in \mathcal{Z}_\varepsilon$ we have $\|u_n^\pm\|_\varepsilon \ge \frac{1}{2}S_\varepsilon^{p/2(p-2)}$. Set $\alpha_n := a \circ \varepsilon$ and $v_n := u_n \circ \varepsilon$. Using Lemma 4.5, we see that the sequences $(\alpha_n)$ and $(v_n)$ satisfy the assumptions (ii), (iii) and (iv) of Lemma 4.4. Therefore, the sequences $\big((-\beta(v_n^+)) * v_n^+\big)$ and $\big((-\beta(v_n^-)) * v_n^-\big)$ are relatively compact in $H^1(\mathbb{R}^N)$. Since $\beta(u_n^\pm) = \varepsilon\beta(v_n^\pm)$ as a consequence of (2.3), the sequences $\big(\beta(v_n^\pm)\big)$ are bounded. It follows that $(u_n)$ is relatively compact in $H^1(\mathbb{R}^N)$. □

*Proof of Proposition* 3.5. Let $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and $u_n \in \mathcal{Z}_{\varepsilon_n}$ be such that

$$J_{\varepsilon_n}(u_n) - 2c(a_0)\varepsilon_n^N = o(\varepsilon_n^N).$$

In order to prove assertion a) via an indirect argument it suffices to show that

$$\frac{1}{\varepsilon_n}\big|\beta(u_n^+) - \beta(u_n^-)\big| \to \infty \quad \text{and} \quad a(\beta(u_n^\pm)) \to a_0. \tag{4.13}$$

By Lemma 3.2 there exist $v_n \in \mathcal{Z}_{\varepsilon_n}$ such that

$$J_{\varepsilon_n}(v_n) \le J_{\varepsilon_n}(u_n), \quad \|u_n - v_n\|_{\varepsilon_n} = o(\varepsilon_n^{N/2}), \quad \text{and} \quad \|\nabla_{\varepsilon_n}J_{\varepsilon_n}(v_n)\|_{\varepsilon_n} = o(\varepsilon_n^{N/2}).$$

Set $\widetilde{u}_n := u_n \circ \varepsilon_n$, $\widetilde{v}_n := v_n \circ \varepsilon_n$, and $\alpha_n := a \circ \varepsilon_n$. Using Lemma 4.5 we obtain

$$I_{\alpha_n}(\widetilde{v}_n) \to 2c(a_0), \quad \|\widetilde{u}_n - \widetilde{v}_n\| \to 0, \quad \text{and} \quad \|\nabla I_{\alpha_n}(\widetilde{v}_n)\| \to 0, \quad \text{as } n \to \infty.$$

Moreover, (3.4) and (4.12) yield

$$\begin{aligned}\max\{1, \alpha_\infty\}\,\|\widetilde{v}_n^\pm\|^2 &\ge \int_{\mathbb{R}^N}(|\nabla\widetilde{v}_n^\pm|^2 + a(\varepsilon_n x)\,|\widetilde{v}_n^\pm|^2) = \varepsilon_n^{-N}\,\|v_n^\pm\|_{\varepsilon_n}^2 \ge \frac{1}{4}\varepsilon_n^{-N}S_{\varepsilon_n}^{p/p-2}\\ &\ge \frac{1}{4}S(a_0)^{p/p-2}.\end{aligned}$$

Hence, by Lemma 4.4, $|\beta(\widetilde{u}_n^+) - \beta(\widetilde{u}_n^-)| \to \infty$ and $\beta(\widetilde{u}_n^\pm) * \alpha_n \to a_0$ uniformly on bounded subsets. By property (2.3), this yields (4.13), and the proof of assertion a) is finished.

We now prove assertion b). Let $r > 0$ and $d \in \big(2c(a_0), \frac{5}{2}c(a_0)\big)$ be given, and set $c := \frac{d}{2} - c(a_0)$. By definition (3.5) of $g_\varepsilon$, it suffices to show that there exists $\varepsilon_0 > 0$ such that

$$J_\varepsilon(t(x * w_\varepsilon)) \le \frac{d\varepsilon^N}{2} \quad \text{for every } \varepsilon \in (0, \varepsilon_0),\ x \in B_{\sqrt{\varepsilon}}(M^0),\ t > 0. \tag{4.14}$$

Note first that, since $W_{\sqrt{\varepsilon}} \to W_0$ as $\varepsilon \to 0$, there exists $t_0 > 0$ such that, for every $t \ge t_0$ and every $\varepsilon \in (0, 1]$,

$$I_{a_\infty}(tW_{\sqrt{\varepsilon}}) \le 0.$$

Hence, for every $t \ge t_0$, $\varepsilon \in (0, 1]$, and $x \in \mathbb{R}^N$,

$$J_\varepsilon(t(x * w_\varepsilon)) = \varepsilon^N I_{a \circ \varepsilon}\Big(t\Big(\frac{x}{\varepsilon} * W_{\sqrt{\varepsilon}}\Big)\Big) \le 0. \tag{4.15}$$

Now fix $a_1 > a_0$ such that

$$\int_{\mathbb{R}^N} (a_1 - a_0)(t_0 W_0)^2 < \frac{c}{2}.$$

Since $a$ is uniformly continuous, we can choose $0 < \varepsilon_0 \le 1$ such that

$$\begin{aligned} &a(y) \le a_1 \quad \text{for every } y \in B_{2\sqrt{\varepsilon_0}}(M^0) \\ &I_{a_0}(\rho_{a_0}(W_{\sqrt{\varepsilon}})) < c(a_0) + \frac{c}{2} \quad \text{for every } \ \varepsilon \in (0, \varepsilon_0). \end{aligned}$$

Here we recall that $\rho_{a_0}$ is the radial projection on the Nehari manifold as defined in (4.2) for $\alpha \equiv a_0$. Hence, for every $t \le t_0$, $\varepsilon \in (0, \varepsilon_0)$, and $x \in B_{\sqrt{\varepsilon_0}}(M^0)$,

$$\begin{aligned} J_\varepsilon(t(x * w_\varepsilon)) &= \varepsilon^N I_{a \circ \varepsilon}\Big(t\Big(\frac{x}{\varepsilon} * W_{\sqrt{\varepsilon}}\Big)\Big) \\ &= \varepsilon^N \left( I_{a_0}\Big(t\Big(\frac{x}{\varepsilon} * W_{\sqrt{\varepsilon}}\Big)\Big) + \int_{\mathbb{R}^N} (a(\varepsilon y) - a_0)\Big(t\Big(\frac{x}{\varepsilon} * W_{\sqrt{\varepsilon}}\Big)\Big)^2 dy \right) \\ &= \varepsilon^N \left( I_{a_0}(tW_{\sqrt{\varepsilon}}) + \int_{B_{\sqrt{\varepsilon}}(0)} (a(\varepsilon z + x) - a_0)\Big(tW_{\sqrt{\varepsilon}}\Big)^2 dz \right) \\ &\le \varepsilon^N \left( I_{a_0}(\rho_{a_0}(W_{\sqrt{\varepsilon}})) + \int_{\mathbb{R}^N} (a_1 - a_0)(t_0 W_0)^2 \right) \\ &\le \varepsilon^N \Big( c(a_0) + \frac{c}{2} + \frac{c}{2} \Big) = \frac{d\varepsilon^N}{2}. \end{aligned}$$

This, together with (4.15), proves (4.14). The proof of assertion b) is finished. □

# 5 The proofs of the main results

## 5.1 The proof of Theorem 2.1

Recall that $a_0 = \inf_{x \in \mathbb{R}^N} a(x)$. Let $C$ be a nonempty compact isolated subset of $M^0 = a^{-1}(a_0)$, and let $V$ be a compact neighborhood of $C$ in $\mathbb{R}^N$ such that $V \cap M^0 = C$. Let

$$\delta := \frac{1}{2}\left( \min_{x \in \partial V} a(x) - a_0 \right) > 0. \tag{5.1}$$

Since $a$ is uniformly continuous, there is a constant $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta)$ such that $a(x) \le a_0 + \delta$ for every $x \in B_{\sqrt{\tilde{\varepsilon}}}(M^0)$. Now let $\beta$ be a generalized barycenter map, and let $d$ and $\varepsilon_0$ be as in Proposition 3.5 for these $\beta$, $\delta$. We may assume that $\varepsilon_0 \le \tilde{\varepsilon}(\delta)$, so that $B_{\sqrt{\varepsilon_0}}(C) \subset V$. We note that $\varepsilon_0$ depends only on $\delta$ and therefore only on the difference $\min_{\partial V} a - a_0$, as claimed in Theorem 2.1(vi). Fix $0 < \varepsilon < \varepsilon_0$ and set

$$\mathcal{Z}_\varepsilon(V) := \left\{ u \in \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N} : \beta(u^+), \beta(u^-) \in V \right\}.$$

**Lemma 5.1.** a) *$\mathcal{Z}_\varepsilon(V) = -\mathcal{Z}_\varepsilon(V)$ is a closed subset of $H^1(\mathbb{R}^N)$ which is positively invariant under the negative gradient flow $\varphi_\varepsilon$.*

b) *$J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{Z}_\varepsilon(V)$.*

c) *Every $u \in \mathcal{Z}_\varepsilon(V)$ changes sign and, if $u$ is a critical point of $J_\varepsilon$, then it has precisely two nodal domains.*

d) *For $u \in \mathcal{Z}_\varepsilon(V)$ the barycenters $\beta(u^+)$ and $\beta(u^-)$ are different and lie in* int$(V)$*, so there is a continuous map*

$$\theta_\varepsilon : \mathcal{Z}_\varepsilon(V) \to F(V), \quad \theta_\varepsilon(u) = \left(\beta(u^+), \beta(u^-)\right).$$

*Proof.* Assertion b) follows from Proposition 3.4 and the fact that $V$ is bounded. Assertion c) follows from Proposition 3.3, and d) is a consequence of Proposition 3.5 and (5.1). We now prove a). Since $\theta_\varepsilon$ is equivariant with respect to the antipodal action $u \mapsto -u$ on $H^1(\mathbb{R}^N)$ and the action $(x, y) \mapsto (y, x)$ on $F(V)$, we have that $\mathcal{Z}_\varepsilon(V) = -\mathcal{Z}_\varepsilon(V)$. Moreover, $\mathcal{Z}_\varepsilon(V) \subset H^1(\mathbb{R}^N)$ is closed, since $\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$ is closed and $\beta$ is continuous. Now let $u \in \mathcal{Z}_\varepsilon(V)$. Since $\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$ is positively invariant under $\varphi_\varepsilon$, Proposition 3.5 guarantees that $a(\beta(\varphi_\varepsilon(t, u)^\pm)) < a_0 + \delta$ for every $t \ge 0$. Hence $\beta(\varphi_\varepsilon(t, u)^\pm) \notin \partial V$ for every $t \ge 0$. Since $\beta(u^\pm) \in$ int$(V)$, we conclude by continuity that $\beta(\varphi_\varepsilon(t, u)^\pm) \in$ int$(V)$ for every $t \ge 0$. Hence $\varphi_\varepsilon(t, u) \in \mathcal{Z}_\varepsilon(V)$ for every $t \ge 0$. So $\mathcal{Z}_\varepsilon(V)$ is positively invariant under $\varphi_\varepsilon$, as claimed. □

Recall that

$$C_\varepsilon = \left\{(x, y) : x, y \in B_{\sqrt{\varepsilon}}(C),\ |x - y| \ge 2\sqrt{\varepsilon}\right\}.$$

Let $H^*$ be Alexander-Spanier or Čech cohomology with coefficients in the field with two elements. Let $G = \mathbb{Z}/2$ and, for $G$-spaces $Y \subset X$, define $H_G^*(X, Y) = H^*(X/G, Y/G)$ as in Section 2, where $X/G$ is the $G$-orbit space of $X$. Since we only deal with free actions, $H_G^*(X, Y)$ is isomorphic to the Borel cohomology $H^*(EG \underset{G}{\times} X, EG \underset{G}{\times} Y)$. In order to conclude the proof of Theorem 2.1 we need to prove the following.

**Proposition 5.2.** *There exists a homomorphism*

$$\tau_\varepsilon : H_G^*(\mathcal{Z}_\varepsilon(V)) \to H_G^*(C_\varepsilon)$$

*such that the composition*

$$\tau_\varepsilon \circ \theta_\varepsilon^* : H_G^*(F(V)) \to H_G^*(C_\varepsilon)$$

*is the homomorphism induced by the inclusion $C_\varepsilon \hookrightarrow F(V)$.*

In order to prove this, we first express a certain subset of $\mathcal{Z}_\varepsilon(V)$ as a fixed point set and then use the fixed point transfer to define $\tau_\varepsilon$. The definition and properties of the transfer are given in the Appendix. We keep on using the $\varepsilon$-dependent notation of the preceeding sections, but omit the $\varepsilon$ in the new notation we introduce from now on. Define $\gamma : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\gamma(u) = \begin{cases} \frac{|u|_p^p}{\|u\|_\varepsilon^2} - 1 & \text{if } u \neq 0 \\ -1 & \text{if } u = 0 \end{cases}$$

This function is continuous because $S_\varepsilon |u|_p^2 \leq \|u\|_\varepsilon^2$ for all $u \in H^1(\mathbb{R}^N)$ and $p > 2$. Note that $\gamma(u) = 0$ if and only if $u \in \mathcal{N}_\varepsilon$. Therefore,

$$\gamma(u^+) = 0 = \gamma(u^-) \quad \text{if and only if} \quad u \in \mathcal{E}_\varepsilon.$$

For $u \in H^1(\mathbb{R}^N)$ we denote by $e(u) \in (0,\infty]$ the entrance time of $u$ into the set $\mathcal{D}_\varepsilon$ defined in (3.3). That is,

$$e(u) := \inf\{t \in (0,\infty] : \varphi_\varepsilon(t,u) \in \mathcal{D}_\varepsilon\}.$$

Since $\mathcal{D}_\varepsilon$ is strictly positively invariant, a standard argument shows that $e : H^1(\mathbb{R}^N) \to [0,\infty]$ is continuous. Moreover, $e(u) = \infty$ if and only if $u \in \mathcal{Z}_\varepsilon$. Consider the retraction

$$\varrho : H^1(\mathbb{R}^N) \smallsetminus \mathcal{Z}_\varepsilon \to \mathcal{D}_\varepsilon, \quad \varrho(u) = \varphi_\varepsilon(e(u),u),$$

and define $\psi : H^1(\mathbb{R}^N) \to \mathbb{R}^2$ by

$$\psi(u) = \begin{cases} 0 & \text{if } u \in \mathcal{Z}_\varepsilon \\ \frac{1}{1+e(u)^p}\big(\gamma(\varrho(u)^+), \gamma(\varrho(u)^-)\big) & \text{if } u \in H^1(\mathbb{R}^N) \smallsetminus \mathcal{Z}_\varepsilon \end{cases}$$

Since $\mathcal{D}_\varepsilon \cap \mathcal{E}_\varepsilon = \emptyset$, $\psi$ satisfies

$$\psi(u) = 0 \text{ if and only if } u \in \mathcal{Z}_\varepsilon. \tag{5.2}$$

We claim that $\psi$ is a continuous function. In order to see this we observe that for a sequence $u_n \in H^1(\mathbb{R}^N) \smallsetminus \mathcal{Z}_\varepsilon$ with $u_n \to u \in \mathcal{Z}_\varepsilon$ we have $e(u_n) \to \infty$. Now

$$\begin{aligned} \|\varrho(u_n) - u_n\|_\varepsilon &\leq \int_0^{e(u_n)} \big\|J_\varepsilon'(\varphi_\varepsilon(t,u_n))\big\|_\varepsilon dt \leq \sqrt{e(u_n)} \cdot \left(\int_0^{e(u_n)} \big\|J_\varepsilon'(\varphi_\varepsilon(t,u_n))\big\|_\varepsilon^2 dt\right)^{1/2} \\ &= \sqrt{e(u_n)} \cdot \big(J(u_n) - J(\varrho(u_n))\big)^{1/2} \leq \sqrt{e(u_n)} \cdot J(u_n)^{1/2} \end{aligned}$$

Since $\|u_n\| = \|u\| + o(1)$ and $J(u_n) = J(u) + o(1)$ we get

$$\|\varrho(u_n)\| \leq \|u\| + o(1) + \sqrt{e(u_n)} \cdot \big(\sqrt{J(u)} + o(1)\big),$$

hence $\|\varrho(u_n)^\pm\| = O\big(\sqrt{e(u_n)}\big)$ as $n \to \infty$. It follows that $\gamma\big(\varrho(u_n)^\pm\big) = O\left([e(u_n)]^{(p-2)/2}\right)$, so $\psi(u_n) \to 0$.

Let $g_\varepsilon : M_\varepsilon^0 \to \mathcal{E}_\varepsilon$ be the map defined in (3.5). Since $C_\varepsilon \subset M_\varepsilon^0$ is compact we may choose $R > 1$ such that

$$J_\varepsilon\big(R(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-)\big) \leq 0 \quad \text{if } (x,y) \in C_\varepsilon \text{ and } \max\{\lambda,\mu\} \geq 1,$$

For $(x,y) \in C_\varepsilon$ we define

$$g_{x,y} : [0,1] \times [0,1] \to \mathbb{R}^2, \quad g_{x,y}(\lambda,\mu) = \psi\big(R(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-)\big).$$

**Lemma 5.3.** *Let* $(x,y) \in C_\varepsilon$.

a) *If* $g_{x,y}(\lambda,\mu) = 0$ *then* $R(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-) \in \mathcal{Z}_\varepsilon(V)$.

b) *If* $(\lambda,\mu) \in \partial\big([0,1]^2\big)$ *then* $g_{x,y}(\lambda,\mu) = \big((R\lambda)^{p-2} - 1, (R\mu)^{p-2} - 1\big)$.

*Proof.* By Proposition 3.5, $g_\varepsilon(x,y) \in \mathcal{E}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N}$ and, therefore, $R\big(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-\big) \in J_\varepsilon^{d\varepsilon^N}$. If $g_{x,y}(\lambda,\mu) = 0$ then, by (5.2), $R\big(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-\big) \in \mathcal{Z}_\varepsilon$. Since $\beta$ is invariant under rotations and $g_\varepsilon(x,y)^+$ and $g_\varepsilon(x,y)^-$ are radial around $x$ and $y$ respectively, we have that $\beta\big(R\lambda g_\varepsilon(x,y)^+\big) = x$ and $\beta\big(R\mu g_\varepsilon(x,y)^-\big) = y$. So, since $x, y \in B_{\sqrt{\varepsilon}}C \subset V$, we conclude that $R\big(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-\big) \in \mathcal{Z}_\varepsilon(V)$. This proves a).

In order to see b) observe that, if $(\lambda,\mu) \in \partial\big([0,1]^2\big)$, then $R\big(\lambda g_\varepsilon(x,y)^+ + \mu g_\varepsilon(x,y)^-\big) \in \mathcal{D}_\varepsilon$. Therefore

$$g_{x,y}(\lambda,\mu) = \big(\gamma\big(R\lambda g_\varepsilon(x,y)^+\big), \gamma\big(R\mu g_\varepsilon(x,y)^-\big)\big) = \big((R\lambda)^{p-2} - 1, (R\mu)^{p-2} - 1\big),$$

as claimed. □

Next we define

$$f : C_\varepsilon \times [0,1]^2 \to C_\varepsilon \times \mathbb{R}^2, \quad f(x,y,\lambda,\mu) = (x, y, (\lambda,\mu) - g_{x,y}(\lambda,\mu)).$$

This map is equivariant with respect to the action

$$(x,y,\lambda,\mu) \mapsto (y,x,\mu,\lambda). \tag{5.3}$$

The projection $\pi : C_\varepsilon \times \mathbb{R}^2 \to C_\varepsilon$ is also equivariant and, since the action is free, the induced map of orbit spaces

$$\widetilde{\pi} : (C_\varepsilon \times \mathbb{R}^2)/G \longrightarrow C_\varepsilon/G$$

is a vector bundle. The map $\widetilde{f} : \big(C_\varepsilon \times [0,1]^2\big)/G \to (C_\varepsilon \times \mathbb{R}^2)/G$ induced by $f$ satisfies the following.

**Lemma 5.4.** a) $\widetilde{\pi} \circ \widetilde{f} = \widetilde{\pi}|_{(C_\varepsilon \times [0,1]^2)/G}$.

b) $\mathrm{Fix}(\widetilde{f}) = \mathrm{Fix}(f)/G \subset \big(C_\varepsilon \times (0,1)^2\big)/G$.

c) *The fixed point transfer* $\tau_{\widetilde{f}} : H^*(\mathrm{Fix}(f)/G) \to H^*(C_\varepsilon/G)$ *satisfies* $\tau_{\widetilde{f}} \circ \widetilde{\pi}^* = id$.

*Proof.* Property a) is immediate, and b) follows from Lemma 5.3b) and the fact that $R > 1$. These properties say that $\widetilde{f}$ is compactly fixed in the sense defined in the Appendix, so it has a fixed point transfer $\tau_{\widetilde{f}}$. The equivariant version of Tietze's extension theorem [26, Proposition 6.10] provides a map

$$g : C_\varepsilon \times (0,1]^2 \times (0,1] \to \mathbb{R}^2$$

such that $g(x,y,\lambda,\mu,0) = g_{x,y}(\lambda,\mu)$,

$$g(x,y,\lambda,\mu,t) = (1-t)\big((R\lambda)^{p-2} - 1, (R\mu)^{p-2} - 1\big) + t\big(\lambda - R^{-1}, \mu - R^{-1}\big)$$

if either $(\lambda,\mu)\in\partial\big([0,1]^2\big)$ or $t=1$, and $g_t := g(\,\cdot\,,t)$ is equivariant with respect to the action (5.3) for every $t\in(0,1]$. Since $R>1$,

$$g^{-1}(0)\subset C_\varepsilon\times(0,1)^2\times(0,1]. \tag{5.4}$$

Now we define $h: C_\varepsilon\times(0,1]^2\times(0,1]\to C_\varepsilon\times\mathbb{R}^2\times(0,1]$ by

$$h(x,y,\lambda,\mu,t)=\big(x,y,(\lambda,\mu)-g(x,y,\lambda,\mu,t)\big).$$

Then $h(x,y,\lambda,\mu,0)=f(x,y,\lambda,\mu)$, $h(x,y,\lambda,\mu,1)=\big(x,y,R^{-1},R^{-1}\big)$, and $h_t := h(\,\cdot\,,t)$ is equivariant for every $t\in(0,1]$. By (5.4) the induced map

$$\widetilde{h}:\big((C_\varepsilon\times(0,1]^2)/G\big)\times(0,1]\to\big((C_\varepsilon\times\mathbb{R}^2)/G\big)\times(0,1]$$

is compactly fixed over $(C_\varepsilon/G)\times(0,1]$. Propositions 7.2 and 7.3 in the Appendix yield

$$\tau_{\widetilde{f}}\circ\widetilde{\pi}^*=\tau_{\widetilde{h}_1}\circ\widetilde{\pi}^*=id.$$

This proves c). □

*Proof of Proposition* 5.2. For $(x,y,\lambda,\mu)\in C_\varepsilon\times[0,1]^2$, Lemma 5.3a) yields:

$$f(x,y,\lambda,\mu)=(x,y,\lambda,\mu)\qquad\Longrightarrow\qquad R\big(\lambda g_\varepsilon(x,y)^+ +\mu g_\varepsilon(x,y)^-\big)\in\mathcal{Z}_\varepsilon(V).$$

Thus the map

$$\iota:\mathrm{Fix}(f)\to\mathcal{Z}_\varepsilon(V),\quad \iota(x,y,\lambda,\mu)=R\big(\lambda g_\varepsilon(x,y)^+ +\mu g_\varepsilon(x,y)^-\big)$$

is well defined, it is equivariant and satisfies $\theta_\varepsilon\circ\iota=i\circ\pi|_{\mathrm{Fix}(f)}$, where $i: C_\varepsilon\hookrightarrow F(V)$ is the inclusion. Define $\tau_\varepsilon$ to be the composition

$$\tau_\varepsilon: H^*_G(\mathcal{Z}_\varepsilon(V))\xrightarrow{\iota^*}H^*_G(\mathrm{Fix}(f))\xrightarrow{\tau\widetilde{f}}H^*_G(C_\varepsilon).$$

Using Lemma 5.4c) we obtain

$$\tau_\varepsilon\circ\theta_\varepsilon^*=\tau_{\widetilde{f}}\circ\iota^*\circ\theta_\varepsilon^*=\tau_{\widetilde{f}}\circ\widetilde{\pi}^*\circ i^*=i^*$$

and our claim follows. □

This completes the proof of Theorem 2.1.

## 5.2 The proof of Corollary 2.2

We first define a suitable version of the equivariant Lusternik-Schnirelmann category for spaces with an action of a group $G$. Recall that a subset $A\subset X$ of a topological space $X$ is said to be locally closed if it is the intersection $A=C\cap O$ of a closed subset $C\subset X$ and an open subset $O\subset X$.

**Definition 5.5.** *The equivariant Lusternik-Schnirelmann category* $\mathrm{cat}_G(X)$ *of a* $G$*-space* $X$ *is the smallest integer* $k$ *such that* $X$ *can be covered by* $k$ *locally closed* $G$*-invariant subsets* $X_1, \dots, X_k$, $X = X_1 \cup \dots \cup X_k$, *where each* $X_j$ *can be equivariantly deformed to an orbit* $Gx_j$ *in* $X$. *If no such covering exists then* $\mathrm{cat}_G(X) := \infty$.

The use of locally closed instead of open or closed subsets is essential because we apply it to the spaces $\mathcal{Z}_\varepsilon(V)$ which may not be ANRs. It is especially suitable for applications in critical point theory; see the proof of Proposition 5.6 below. If $X$ is an ANR then one may replace locally closed in Definition 5.5 by open or closed. We shall only deal with free actions of $G = \mathbb{Z}/2$, so $\mathrm{cat}_G(X)$ is the category of the orbit space $X/G$ defined with locally closed sets, of course. Clearly, one can also define various relative versions of $\mathrm{cat}_G$.

**Proposition 5.6.** $\mathcal{Z}_\varepsilon(V)$ *contains at least* $\mathrm{cat}_G\big(\mathcal{Z}_\varepsilon(V)\big)$ *pairs of critical points of* $J_\varepsilon$.

*Proof.* Suppose $\mathcal{Z}_\varepsilon(V)$ contains precisely $k$ pairs $\pm u_1, \dots, \pm u_k$ of critical points of $J_\varepsilon$. We may assume that $J(u_1) \le \dots \le J(u_k)$. By Lemma 5.1, $\mathcal{Z}_\varepsilon(V)$ is symmetric and positively invariant for the negative gradient flow $\varphi_\varepsilon$ of $J_\varepsilon$, and $J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{Z}_\varepsilon(V)$. It follows that for each $u \in \mathcal{Z}_\varepsilon(V)$ there exists $j$ with $\varphi(t,u) \to \pm u_j$ as $t \to \infty$. Thus the sets

$$X_j := \{u \in \mathcal{Z}_\varepsilon(V) : \varphi_\varepsilon(t,u) \to \pm u_j \text{ as } t \to \infty\}$$

are pairwise disjoint and cover $\mathcal{Z}_\varepsilon(V)$. Since $\varphi_\varepsilon$ is equivariant, $X_j$ is a symmetric subset of $\mathcal{Z}_\varepsilon(V)$. Using the Palais-Smale condition for $J_\varepsilon$ on $\mathcal{Z}_\varepsilon(V)$ once more it is easy to see that for every $j = 1, \dots, k$ the union $X_1 \cup \dots \cup X_j$ is an open subset of $\mathcal{Z}_\varepsilon(V)$. This implies that $X_j$ is a locally closed subset of $\mathcal{Z}_\varepsilon(V)$. Finally, using the flow $\varphi_\varepsilon$ one sees that $X_j$ can be equivariantly deformed to $\pm u_j$ in $\mathcal{Z}_\varepsilon(V)$, in fact, even in itself. It follows that $\mathrm{cat}_G\big(\mathcal{Z}_\varepsilon(V)\big) \le k$. □

**Remark 5.7.** *Let* $\mathrm{cat}^o_G(X)$ *denote the equivariant category of* $X$ *using open coverings. Then* $\mathrm{cat}_G(X) \le \mathrm{cat}^o_G(X)$ *and equality holds if* $X$ *is an ANR. The same is true for the equivariant category defined via closed coverings. However, we do not know whether* $\mathcal{Z}_\varepsilon(V)$ *is an ANR. Thus it may be that* $\mathrm{cat}_G(X) < \mathrm{cat}^o_G(X)$. *We do not know whether Proposition* 5.6 *is true with* $\mathrm{cat}^o_G(X)$ *instead of* $\mathrm{cat}_G(X)$.

**Proposition 5.8.** *For every metric free* $G$*-space* $X$ *we have* $\mathrm{cat}_G(X) \ge H^*_G\text{-cupl}(X)$.

*Proof.* Let $X = X_1 \cup \dots \cup X_k$ be a covering of $X$ as in Definition 5.5. Consider $k$ cohomology classes $\zeta_1, \dots, \zeta_k \in \widetilde{H}^*_G(X)$. We have to show that their cup product is trivial: $\zeta_1 \smile \dots \smile \zeta_k = 0$. Since $X_j/G$ is contractible in $X/G$ the restriction $\zeta_j|_{X_j} \in \widetilde{H}^*_G(X_j)$ of $\zeta_j$, that is the image of $\zeta_j$ under the homomorphism induced by the inclusion $X_j \hookrightarrow X$, is trivial. Now $X_j = C_j \cap O_j$ is the intersection of a closed subset $C_j \subset X$ and an open subset $O_j \subset X$. We may assume that $C_j$ and $O_j$ are symmetric. Now $X_j$ is a relatively closed subset of the metric space $O_j$. Then the continuity property of Alexander-Spanier cohomology yields an open symmetric neighborhood $U_j \subset O_j$ of $X_j$ such that $\zeta_j|_{U_j} = 0 \in \widetilde{H}^*_G(U_j)$. The exact sequence of the pair $(X, U_j)$ implies that $\zeta_j$ is the image of some element $\xi_j \in H^*_G(X, U_j)$. It follows that the cup product $\zeta_1 \smile \dots \smile \zeta_k$ is the image of $\xi_1 \smile \dots \smile \xi_k \in H^*_G(X, U_1 \cup \dots, U_k) = H^*_G(X, X) = 0$. □

*Proof of Corollary* 2.2. Set $k := H^*_G\text{-cupl}_{F(V)}C_\varepsilon$, and let $\zeta_1, \dots, \zeta_{k-1} \in \widetilde{H}^*_G(F(V))$ be cohomology classes such that $i^*(\zeta_1 \smile \cdots \smile \zeta_{k-1}) \neq 0 \in H^*_G(C_\varepsilon)$. Here $i : C_\varepsilon \hookrightarrow F(V)$ is the inclusion. By Theorem 2.1 there exists a homomorphism $\tau_\varepsilon : H^*_G(\mathcal{Z}_\varepsilon(V)) \to H^*_G(C_\varepsilon)$ such that $\tau_\varepsilon \circ \theta^*_\varepsilon = i^*$. It follows that

$$\theta^*_\varepsilon(\zeta_1) \smile \cdots \smile \theta^*_\varepsilon(\zeta_{k-1}) = \theta^*_\varepsilon(\zeta_1 \smile \cdots \smile \zeta_{k-1}) \neq 0 \in H^*_G(\mathcal{Z}_\varepsilon(V)).$$

This implies $H^*_G\text{-cupl}(\mathcal{Z}_\varepsilon(V)) \geq k$ and the corollary follows from Theorem 2.1 and the Propositions 5.6 and 5.8. □

We have to work with Alexander-Spanier cohomology because the homomorphism $\tau_\varepsilon$ has only been defined for this cohomology. We could pass to singular cohomology if $\mathcal{Z}_\varepsilon(V)$ would be an ANR.

### 5.3 The proof of Theorem 2.4 and Corollary 2.5

Let $C$ and $C'$ be disjoint nonempty compact isolated subsets of $M^0$, and let $V$ and $V'$ be disjoint compact neighborhoods of $C$ and $C'$ respectively with $V \cap M^0 = C$ and $V' \cap M^0 = C'$. Let

$$\delta := \frac{1}{2}\left(\min_{x \in \partial V \cup \partial V'} a(x) - a_0\right) > 0.$$

As in the proof of Theorem 2.1, let $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta)$ be such that $a(x) \leq a_0 + \delta$ for every $x \in B_{\sqrt{\tilde{\varepsilon}}}(M^0)$. Now let $\beta$ be a generalized barycenter map, and let $d$ and $\varepsilon_0$ be as in Proposition 3.5 for these $\beta$, $\delta$. We may assume that $\varepsilon_0 \leq \tilde{\varepsilon}(\delta)$, so that $B_{\sqrt{\varepsilon_0}}(C) \subset V$ and $B_{\sqrt{\varepsilon_0}}(C') \subset V'$. We note that $\varepsilon_0$ depends only on $\delta$ and therefore only on the difference $\min_{\partial V \cup \partial V'} a - a_0$, as claimed in Theorem 2.4(vi). Fix $0 < \varepsilon < \varepsilon_0$ and set

$$\mathcal{Z}_\varepsilon(V, V') := \left\{ \pm u \in \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d\varepsilon^N} : \beta(u^+) \in V,\ \beta(u^-) \in V' \right\}.$$

The proof of Theorem 2.4 is now completely analogous to that of Theorem 2.1.

The proof of Corollary 2.5 is completely analogous to that of Corollary 2.2 using Theorem 2.4 instead of Theorem 2.1.

## 6 The cuplength of configuration spaces

Due to their many applications, configuration spaces have been extensively studied during the last fifty years. However, not much is known about the multiplicative structure of their cohomology. In this section we obtain lower bounds for the cuplength of configuration spaces of unordered pairs in some special cases. Let us recall again the definitions.

$$F(A) := \{(x, y) \in A \times A : x \neq y\}$$

is the configuration space of ordered pairs in a topological space $A$. The group $G = \mathbb{Z}/2$ acts on $F(A)$ via permutation of the coordinates $(x, y) \mapsto (y, x)$. The orbit space

$$K(A) := F(A)/G = \{\{x, y\} \in A \times A : x \neq y\}$$

is the configuration space of unordered pairs in $A$.

Since in this section we shall mostly work with singular cohomology we write $H^*$ for singular cohomology and $\check{H}^*$ for Alexander-Spanier cohomology, both with coefficients in the field of two elements. The reason for the use of singular cohomology is that we use characteristic classes and the $\times$-product which are well known for singular cohomology and are more complicated in Alexander-Spanier cohomology; cf. [24]. Recall that the cuplength $H^*$-cupl of a continuous map $f : X \to Y$ is the smallest integer $k$ such that for any $k$ cohomology classes $\zeta_1, \dots, \zeta_k \in \widetilde{H}^*(Y)$ one has $f^*(\zeta_1 \smile \cdots \smile \zeta_k) = 0 \in \widetilde{H}^*(X)$. For an equivariant map $f : X \to Y$ between free $G$-spaces we define $H^*_G$-cupl$(f) = H^*$-cupl$(f/G)$ where $f/G : X/G \to Y/G$ is the induced map on the orbits spaces.

**Proposition 6.1.** *Let $C \neq \emptyset$ be a closed subset of $\mathbb{R}^N$ and let $V$ be a neighborhood of $C$. Then there is a continuous map $\alpha : C \times \mathbb{R}P^{N-1} \hookrightarrow K(V)$, where $\mathbb{R}P^{N-1}$ is the $(N-1)$-dimensional real projective space, such that*

$$H^*\text{-cupl}(K(V)) \geq H^*\text{-cupl}(\alpha) \geq N.$$

*In particular, $H^*_G$-cupl$(F(V)) = H^*$-cupl$(K(V)) \geq N$ for every subset $V$ of $\mathbb{R}^N$ with* int$(V) \neq \emptyset$.

*Proof.* Choose a continuous map $\gamma : C \to (0, \infty)$ with $\gamma(x) < \text{dist}(x, \mathbb{R}^N \setminus V)$ for every $x \in C$. The map

$$\kappa : C \times \mathbb{S}^{N-1} \to F(V), \quad (x, z) \mapsto (x + \gamma(x)z, x - \gamma(x)z) \tag{6.1}$$

is $G$-equivariant for the action $(x, z) \mapsto (x, -z)$ on $C \times \mathbb{S}^{N-1}$. Hence it induces a map of the orbit spaces $\alpha : C \times \mathbb{R}P^{N-1} \to K(V)$, which maps the first Stiefel-Whitney class $\widetilde{\omega} \in H^1(K(V))$ of the 0-sphere bundle $F(V) \to K(V)$ to the first Stiefel-Whitney class of the bundle $C \times \mathbb{S}^{N-1} \to C \times \mathbb{R}P^{N-1}$ [21], that is, $\alpha^*(\widetilde{\omega}) = 1 \times \omega$, where $\omega \in H^1(\mathbb{R}P^{N-1})$ is the generator. Therefore $\alpha^*(\widetilde{\omega})^{N-1} \neq 0$ and, consequently, $\widetilde{\omega}^{N-1} \neq 0$. □

*Proof of Proposition* 2.3. If $0 < \delta := \sqrt{\varepsilon} < \text{dist}(C, \mathbb{R}^N \setminus V)$, the image of the embedding (6.1) with $\gamma(x) \equiv \delta$ is contained in $C_\varepsilon = \{(x, y) : x, y \in B_{\sqrt{\varepsilon}}(C),\ |x - y| \geq 2\sqrt{\varepsilon}\}$. Therefore

$$H^*_G\text{-cupl}_{F(V)}(C_\varepsilon) = H^*\text{-cupl}_{K(V)}(C_\varepsilon/G) \geq H^*\text{-cupl}(\alpha) \geq N.$$

where $\alpha : C \times \mathbb{R}P^{N-1} \hookrightarrow K(V)$ is the embedding from Proposition 6.1. We need this estimate also for $\check{H}^*$. For this we may assume that $V$ is a regular neighborhood so that $\check{H}^*_G(F(V)) \cong H^*_G(F(V))$. Let $\zeta_1, \dots, \zeta_{k-1} \in \widetilde{H}^*(K(V))$ be cohomology classes with $\eta := \alpha^*(\zeta_1 \smile \cdots \smile \zeta_{k-1}) \neq 0 \in H^*(C \times \mathbb{R}P^{N-1})$. Then $\check{\eta} := \check{\alpha}(\zeta_1 \smile \cdots \smile \zeta_{k-1}) \neq 0 \in \check{H}^*(C \times \mathbb{R}P^{N-1})$ because it maps onto $\eta$ under the natural homomorphism $\check{H}^* \to H^*$. □

The estimate given by Proposition 6.1 is not optimal as the following proposition shows. Its proof uses the Leray-Serre spectral sequence. For details on this and other spectral sequences we refer to [25].

**Proposition 6.2.** *Let* $C \neq \emptyset$ *be a closed subset of* $\mathbb{R}^N$ *which is a deformation retract of a neighborhood* $V \subset \mathbb{R}^N$. *If* $H^i(C) = 0$ *for* $0 < i < m$, *some* $m < N$, *and if there are* $k$ *cohomology classes* $\zeta_1, \dots, \zeta_k \in H^m(C)$ *whose cup-product is nontrivial then*

$$H^*\text{-cupl}(K(V)) \geq H^*\text{-cupl}(\alpha) \geq k + N.$$

*where* $\alpha : C \times \mathbb{R}P^{N-1} \to K(V)$ *is the map from Proposition* 6.1.

*Proof.* We consider the map $\kappa : C \times \mathbb{S}^{N-1} \to F(V)$ and the Stiefel-Whitney classes $\widetilde{\omega} \in H^1(K(V))$ and $\omega \in H^1(\mathbb{R}P^{N-1})$ from the proof of Proposition 6.1 above. Recall that $\alpha^*(\widetilde{\omega}) = 1 \times \omega$. We shall show that there are cohomology classes $\widetilde{\zeta_j} \in H^m(K(V))$ such that

$$\alpha^*(\widetilde{\zeta_j}) = \zeta_j \times 1.$$

If this is true then $\alpha^*(\widetilde{\zeta_1} \smile \cdots \smile \widetilde{\zeta_k} \smile \widetilde{\omega}^{N-1}) = (\zeta_1 \smile \cdots \smile \zeta_k) \times \omega^{N-1} \neq 0$, and therefore $\widetilde{\zeta_1} \smile \cdots \smile \widetilde{\zeta_k} \smile \widetilde{\omega}^{N-1} \neq 0$, which proves our claim.

We consider the diagram

$$\begin{tikzcd}
C \times \mathbb{S}^{N-1} \arrow[r, "\beta"] \arrow[d, "p_C"'] & F(V) \arrow[d, "p_V"'] & \\
C \arrow[r, "i"] & V \arrow[r, "r"] & C
\end{tikzcd}$$

where the vertical arrows are the projections onto the first factor, $i$ is the inclusion, $r$ the retraction, $r \circ i \simeq id$. The diagram commutes up to homotopy. For $\zeta \in H^*(C)$ let $\overline{\zeta} := p_V^* \circ r^*(\zeta) \in H^*(F(V))$, so that

$$\kappa^*(\overline{\zeta}) = p_C^* \circ i^* \circ r^*(\zeta) = p_C^*(\zeta) = \zeta \times 1 \in H^*(C \times \mathbb{S}^{N-1}). \tag{6.2}$$

We shall show that, if $\zeta \in H^m(C)$, then $\overline{\zeta}$ is in the image of $\pi^* : H^m(K(V)) \to H^m(F(V))$; here $\pi : F(V) \to K(V)$ is the natural projection. We consider the Leray-Serre spectral sequences of the Borel fibrations

$$F(V) \underset{G}{\times} EG \to BG$$
$$(C \times \mathbb{S}^{N-1}) \underset{G}{\times} EG \to BG$$

Note that, since $G$ acts freely on $F(V)$ and on $C \times \mathbb{S}^{N-1}$, the projections $F(V) \times EG \to F(V)$ and $C \times \mathbb{S}^{N-1} \times EG \to C \times \mathbb{S}^{N-1}$ induce homotopy equivalences between the $G$-orbit spaces

$$F(V) \underset{G}{\times} EG \simeq K(V), \quad (C \times \mathbb{S}^{N-1}) \underset{G}{\times} EG \simeq C \times \mathbb{R}P^{N-1}.$$

The map $\kappa : C \times \mathbb{S}^{N-1} \to F(V)$ induces a map of spectral sequences

$$\kappa^* : E_r^{p,q} \to \widetilde{E}_r^{p,q}.$$

The $E_2$-term of the spectral sequence associated to $F(V) \underset{G}{\times} EG \to BG$ is

$$E_2^{p,q} = H^p(BG; H^q(F(V))).$$

Our assumptions on $C$ imply that $H^q(F(V)) = 0$ if $0 < q < m$. Indeed, using excision and the suspension isomorphism we have

$$H^q(V \times V, F(V)) \cong H^q(V \times \mathbb{B}^N, V \times (\mathbb{B}^N \setminus \{0\})) \cong \widetilde{H}^{q-N}(V) = 0 \quad \text{for } q \leq N,$$

so from the exact cohomology sequence of the pair $(V \times V, F(V))$ we conclude that

$$H^q(F(V)) \cong H^q(V \times V) \cong H^q(C \times C) \quad \text{for } q < N.$$

We conclude that $H^q(F(V)) = 0$ if $0 < q < m$, as claimed. It follows that

$$\begin{aligned} H^m(F(V)) &\cong E_2^{0,m} = \cdots = E_{m+1}^{0,m}, \\ H^{m+1}(BG) &\cong E_2^{m+1,0} = \cdots = E_{m+1}^{m+1,0}. \end{aligned}$$

Now we fix $\zeta \in H^m(C)$, $\overline{\zeta} = p_V^* \circ r^*(\zeta) \in H^m(F(V)) \cong E_2^{0,m}$, and recall that $\kappa^*(\overline{\zeta}) = \zeta \times 1 \in H^m(C \times \mathbb{S}^{N-1}) \cong \widetilde{E}_2^{0,m}$. Since $\zeta \times 1$ is a permanent cycle,

$$\kappa^* d_{m+1}(\overline{\zeta}) = \widetilde{d}_{m+1}(\zeta \times 1) = 0.$$

Now $\kappa^*$ is the identity on $H^{m+1}(BG) \cong E_2^{m+1,0} = \widetilde{E}_2^{m+1,0}$, hence $d_{m+1}(\overline{\zeta}) = 0$ and, therefore, $\overline{\zeta}$ is a permanent cycle too. Thus, there exists $\widetilde{\zeta} \in H^m(F(V) \underset{G}{\times} EG) \cong H^m(K(V))$ which restricts to $\overline{\zeta} \in H^m(F(V))$ and, from (6.2), we obtain

$$\alpha^*(\widetilde{\zeta}) = \zeta \times 1 \in H^m(M \times \mathbb{R}P^{N-1}).$$

This finishes the proof. □

Special cases where the assumptions of Proposition 6.2 hold are the following.

**Corollary 6.3.** *Let $C \subset \mathbb{R}^N$ be homeomorphic to a sphere $\mathbb{S}^m$, or to a torus $(\mathbb{S}^m)^k$, or to a product of projective spaces $\Pi_{j=1}^k \mathbb{R}P^{m_j}$, $\Pi_{j=1}^k \mathbb{C}P^{m_j}$, $\Pi_{j=1}^k \mathbb{H}P^{m_j}$. If $V$ is a tubular neighborhood of $C$ in $\mathbb{R}^N$ with $C \hookrightarrow V$ a homotopy equivalence, then*

$$H^*\text{-cupl}(K(V)) \geq H^*\text{-cupl}(\alpha) \geq H^*\text{-cupl}(C) + N - 1.$$

*Here $\alpha : C \times \mathbb{R}P^{N-1} \to K(V)$ is the map from Proposition* 6.1.

# 7 Appendix: The fixed point transfer

We discuss a version of Dold's fixed point transfer [19] for vector bundles, which we use in Section 5. Let $\pi : E \to B$ be an $n$-dimensional real riemannian vector bundle over a metric space $B$. We denote by $0 \subset E$ its zero section. Let $H^*(\,\cdot\,)$ be singular cohomology with coefficients in the field $\mathbb{F}_2$ of two elements. A Thom class for $\pi$ is an element $\theta_\pi \in H^n(E, E \smallsetminus 0)$ whose image under the inclusion generates $H^n(\pi^{-1}(b), \pi^{-1}(b) \smallsetminus \{0\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}) \cong \mathbb{F}_2$ for every $b \in B$. Since we use $\mathbb{F}_2$-coefficients, a Thom class exists for every vector bundle. The following result is classical, see for example [21, Theorem 10.3].

**Theorem 7.1 (Thom isomorphism).** *The homomorphism*

$$H^i(B) \to H^{i+n}(E, E \smallsetminus 0), \quad \zeta \mapsto \theta_\pi \smile \pi^*(\zeta),$$

*is an isomorphism.*

The pullback of $\pi$ over a map $\eta : X \to B$ is defined as

$$E \underset{B}{\times} X := \big\{(y,x) \in E \times X : \pi(y) = \eta(x)\big\}.$$

We write

$$\widetilde{\pi} : E \underset{B}{\times} X \to X \quad \text{and} \quad \widetilde{\eta} : E \underset{B}{\times} X \to E$$

for the projections. $\widetilde{\pi}$ is again an $n$-dimensional vector bundle. If $\theta_\pi$ is a Thom class for $\pi$ then $\theta\widetilde{\pi} := \widetilde{\eta}^*(\theta_\pi)$ is a Thom class for $\widetilde{\pi}$.

A map $f : U \to E$ is *compactly fixed over* $B$ if $U$ is an open subset of $E$, $\pi \circ f = \pi|_U$, $\mathrm{Fix}(f) := \{x \in U : f(x) = x\}$ is closed in $E$, and there is a continuous function $\varrho : B \to (0, \infty)$ such that

$$\mathrm{Fix}(f) \subset \{x \in E : |x| \le \varrho(\pi(x))\} =: E_\varrho.$$

Here $|\cdot|$ stands for the riemannian metric of $\pi$. This last property is equivalent to saying that $\pi|_{\mathrm{Fix}(f)} : \mathrm{Fix}(f) \to B$ is a proper map [19]. Following Dold [19] we define the fixed point transfer $\tau_f$ of a compactly fixed map $f$ over $B$ as follows. Let $X \subset U$ be an open neighborhood of $\mathrm{Fix}(f)$ and consider the following sequence of maps:

$$\begin{tikzcd}
(E \underset{B}{\times} X, E \underset{B}{\times} X \smallsetminus 0 \underset{B}{\times} X) & (X, X \smallsetminus \mathrm{Fix}(f)) \arrow[l, "(i-f,id)"'] \arrow[r, hook, "i_1"] & (E, E \smallsetminus \mathrm{Fix}(f)) \\
{} & (E, E \smallsetminus E_\varrho) \arrow[l, hook', "i_2"'] \arrow[r, hook, "i_3"] & (E, E \smallsetminus 0),
\end{tikzcd} \tag{7.1}$$

where $(i - f, id)(x) = (x - f(x), x)$. Note that the difference $x - f(x)$ makes sense because $\pi(x) = \pi(f(x))$ and $\pi^{-1}(b)$ is a vector space for every $b \in B$. Note also that, by excision, the inclusion $i_1$ induces an isomorphism in cohomology. The homotopy and exactness properties of cohomology ensure that the inclusion $i_3$ is an isomorphism too. Therefore, applying $H^*(\,\cdot\,)$ and composing both ends with the Thom isomorphism we obtain a homomorphism

$$\tau_f^X : H^*(X) \to H^*(B)$$

called the *trace* of $f$. If $Y$ is an open subset of $X$ and $j : Y \hookrightarrow X$ is the inclusion, then $\tau_f^Y \circ j^* = \tau_f^X$. Passing to the direct limit

$$\check{H}(\mathrm{Fix}(f)) \cong \varinjlim \{H^*(X) : X \text{ open in } E,\ X \supset \mathrm{Fix}(f)\}$$

(cf. [18, VIII.6], [19]) we obtain a homomorphism

$$\tau_f : \check{H}(\mathrm{Fix}(f)) \to H^*(B)$$

called the *fixed point transfer of* $f$. It has many useful properties [19]. We mention only those which we need for our purposes.

**Proposition 7.2 (Units).** *If* $\vartheta : B \to E$ *is a section of* $\pi$ *(i. e.* $\pi \circ \vartheta = id$*) and* $f = \vartheta\pi : E \to E$, *then* $\mathrm{Fix}(f) = \vartheta(B)$ *and* $\tau_f = \vartheta^* : H^*(\vartheta(B)) \to H^*(B)$.

*Proof.* The restriction to $(E, E \setminus E_\varrho)$ of the map $id - \vartheta\pi : (E, E \setminus \vartheta(B)) \to (E, E \setminus 0)$ is homotopic to the inclusion $i_3 : (E, E \setminus E_\varrho) \hookrightarrow (E, E \setminus 0)$. Moreover, if $X = \{x \in E : |x - \vartheta\pi(x)| \leq \gamma(\pi(x))\}$ for some continuous function $\gamma : B \to (0, \infty)$, then $(i - \vartheta\pi, id)$ is homotopic to $(i - \vartheta\pi, \vartheta\pi) : (X, X \setminus \vartheta(B)) \to (E \underset{B}{\times} X, E \underset{B}{\times} X \setminus 0 \underset{B}{\times} X)$. This last map can be written as the composition

$$(X, X \setminus \vartheta(B)) \xrightarrow{i-\vartheta\pi} (E, E \setminus 0) \xrightarrow{(id,\vartheta\pi)} (E \underset{B}{\times} X, E \underset{B}{\times} X \setminus 0 \underset{B}{\times} X).$$

Therefore the sequence of maps (7.1) induces the same homomorphism in cohomology as $(id, \vartheta\pi)$ does. Set $\eta := \pi|_X : X \to B$. Then $\eta \circ \vartheta = id$ and $\widetilde{\eta} \circ (id, \vartheta\pi) = id$. Hence, for every $\zeta \in H^*(X)$,

$$(id, \vartheta\pi)^*(\theta_{\widetilde{\pi}} \smile \widetilde{\pi}^*(\zeta)) = (id, \vartheta\pi)^*\big(\widetilde{\eta}^*(\theta_\pi) \smile \widetilde{\pi}^*(\zeta)\big) = (\theta_\pi \smile \pi^*(\vartheta^*\zeta)).$$

In other words, the trace homomorphism is $\tau_f^X = \vartheta^* : H^*(X) \to H^*(B)$. Since every neighborhood of $\mathrm{Fix}(f) = \vartheta(B)$ contains a neighborhood of the form $\{x \in E : |x - \vartheta\pi(x)| \leq \gamma(\pi(x))\}$ for some continuous function $\gamma : B \to (0, \infty)$, our claim follows. ☐

**Proposition 7.3 (Homotopy).** *Let* $\pi : F \to B \times (0, 1]$ *be a vector bundle and let* $h : W \to F$ *be a compactly fixed map over* $B \times (0, 1]$. *For each* $t \in [0, 1]$ *set* $F_t := \pi^{-1}(B \times \{t\})$, $W_t := W \cap F_t$, *and let* $\pi_t : F_t \to B \times \{t\} \equiv B$ *and* $h_t : W_t \to F_t$ *be the restrictions of* $\pi$ *and* $h$ *respectively, and* $j_t : \mathrm{Fix}(h_t) \hookrightarrow \mathrm{Fix}(h)$ *be the inclusion. Then the following diagram commutes:*

$$\begin{tikzcd}
\check{H}^*(\mathrm{Fix}(h_0)) \arrow[d, "\tau_{h_0}"'] & \check{H}^*(\mathrm{Fix}(h)) \arrow[l, "\check{j}_0"'] \arrow[r, "\check{j}_1"] \arrow[d, "\tau_h"'] & \check{H}^*(\mathrm{Fix}(h_1)) \arrow[d, "\tau_{h_1}"'] \\
H^*(B) \arrow[r, "\cong"] & H^*(B \times (0, 1]) \arrow[r, "\cong"] & H^*(B)
\end{tikzcd}$$

*Consequently,*

$$\tau_{h_0} \circ \check{\pi}_0 = \tau_{h_1} \circ \check{\pi}_1.$$

The proof is straightforward.

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