A global compactness result for elliptic problems with critical nonlinearity on symmetric domains

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Abstract

We give a precise description of all G-invariant Palais-Smale sequences for the variational problem associated with an elliptic Dirichlet problem at critical growth on a bounded domain which is invariant under the action of a group G of orthogonal transformations.

1 Introduction and statement of results

Lack of compactness in elliptic problems which are invariant under translations or dilations has been extensively studied. It is known to produce quite interesting phenomena. In particular, it gives rise to an effect of the topology of the domain on the number of solutions of suitable perturbations of such problems (for a detailed discussion see for example [2], [11], [13]). If the domain is invariant under the action of some group of orthogonal transformations of $\mathbb{R}^N$ then there is an influence of these symmetries as well. The purpose of this note is to give a precise description of the way the symmetries of the domain affect the lack of compactness.

We consider the problem

$$(P) \qquad \begin{cases} -\Delta u = Q(x) \mid u \mid^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, Q is continuous and strictly positive in $\overline{\Omega}$.

M. Struwe [10] gave a global compactness result for this problem when $Q \equiv 1$. That is, he gave a complete description of the Palais-Smale sequences of the associated variational problem. He showed that the lack of compactness is produced by solutions of the limiting problem

$$(P_\infty) \qquad \begin{cases} -\Delta u = \mid u \mid^{2^*-2} u & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to 0 \end{cases}$$

concentrating at points of the domain.

Here we shall consider domains Ω which are invariant under the action of some closed subgroup G of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$, that is, domains Ω such that $gx \in \Omega$ for each $x \in \Omega$, $g \in G$. We also assume that the function Q is G-invariant, that is, $Q(gx) = Q(x)$ for each $x \in \Omega$, $g \in G$, and consider the problem

$$(P^G) \qquad \begin{cases} -\Delta u = Q(x) \mid u \mid^{2^*-2} u & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \\ u(gx) = u(x) & \text{for all } g \in G \end{cases}$$

Our aim is to give a global compactness result for this problem. Roughly speaking, we will show that lack of compactness is produced by solutions of limiting problems of the form

$$(P_\infty^\Gamma) \qquad \begin{cases} -\Delta u = \mid u \mid^{2^*-2} u & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to 0 \\ u(gx) = u(x) & \text{for all } g \in \Gamma \end{cases}$$

concentrating at G-orbits of Ω with orbit type G/Γ for some closed subgroup Γ of finite index in G.

Before giving a precise statement we recall some basic notions. The *G-orbit* of a point $y \in \mathbb{R}^N$ is the set

$$Gy := \{gy \in \mathbb{R}^N : g \in G\}$$

and the *G-isotropy group* of y is the subgroup

$$G_y := \{g \in G : gy = y\}$$

of G. The G-orbit Gy is G-homeomorphic to the homogeneous G-space of right cosets G/G_y. Observe that isotropy subgroups satisfy that $G_{gy} = gG_yg^{-1}$. Therefore, the set of isotropy subgroups of the points of the G-orbit Gy is the whole conjugacy class of the subgroup G_y in G. The G-isomorphism class of G/G_y is called the G-orbit type of the G-orbit Gy [7]. We denote by $|G/G_y|$ the index of G_y in G, that is, $|G/G_y| = \#Gy$ is the cardinality of Gy.

The action of G on Ω induces an orthogonal G-action on the Sobolev space $H_0^1(\Omega)$ given by

$$(gu)(x) := u(g^{-1}x).$$

The energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega Q\,|u|^{2^*}$$

defined on $H_0^1(\Omega)$ is G-invariant, that is, $E(gu) = E(u)$ for every $u \in H_0^1(\Omega)$, $g \in G$. The weak solutions of problem (P^G) are the critical points of the restriction of E to the subspace of G-fixed points

$$H_0^1(\Omega)^G := \left\{u \in H_0^1(\Omega) : u(gx) = u(x) \text{ for all } g \in G\right\}$$

of $H_0^1(\Omega)$. A sequence (u_k) such that

$$u_k \in H_0^1(\Omega)^G, \qquad E(u_k) \to c, \qquad \text{and} \qquad \|DE(u_k)\| \to 0 \text{ in } H^{-1}(\Omega)$$

will be called a G-invariant Palais-Smale sequence for E, or a *G-PS-sequence* for short. We shall prove the following.

Theorem 1 *Let (u_k) be a G-PS-sequence for E. Then, replacing (u_k) by a subsequence if necessary, there exist a solution u of problem (P^G), m closed subgroups $\Gamma_1, ..., \Gamma_m$ of finite index in G and, for each $i = 1, ..., m$, a sequence $(y_{i,k})$ in Ω, a sequence $(\varepsilon_{i,k})$ in $(0,\infty)$, and a Γ_i-invariant solution $(\widetilde{u}_i)$ of the limiting problem $(P_\infty^{\Gamma_i})$, such that,*

(i) $G_{y_{i,k}} = \Gamma_i$ *for all* $k \geq 1$, *and* $y_{i,k} \to y_i$ *as* $k \to \infty$,

(ii) $\varepsilon_{i,k}^{-1}\,dist(y_{i,k}, \partial\Omega) \to \infty$ *and* $\varepsilon_{i,k}^{-1}\,|gy_{i,k} - g'y_{i,k}| \to \infty$ *as* $k \to \infty$ *for all* $[g] \neq [g'] \in G/\Gamma_i$,

(iii) $\left\| u_k - u - \sum_{i=1}^{m} \sum_{[g]\in G/\Gamma_i} \varepsilon_{i,k}^{\frac{2-N}{2}} Q(y_i)^{\frac{2-N}{4}}\, \widetilde{u}_i\left(g^{-1}\left(\frac{\cdot\, - g y_{i,k}}{\varepsilon_{i,k}}\right)\right)\right\| \longrightarrow 0$
in $D^{1,2}(\mathbb{R}^N)$ *as* $k \to \infty$,

(iv) $E(u_k) \to E(u) + \sum_{i=1}^{m} \left(\frac{|G/\Gamma_i|}{Q(y_i)^{\frac{N-2}{2}}}\right) E(\widetilde{u}_i)$ *as* $k \to \infty$.

Let us look at some consequences of this result. We write

$$\mu_Q^G := \left(\min_{x\in\overline{\Omega}} \frac{\#Gx}{Q(x)^{\frac{N-2}{2}}}\right) \frac{1}{N} S^{\frac{N}{2}} \leq \infty$$

where $\#Gx$ is the cardinality of the G-orbit of x, and S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$. So $\frac{1}{N}S^{\frac{N}{2}}$ is the least energy of a nontrivial solution of the limiting problem (P_∞).

We say that E is satisfies the G-Palais-Smale condition $(PS)_c^G$ at c if every G-PS-sequence for E such that $E(u_k) \to c$ has a convergent subsequence. An immediate consequence of Theorem 1 is the following.

Corollary 2 *E satisfies $(PS)_c^G$ at every $c < \mu_Q^G$. In particular, if every G-orbit in Ω is infinite, then E satisfies $(PS)_c^G$ at every $c \in \mathbb{R}$.*

For $Q \equiv 1$ this result is due to P.L. Lions [8]. For arbitrary Q this has been shown in [4]. Corollary 2 says, in particular, that lack of compactness can only occur if Ω contains some finite G-orbit. It implies that problem (P^G) has infinitely many solutions if every G-orbit of Ω is infinite.

Let us take a closer look at the level μ_Q^G. The nontrivial least energy solutions of (P_∞), up to sign, are the instantons

$$U_{\varepsilon,z}(x) = a_N \left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N-2}{2}}, \qquad a_N = [N(N-2)]^{\frac{N-2}{4}},\ \varepsilon > 0,\ z \in \mathbb{R}^N.$$

cf. [1],[12]. They satisfy

$$\int_{\mathbb{R}^N} |\nabla U_{\varepsilon,z}|^2 = S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{2^*}.$$

Theorem 1 implies, in particular, the following.

Corollary 3 *Let* (u_k) *be a* G*-PS-sequence for* E *such that* $E(u_k) \to \mu_Q^G$. *Then a subsequence of* (u_k) *either converges strongly to a nontrivial solution of* u *of problem* (P^G), *or there exist* $\nu = \pm 1$, *and sequences* (y_k) *in* Ω *and* ε_k *in* $(0, \infty)$, *such that*

(i) $y_k \to y \in \overline{\Omega}$ *as* $k \to \infty$, $G_{y_k} = G_y$ *for all* k, *and*

$$\frac{\#Gy}{Q(y)^{\frac{N-2}{2}}} = \min_{x \in \overline{\Omega}} \frac{\#Gx}{Q(x)^{\frac{N-2}{2}}} < \infty,$$

(ii) $\varepsilon_k^{-1} dist(y_k, \partial\Omega) \to \infty$ *and* $\varepsilon_k^{-1} |gy_k - g'y_k| \to \infty$ *as* $k \to \infty$ *for all* $[g] \neq [g'] \in G/G_y$,

(iii) $\left\| u_k - (-1)^\nu \sum_{[g] \in G/G_y} Q(y)^{\frac{2-N}{4}} U_{\varepsilon_k, gy_k} \right\| \longrightarrow 0$ *in* $D^{1,2}(\mathbb{R}^N)$ *as* $k \to \infty$.

Corollary 3 says that least energy nonconvergent G-PS-sequences must concentrate at G-orbits of the set

$$M := \left\{ y \in \overline{\Omega} : \frac{\#Gy}{Q(y)^{\frac{N-2}{2}}} = \min_{x \in \overline{\Omega}} \frac{\#Gx}{Q(x)^{\frac{N-2}{2}}} \right\}.$$

This hints towards the fact that the topology of M must have an effect on the number of low energy solutions of suitable perturbations of problem (P^G). This is, in fact, true. Examples of this behavior can be found in [4] and [6]. But it says more than that: It says that concentration occurs along G-orbits Gy_k with the same orbit type as Gy. This has proved to be quite useful in applications, see for example [5], [6].

The main step in the proof of Theorem 1 consists in showing that concentration occurs along G-orbits with the same orbit type. This allows us to proceed inductively to obtain a global compactness result for problem (P^G), that is, to give a description of all G-invariant PS-sequences, and not only of G-minimizing sequences as was done in [4].

2 Proof of Theorem 1

As in the non-symmetric case [11], [13], Theorem 1 follows inductively from the following proposition.

Proposition 4 *Let* (u_k) *be a G-PS-sequence for* E *such that* $u_k \rightharpoonup 0$ *weakly in* $H_0^1(\Omega)^G$ *and* $E(u_k) \to c > 0$. *Then, replacing* (u_k) *by a subsequence if necessary, there exist a closed subgroup* Γ *of finite index in* G, *a sequence* (y_k) *in* Ω, *a sequence* ε_k *in* $(0, \infty)$, *a* Γ*-invariant solution* $\widetilde{u}$ *of the limiting problem* (P_∞^Γ), *and a G-PS-sequence* (v_k) *for* E *such that*

(i) $G_{y_k} = \Gamma$ *for all* k, *and* $y_k \to y$ *as* $k \to \infty$,

(ii) $\varepsilon_k^{-1} |gy_k - g'y_k| \to \infty$ *and* $\varepsilon_k^{-1} \mathit{dist}(y_k, \partial\Omega) \to \infty$ *as* $k \to \infty$ *for all* $[g] \neq [g'] \in G/\Gamma$,

(iii) $v_k = u_k - \sum_{[g] \in G/\Gamma} \varepsilon_k^{\frac{2-N}{2}} Q(y)^{\frac{2-N}{4}} \widetilde{u} \left(g^{-1} \left(\frac{\cdot\, - g y_k}{\varepsilon_k} \right) \right) + o(1)$ *in* $D^{1,2}(\mathbb{R}^N)$,

(iv) $E(v_k) \to c - \left(\frac{|G/\Gamma|}{Q(y)^{\frac{N-2}{2}}} \right) E(\widetilde{u})$ *as* $k \to \infty$.

Proof. The proof will follow in several steps:

1) Since PS-sequences for E are bounded in $H_0^1(\Omega)$,

$$\int_\Omega Q(x) |u_k|^{2^*} \, dx = NE(u_k) - \frac{N}{2} DE(u_k) u_k \to Nc > 0.$$

Let $\delta := \min\{\frac{Nc}{2}, (\max_{\overline{\Omega}} Q)^{\frac{2-N}{2}} (\frac{S}{2})^{\frac{N}{2}}\}$ where S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$. Let $B(x, r)$ denote the closed ball in $\mathbb{R}^N$ with center x and radius r. The Levy concentration function

$$\Phi_k(r) := \sup_{x \in \mathbb{R}^N} \int_{B(x,r)} Q \, |u_k|^{2^*}$$

satisfies that $\Phi_k(0) = 0$ and $\Phi_k(\infty) > \delta$ for k large enough. Hence we may choose $\xi_k \in \Omega$ and $\varepsilon_k > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \int_{B(x,\varepsilon_k)} Q \, |u_k|^{2^*} = \int_{B(\xi_k,\varepsilon_k)} Q \, |u_k|^{2^*} = \delta, \tag{1}$$

Observe that, since Ω is bounded, the sequence (ε_k) is bounded.

2) Let $V = \mathbb{R}^N$ and, for each closed subgroup H of G, let

$$V^H = \{x \in V : gx = x \text{ for all } g \in H\}$$

be its H-fixed point subspace. If $\xi \in V$ we write ξ^H for the orthogonal projection of ξ onto V^H.

We shall show that there is a closed subgroup Γ of G such that, up to a subsequence,

a) Γ has finite index in G,
b) $G_{\xi_k^\Gamma} = \Gamma$ for all k,
c) $\varepsilon_k^{-1}\left|g\xi_k^\Gamma - g'\xi_k^\Gamma\right| \to \infty$ as $k \to \infty$ for all $[g] \neq [g'] \in G/\Gamma$,
d) $\varepsilon_k^{-1}\left|\xi_k - \xi_k^\Gamma\right| < C < \infty$ for all k.

If $\varepsilon_k^{-1}\left|\xi_k - \xi_k^G\right| < C < \infty$ for all k, then $\Gamma = G$ has the desired properties.

If $\varepsilon_k^{-1}\left|\xi_k - \xi_k^G\right| \to \infty$, let V_1 be the orthogonal complement of V^G in V and write $\xi_k = \xi_k^G + \xi_k^1$. Since $\varepsilon_k^{-1}\left|\xi_k^1\right| \to \infty$, up to a subsequence, $\xi_k^1 \neq 0$ and

$$\eta_k^1 = \frac{\xi_k^1}{\left|\xi_k^1\right|} \to \eta^1 \in V_1.$$

Let $G^1 := G_{\eta^1}$. For every closed subgroup H of G^1, $(\eta_k^1)^H \to (\eta^1)^H = \eta^1$. Hence,

$$G_{\xi_k^H} = G_{(\xi_k^1)^H} = G_{(\eta_k^1)^H} \subset G_{\eta^1} = G^1.$$

for k large enough and, in particular, $G_{\xi_k^{G^1}} = G^1$. We now show that $\left|G/G^1\right| < \infty$. For every finite subset $\{g_1, ..., g_m\}$ of G representing distinct cosets $[g_1], ..., [g_m] \in G/G^1$ let $\rho > 0$ be such that

$$\left|g_i\eta^1 - g_j\eta^1\right| > 2\rho \qquad \text{for } i \neq j.$$

Then, for every closed subgroup H of G^1,

$$\left|g_i(\eta_k^1)^H - g_j(\eta_k^1)^H\right| > \rho \qquad \text{for } i \neq j \text{ and } k \text{ large enough}$$

and, since $\varepsilon_k^{-1}\left|\xi_k^1\right| \to \infty$,

$$\varepsilon_k^{-1}\left|\xi_k^1\right|\rho < \varepsilon_k^{-1}\left|g_i(\xi_k^1)^H - g_j(\xi_k^1)^H\right| = \varepsilon_k^{-1}\left|g_i\xi_k^H - g_j\xi_k^H\right| \to \infty \qquad \text{for } i \neq j.$$

In particular, $B(g_i\xi_k, \varepsilon_k) \cap B(g_j\xi_k, \varepsilon_k) = \emptyset$ for $i \neq j$ and k large enough so, since u_k and Q are G-invariant,

$$m\delta = \sum_{j=1}^{m} \int_{B(g_j\xi_k, \varepsilon_k)} Q\left|u_k\right|^{2^*} \leq \int_\Omega Q\left|u_k\right|^{2^*} = Nc + o(1).$$

It follows that $\left|G/G^1\right| < \infty$ and that

$$\varepsilon_k^{-1}\left|g_i(\xi_k^1)^H - g_j(\xi_k^1)^H\right| = \varepsilon_k^{-1}\left|g\xi_k^H - g'\xi_k^H\right| \to \infty \qquad \text{if } [g] \neq [g'] \in G/G^1$$

for every closed subgroup H of G^1. So, if $\varepsilon_k^{-1}\left|\xi_k - \xi_k^{G^1}\right| < C < \infty$ for all k, then $\Gamma = G^1$ has the desired properties.

If not, we proceed inductively as above to obtain a set of closed subgroups $G = G^0 \supset G^1 \supset \cdots \supset G^n = \Gamma$, a set of linear subspaces $V = V_0 \supset V_1 \supset \cdots \supset V_n$, and a set of points $\xi_k = \xi_k^0,\ \xi_k^1, ..., \xi_k^n$, such that $|G^i/G^{i+1}| < \infty$, $V^{G^i} \subsetneq V^{G^{i+1}}$, $V_i = V_i^{G^i} \oplus V_{i+1}$, $\xi_k^{i+1} = \xi_k^i - (\xi_k^i)^{G^i} \in V_{i+1}$,

$$\begin{aligned} G^i_{(\xi_k^i)^\Gamma} &= G^i_{(\xi_k^{i+1})^\Gamma} = G^{i+1}_{(\xi_k^{i+1})^\Gamma}, \\ \varepsilon_k^{-1}\left|g\xi_k^\Gamma - g'\xi_k^\Gamma\right| &\to \infty \quad \text{for } [g] \neq [g'] \in G^i/G^{i+1}, \end{aligned}$$

and $\varepsilon_k^{-1}\left|\xi_k - \xi_k^{G^i}\right| \to \infty$, for each $i = 0, ..., n-1$, but $\varepsilon_k^{-1}\left|\xi_k - \xi_k^\Gamma\right| < C < \infty$ for all k. Therefore, Γ has the desired properties.

3) We write $y_k := \xi_k^\Gamma$ and define

$$\overline{u}_k(z) := \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k z + y_k) \quad \text{and} \quad \overline{Q}_k(z) := Q(\varepsilon_k z + y_k).$$

Thus, $\overline{u}_k$ and $\overline{Q}_k$ are Γ-invariant,

$$\int |\nabla \overline{u}_k|^2 = \int |\nabla u_k|^2 \quad \text{and} \quad \int Q_k |\overline{u}_k|^{2^*} = \int Q\, |u_k|^{2^*}.$$

In particular, $(\overline{u}_k)$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)^\Gamma$. Hence, up to a subsequence, $\overline{u}_k \rightharpoonup \overline{u}$ weakly in $D^{1,2}(\mathbb{R}^N)^\Gamma$, $\overline{u}_k \to \overline{u}$ a.e. on $\mathbb{R}^N$ and $\overline{u}_k \to \overline{u}$ in $L^2_{loc}(\mathbb{R}^N)$. If $\overline{u} \equiv 0$ then, for every $z \in \mathbb{R}^N$ and every $h \in C_c^\infty(B(z,1))$,

$$\begin{aligned} S\left(\int |h\overline{u}_k|^{2^*}\right)^{\frac{2}{2^*}} &\leq \int |\nabla(h\overline{u}_k)|^2 = \\ &= \int \nabla\overline{u}_k \cdot \nabla(h^2\overline{u}_k) + \int |\nabla h|^2\, \overline{u}_k^2 \\ &= \int h^2 Q_k |\overline{u}_k|^{2^*} - DE(u_k)\left(h^2\left(\frac{\cdot - y_k}{\varepsilon_k}\right)u_k\right) + o(1) \\ &\leq (\max_{\overline{\Omega}} Q)^{\frac{N-2}{N}}\left(\int_{B(z,1)} Q_k |\overline{u}_k|^{2^*}\right)^{\frac{2}{N}}\left(\int |h\overline{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1) \\ &\leq (\max_{\overline{\Omega}} Q)^{\frac{N-2}{N}}\delta^{\frac{2}{N}}\left(\int |h\overline{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1) \\ &= \frac{S}{2}\left(\int |h\overline{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1), \end{aligned}$$

where the first inequality is Sobolev's inequality, the second one follows from the fact that (u_k) is a PS-sequence and from Hölder's inequality, and the third one uses (1). It follows that $\overline{u}_k \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$. On the other hand, since $\varepsilon_k^{-1} |\xi_k - y_k| < C < \infty$ for all k,

$$\begin{aligned} \delta &= \int_{B(\xi_k,\varepsilon_k)} Q\,|u_k|^{2^*} \le \int_{B(y_k,\varepsilon_k(C+1))} Q\,|u_k|^{2^*} \\ &= \int_{B(0,C+1)} Q_k\,|\overline{u}_k|^{2^*} \le (\max_{\overline{\Omega}} Q) \int_{B(0,C+1)} |\overline{u}_k|^{2^*}. \end{aligned}$$

This is a contradiction. Therefore, $\overline{u} \not\equiv 0$.

4) Since Ω is bounded and $u_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, up to a subsequence, $y_k \to y \in \overline{\Omega}$ and $\varepsilon_k \to 0$.

If $(\varepsilon_k^{-1} dist(y_k, \partial\Omega))$ is bounded, we may assume that

$$\lim_{k\to\infty} \varepsilon_k^{-1} dist(y_k, \partial\Omega) = d.$$

It is then easy to verify that, up to a rotation of $\mathbb{R}^N$, the sets $\Omega_k := \{z \in \mathbb{R}^N : \varepsilon_k z + y_k \in \Omega\}$ satisfy

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} \Omega_k \right) = \mathbb{H}^N := \{(z_1, ..., z_N) \in \mathbb{R}^N : z_N \ge -d\}.$$

Hence, $\overline{u}$ is a solution of the autonomous equation

$$-\Delta u = Q(y)\,|u|^{2^*-2}\,u \quad \text{in } \mathbb{H}^N$$

and, by Pohozaev's identity [9], $\overline{u} \equiv 0$. This is a contradiction.

Therefore,

$$\varepsilon_k^{-1} dist(y_k, \partial\Omega) \to \infty$$

and $\widetilde{u} := Q(y)^{\frac{N-2}{4}}\overline{u}$ is a nontrivial solution of the limiting problem (P^{Γ}_{∞}) in $\mathbb{R}^N$. Moreover, since $\varepsilon_k^{-1} |\xi_k - y_k| < C < \infty$ for all k, it follows also that $y_k \in \Omega$.

5) We define $v_k \in H_0^1(\Omega)^G$ as follows: Let $\varphi \in C^\infty(\mathbb{R}^N)$ be radially symmetric and such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B(0,1)$ and $\varphi \equiv 0$ outside of $B(0,2)$. Let

$$4\rho_k := \min\{dist(y_k, \partial\Omega), |gy_k - g'y_k| : [g] \ne [g'] \in G/\Gamma\}.$$

Thus, $\varepsilon_k^{-1}\rho_k \to \infty$. Since $G_{y_k} = \Gamma$ and since $\widetilde{u}$ is Γ-invariant, the function

$$w_k := \sum_{[g]\in G/\Gamma} \varepsilon_k^{\frac{2-N}{2}} Q(y)^{\frac{2-N}{4}}\ \widetilde{u}(\varepsilon_k^{-1}g^{-1}(\cdot - gy_k))\ \varphi(\rho_k^{-1}(\cdot - gy_k)) \in H_0^1(\Omega)$$

does not depend on the choice of g in $[g]$, and it is G-invariant. We define

$$v_k := u_k - w_k \in H_0^1(\Omega)^G.$$

Property *iii)* above follows from the fact that $\varepsilon_k^{-1}\rho_k \to \infty$. We verify property *iv)*. We write

$$\|u\|^2 := \int |\nabla u|^2 .$$

Let $G/\Gamma = \{[g_1], ..., [g_n]\}$. Since $\varepsilon_k^{-1} |g_i y_k - g_j y_k| \to \infty$ as $k \to \infty$ for each $i \neq j$, $\overline{u}_k \rightharpoonup \overline{u} = Q(y)^{\frac{2-N}{4}}\ \widetilde{u}$ weakly in $D^{1,2}(\mathbb{R}^N)$, and u_k is G-invariant, it follows that

$$\left\| u_k - \sum_{i=1}^{n} \varepsilon_k^{\frac{2-N}{2}} Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_i^{-1} \left(\frac{\cdot\ - g_i y_k}{\varepsilon_k} \right) \right\|^2 =$$

$$= \left\| \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k\ \cdot\ + g_1 y_k) - \sum_{i=1}^{n} Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_i^{-1} \left(\cdot\ + \frac{g_1 y_k - g_i y_k}{\varepsilon_k} \right) \right\|^2$$

$$= \left\| \overline{u}_k g_1^{-1} - Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_1^{-1} - \sum_{i\neq 1} Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_i^{-1} \left(\cdot\ + \frac{g_1 y_k - g_i y_k}{\varepsilon_k} \right) \right\|^2$$

$$= \left\| \overline{u}_k g_1^{-1} - \sum_{i\neq 1} Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_i^{-1} \left(\cdot\ + \frac{g_1 y_k - g_i y_k}{\varepsilon_k} \right) \right\|^2 -$$

$$- \left\| Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_1^{-1} \right\|^2 + o(1)$$

$$= \left\| u_k - \sum_{i\neq 1} \varepsilon_k^{\frac{2-N}{2}}\ Q(y)^{\frac{2-N}{4}}\ \widetilde{u} g_i^{-1} \left(\frac{\cdot\ - g_i y_k}{\varepsilon_k} \right) \right\|^2 - Q(y)^{\frac{2-N}{2}} \|\ \widetilde{u}\|^2 + o(1)$$

and, inductively,

$$\begin{aligned}\|u_k\|^2 &= \left\| u_k - \sum_{i=1}^{n} \varepsilon_k^{\frac{2-N}{2}} Q(y)^{\frac{2-N}{4}}\, \widetilde{u} g_i^{-1}\left(\frac{\cdot\ - g_i y_k}{\varepsilon_k}\right)\right\|^2 + \\ &\quad + \frac{n}{Q(y)^{\frac{N-2}{2}}} \|\widetilde{u}\|^2 + o(1) \\ &= \|v_k\|^2 + \frac{|G/\Gamma|}{Q(y)^{\frac{N-2}{2}}} \|\widetilde{u}\|^2 + o(1)\end{aligned}$$

Similarly, using the Brézis-Lieb Lemma [3], we obtain

$$\int Q\, |u_k|^{2^*} = \int Q\, |v_k|^{2^*} + \frac{|G/\Gamma|}{Q(y)^{\frac{N-2}{2}}} \int |\widetilde{u}|^{2^*}$$

and property *iv)* follows. A similar argument using Lemma 8.9 in [13] shows that $\|DE(v_k)\|_{H^{-1}} \to 0$. ■

As in [10], [11], [13], Theorem 1 follows inductively from this proposition. We sketch the proof for the reader's convenience.

Proof of Theorem 1.

Since PS-sequences for E are bounded in $H_0^1(\Omega)$,

$$\int_\Omega |\nabla u_k|^2\, dx = NE(u_k) - \frac{N-2}{2} DE(u_k)u_k \to Nc.$$

Therefore $c \geq 0$. We may assume that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)^G$ and $u_k \to u$ a.e. in Ω. It is easy to see that $DE(u) = 0$ and that $u_k^1 := u_k - u$ is a G-PS-sequence such that $u_k^1 \rightharpoonup 0$ weakly in $H_0^1(\Omega)^G$ and $E(u_k^1) = E(u_k) - E(u) = c - E(u) + o(1)$. The result now follows inductively from Proposition 4. ■

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