MULTIPLE POSITIVE AND 2-NODAL SYMMETRIC SOLUTIONS OF ELLIPTIC PROBLEMS WITH CRITICAL NONLINEARITY

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Abstract. We consider the problem $-\Delta u + a(x)u = f(x)\,|u|^{2^*-2}\,u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω a bounded smooth domain in $\mathbb{R}^N$, $N \geq 4$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, and a, f are continuous functions. We assume that Ω, a and f are invariant under the action of a group of orthogonal transformations. We obtain multiplicity results which contain information about the symmetry and symmetry-breaking properties of the solutions, and about their nodal domains. Our results include new multiplicity results for the Brezis-Nirenberg problem $-\Delta u + \lambda u = |u|^{2^*-2}\,u$ in Ω, $u = 0$ on $\partial\Omega$.

1. Introduction

Consider the model problem

$$(\wp_\lambda) \quad \begin{cases} -\Delta u + \lambda u = |u|^{2^*-2}\,u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 4$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent and $\lambda \in \mathbb{R}$.

This problem has been subject of extensive research for more than two decades. It is well known that it does not have a nontrivial solution if Ω is strictly starshaped and $\lambda \geq 0$ [17]. In contrast with this situation Brezis and Nirenberg [4] showed that there is a solution of $(\wp_\lambda)$ if $N \geq 4$ and $\lambda \in (-\lambda_1, 0)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ on Ω. Furthermore, Rey [18] and Lazzo [14] showed that there is an effect of the domain topology on the number of low energy positive solutions of this problem for λ close enough to 0.

Cerami, Solimini and Struwe [6] showed the existence of a sign changing solution if $N \geq 6$ and $\lambda \in (-\lambda_1, 0)$. This solution has precisely two nodal domains. If the domain is invariant with respect to an orthogonal involution (for example, if it is symmetric with respect to the origin) and $N \geq 4$ Castro and Clapp [5] showed that there is an effect of the equivariant topology of the domain on the number of solutions with precisely two nodal domains for λ sufficiently close to 0.

Here we shall consider domains which are invariant under some group of symmetries. Our results will give information not only on the number of solutions, but on their symmetry and nodal properties as well. To illustrate our results we state here the following special case. We write $\mathbb{Z}/2$-cat(Ω) for the equivariant Lusternik-Schnirelmann category and cat(Ω) for the usual one.

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Theorem 1. *Let $N \geq 4$. If Ω is symmetric with respect to the origen, i.e. $\Omega = -\Omega$, then there exists a $\lambda^* < 0$ for which the following holds:*
(a) If $0 \in \Omega$ then, for every $\lambda \in (\lambda^*, 0)$, problem $(\wp_\lambda)$ has at least $\operatorname{cat}(\Omega)$ *positive solutions, one of which is an even function, and* $\mathbb{Z}/2\text{-}\operatorname{cat}(\Omega \smallsetminus \{0\}) \geq N$ *pairs $\pm u$ of odd solutions with precisely two nodal domains.*
(b) If $0 \notin \Omega$ and Ω admits an odd map $\mathbb{S}^{k-1} \to \Omega$ then, for every $\lambda \in (\lambda^*, 0)$, problem $(\wp_\lambda)$ has at least $\operatorname{cat}(\Omega) \geq 2$ *positive solutions which are not even, one even positive solution, and* $\mathbb{Z}/2\text{-}\operatorname{cat}(\Omega) \geq k$ *pairs $\pm u$ of odd solutions with precisely two nodal domains.*

A better result can be obtained if Ω is thin enough (see Theorem 9).

Theorem 1 improves a result of Castro and Clapp. For example, if Ω is symmetric and starshaped with respect to the origin, Theorem 2 in [5] yields only one odd solution with precisely two nodal domains, whereas the result above yields at least N of them. Our methods should allow similar improvements of the results in [11, 12].

For arbitrary domains Devillanova and Solimini [10] showed that, if $N \geq 4$ and $\lambda \in (-\lambda_1, 0)$, problem $(\wp_\lambda)$ has at least $\frac{N}{2} + 1$ solutions with small energy. Similar results for any $\lambda < 0$ were obtained by Clapp and Weth [9]. But these results provide no information on whether the solutions change sign or not.

We shall consider, in fact, a more general problem and arbitrary symmetries. We give now a precise statement of our results.

2. Statement of results

Let Γ be a closed subgroup of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$. Consider the problem

$$(\wp^\Gamma_{a,f}) \qquad \begin{cases} -\Delta u + a(x)u = f(x)\left|u\right|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\gamma x) = u(x) & \forall x \in \Omega,\ \gamma \in \Gamma \end{cases}$$

where Ω is a Γ-invariant bounded smooth domain in $\mathbb{R}^N$, $N \geq 4$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $a, f : \mathbb{R}^N \to \mathbb{R}$ are continuous Γ-invariant functions.

Recall that a subset X of $\mathbb{R}^N$ is Γ-invariant if $\gamma x \in X$ for all $x \in X$, $\gamma \in \Gamma$, and a function $h : X \to \mathbb{R}$ is Γ-invariant if $h(\gamma x) = h(x)$ for all $x \in X$, $\gamma \in \Gamma$. We write $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ for the Γ-orbit of a point $x \in \mathbb{R}^N$ and denote its cardinality by $\#\Gamma x$. We write $X/\Gamma := \{\Gamma x : x \in X\}$ for the Γ-orbit space of X with the quotient topology.

Consider the set

$$M := \left\{ y \in \overline{\Omega} : \frac{\#\Gamma y}{f(y)^{\frac{N-2}{2}}} = \min_{x \in \overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} \right\}. \tag{2.1}$$

We shall assume that a, f satisfy the following.

- (a_1)**:** $\min_{\overline{\Omega}} a > -\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ in Ω.
- (a_2)**:** $a(x) < 0$ for every $x \in M$.
- (f_1)**:** $f(x) > 0$ for every x in $\overline{\Omega}$.
- (f_2)**:** f is locally flat at M, that is, there exist $r > 0$, $\nu > N$ and $A > 0$ such that $\left|f(x) - f(y)\right| \leq A\left|x - y\right|^{\nu}$ if $y \in M$ and $\left|x - y\right| < r$.

Set

$$\|u\|^2 := \int_\Omega |\nabla u|^2, \qquad \|u\|_a^2 := \int_\Omega \left(|\nabla u|^2 + a(x)u^2\right). \tag{2.2}$$

As usual, we denote by S the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \smallsetminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2$, and define

$$\ell_f^\Gamma := \left(\min_{x \in \overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}}\right) S^{\frac{N}{2}}. \tag{2.3}$$

Our multiplicity results will require the following nonexistence assumption.

$(\mathbf{A}_f^\Gamma)$: Problem

$$(\wp_{0,f}^\Gamma) \qquad \begin{cases} -\Delta u = f(x)\,|u|^{2^*-2}\,u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \\ u(\gamma x) = u(x) & \forall x \in \Omega,\ \gamma \in \Gamma \end{cases}$$

does not have a positive solution u which satisfies $\|u\|^2 \leq \ell_f^\Gamma$.

It is well known that this assumption holds for every domain Ω if Γ is the trivial group and f is constant. Here we shall provide another condition for it to hold (see Theorem 7). We write $\operatorname{cat}_X(A)$ for the Lusternik-Schnirelmann category of A in X, that is, for the smallest number of open subsets of A which cover A and are contractible in X. We shall prove the following results.

2.1. **Existence and multiplicity of positive solutions.**

Theorem 2. *If $N \geq 4$, assumptions $(a_1), (a_2), (f_1), (f_2)$ hold, and $\Omega \cap M \neq \emptyset$, then problem $(\wp_{a,f}^\Gamma)$ has at least one positive solution which satisfies $\|u\|_a^2 < \ell_f^\Gamma$.*

For $\delta > 0$ set

$$M_\delta^- := \{y \in M : \operatorname{dist}(y, \partial\Omega) \geq \delta\}, \qquad B_\delta(M) := \{z \in \mathbb{R}^N : \operatorname{dist}(z, M) \leq \delta\}. \tag{2.4}$$

Theorem 3. *Let $N \geq 4$ and assume that $(a_1), (a_2), (f_1), (f_2)$ and (A_f^Γ) hold. Given $\delta, \delta' > 0$, there exists $a_0 \in (-\lambda_1, 0)$ such that, if $\min_{\overline{\Omega}} a \geq a_0$, then problem $(\wp_{a,f}^\Gamma)$ has at least*

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$$

positive solutions which satisfy $\ell_f^\Gamma - \delta' \leq \|u\|_a^2 < \ell_f^\Gamma$.

If Γ is the trivial group, $a \equiv \lambda \in (-\lambda_1, 0)$ and $f \equiv 1$, then $(a_1), (a_2), (f_1), (f_2)$ and (A_f^Γ) clearly hold. Theorem 2 is the celebrated Brezis-Nirenberg theorem [4], and Theorem 3 is due to Rey and Lazzo [18, 14] in this particular case.

2.2. **Existence and multiplicity of nodal solutions.** Assume now that Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2 := \{1, -1\}$ defined on a closed subgroup G of $O(N)$ for which Ω, a and f are G-invariant. A function u which satisfies

$$u(gx) = \tau(g)u(x) \qquad \forall x \in \Omega,\ g \in G,$$

will be called *τ-equivariant.* Every τ-equivariant function is Γ-invariant. Moreover, it satisfies $u(gx) = -u(x)$ for all $x \in \Omega$, $g \in \tau^{-1}(-1)$. Thus, every τ-equivariant nontrivial solution of $(\wp^{\Gamma}_{a,f})$ changes sign.

Definition 1. *A Γ-invariant subset X of $\mathbb{R}^N$ will be called Γ-connected if it cannot be written as the union of two open disjoint Γ-invariant subsets, and a solution u of $(\wp^{\Gamma}_{a,f})$ will be called $(\Gamma, 2)$-nodal if the sets*

$$\{x \in \Omega : u(x) > 0\} \quad \text{ and } \quad \{x \in \Omega : u(x) < 0\}$$

are nonempty and Γ-connected.

Note that a Γ-connected set is not necessarily connected, hence a $(\Gamma, 2)$-nodal solution may have more than two nodal domains.

Theorem 4. *Let $N \geq 4$ and assume that $(a_1), (a_2), (f_1), (f_2)$ hold. If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ defined on a closed subgroup G of $O(N)$ for which Ω, a and f are G-invariant, and $Gx \neq \Gamma x$ for some $x \in \Omega \cap M$, then problem $(\wp^{\Gamma}_{a,f})$ has at least one pair of τ-equivariant $(\Gamma, 2)$-nodal solutions $\pm u$ which satisfy $\|u\|_a^2 < 2\ell_f^{\Gamma}$.*

For each G-invariant subset X of $\mathbb{R}^N$, set

$$X^{\tau} := \{x \in X : Gx = \Gamma x\}.$$

Given $\delta > 0$, set

$$M^{-}_{\tau,\delta} := \{y \in M : \operatorname{dist}(y, \partial\Omega \cup \Omega^{\tau}) \geq \delta\},$$

and let $B_\delta(M)$ be as in (2.4).

Theorem 5. *Let $N \geq 4$ and assume that $(a_1), (a_2), (f_1), (f_2)$ and (A^{Γ}_f) hold. Assume further that Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ defined on a closed subgroup G of $O(N)$ for which Ω, a and f are G-invariant. Given $\delta, \delta' > 0$ there exists $a_0 \in (-\lambda_1, 0)$ such that, if $\min_{\overline{\Omega}} a \geq a_0$, then problem $(\wp^{\Gamma}_{a,f})$ has at least*

$$\operatorname{cat}_{(B_\delta(M) \smallsetminus B_\delta(M)^{\tau})/G}(M^{-}_{\tau,\delta}/G)$$

pairs $\pm u$ of τ-equivariant $(\Gamma, 2)$-nodal solutions which satisfy $2\ell^{\Gamma}_f - \delta' \leq \|u\|_a^2 < 2\ell^{\Gamma}_f$.

Theorem 5 extends and improves Theorem 2 in [5]. Indeed, if $a \equiv \lambda \in (-\lambda_1, 0)$, $f \equiv 1$, and $\tau : G \cong \mathbb{Z}/2$ is an isomorphism, we obtain at least $\operatorname{cat}[(\Omega \smallsetminus \Omega^{\tau})/G] = \mathbb{Z}/2\text{-}\operatorname{cat}(\Omega \smallsetminus \Omega^{\tau})$ pairs $\pm u$ of τ-equivariant 2-nodal solutions, instead of just $\mathbb{Z}/2\text{-}\operatorname{cat}_{\Omega}(\Omega \smallsetminus \Omega^{\tau})$ of them as asserted in [5]. For example, if Ω is symmetric and starshaped with respect to the origin, then $\Omega^{\tau} = \{0\}$ and $\mathbb{Z}/2\text{-}\operatorname{cat}(\Omega \smallsetminus \{0\}) = N$, whereas $\mathbb{Z}/2\text{-}\operatorname{cat}_{\Omega}(\Omega \smallsetminus \{0\}) = 1$. (See Section 7, Definition 4, for the definition of $\mathbb{Z}/2$-cat).

2.3. **Symmetry breaking properties of the solutions.** The following result restricts the symmetries of the solutions provided by Theorem 3.

Theorem 6. *Let* $N \geq 4$ *and assume that* $(a_1), (a_2), (f_1), (f_2)$ *and* $(\mathrm{A}^{\Gamma}_{f})$ *hold. Let* $\widetilde{\Gamma}$ *be a closed subgroup of* $O(N)$ *which contains* Γ*, for which* Ω*,* a *and* f *are* $\widetilde{\Gamma}$*-invariant, and the inequality*

$$\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} < \min_{x\in\overline{\Omega}} \frac{\#\widetilde{\Gamma} x}{f(x)^{\frac{N-2}{2}}} \tag{2.5}$$

holds. Given $\delta, \delta' > 0$ *there exists* $a_0 \in (-\lambda_1, 0)$ *such that, if* $\min_{\overline{\Omega}} a \geq a_0$*, then problem* $(\wp^{\Gamma}_{a,f})$ *has at least*

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M^-_\delta/\Gamma)$$

positive solutions which are not $\widetilde{\Gamma}$*-invariant and satisfy* $\ell^{\Gamma}_{f} - \delta' \leq \|u\|^2_a < \ell^{\Gamma}_{f}$.

A similar result holds for τ-equivariant solutions.

2.4. **A nonexistence result.** We now give some conditions on the domain Ω which guarantee that condition $(\mathrm{A}^{\Gamma}_{f})$ holds for $f \equiv 1$. Let $\varphi : \mathbb{S}^{N-1} \to [1,2]$ be a C^∞-function on the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$, and let $S_\varphi := \{\varphi(z)z : z \in \mathbb{S}^{N-1}\}$. For $\kappa > 0$ consider the annular domain

$$A_{\varphi,\kappa} := \{z + tn_z : z \in S_\varphi,\ 0 < t < \kappa\},$$

where n_z is the outward unit normal to S_φ at z. Using a result of Ben Ayed, El Mehdi y Hammami [2] we shall prove the following.

Theorem 7. *For every* Γ*-invariant* C^∞*-function* $\varphi : \mathbb{S}^{N-1} \to [1,2]$*,* $N \geq 4$*, there exists* $\kappa > 0$ *with the following property: If* $\Omega \subset A_{\varphi,\kappa}$ *and*

$$\min_{x\in\overline{\Omega}} \#\Gamma x = \min_{z\in\mathbb{S}^{N-1}} \#\Gamma z, \tag{2.6}$$

then problem

$$(\wp^{\Gamma}_{0}) \qquad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\gamma x) = u(x) & \forall x \in \Omega,\ \gamma \in \Gamma \end{cases}$$

does not have a nontrivial solution u *which satisfies* $\|u\|^2 \leq \left(\min_{x\in\overline{\Omega}} \#\Gamma x\right) S^{\frac{N}{2}}$.

3. The variational problem

Let $\tau : G \to \mathbb{Z}/2$ be a homomorphism on a closed subgroup G of $O(N)$, and let $\Gamma := \ker \tau$. Consider the problem

$$(\wp^{\tau}_{a,f}) \qquad \begin{cases} -\Delta u + a(x)u = f(x)\,|u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(gx) = \tau(g)u(x) & \forall x \in \Omega,\ g \in G \end{cases}$$

where Ω is a G-invariant bounded smooth domain in $\mathbb{R}^N$, and $a, f : \mathbb{R}^N \to \mathbb{R}$ are G-invariant continuous functions which satisfy (a_1) and (f_1).

If $\tau \equiv 1$ is the trivial homomorphism then problems $(\wp^{\Gamma}_{a,f})$ and $(\wp^{\tau}_{a,f})$ are the same. If τ is an epimorphism, then a solution of $(\wp^{\tau}_{a,f})$ is a solution of $(\wp^{\Gamma}_{a,f})$ with the additional property that $u(gx) = -u(x)$ for all $x \in \Omega$, $g \in \tau^{-1}(-1)$. Thus, every nontrivial solution of $(\wp^{\tau}_{a,f})$ is a sign changing solution of $(\wp^{\Gamma}_{a,f})$ in this case.

τ induces an action of G on $H_0^1(\Omega)$ as follows:

$$(gu)(x) := \tau(g)u(g^{-1}x). \tag{3.1}$$

The fixed point space of this action,

$$\begin{aligned} H_0^1(\Omega)^\tau \quad &:= \{u \in H_0^1(\Omega) : gu = u \ \ \forall g \in G\} \\ &= \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x) \ \ \forall g \in G, \ x \in \Omega\}, \end{aligned}$$

is the space of τ-equivariant functions. The fixed point space of the restriction of this action to Γ is the space

$$H_0^1(\Omega)^\Gamma := \{u \in H_0^1(\Omega) : u(gx) = u(x) \ \ \forall g \in \Gamma, \ x \in \Omega\}$$

of Γ-invariant functions. Note that $H_0^1(\Omega)^\tau \subset H_0^1(\Omega)^\Gamma$. But, whereas $H_0^1(\Omega)^\Gamma$ is always infinite dimensional, $H_0^1(\Omega)^\tau$ might be trivial. For example, if Ω is a ball or an annulus and $\tau : O(N) \to \mathbb{Z}/2$ is the determinant map, then $\ker \tau = SO(N)$, $H_0^1(\Omega)^{SO(N)}$ is the space of radial functions, and $H_0^1(\Omega)^\tau = \{0\}$. As in the Introduction (2.2) we set

$$\|u\|_a^2 := \int_\Omega \left(|\nabla u|^2 + a(x)u^2 \right), \qquad |u|_{f,2^*}^{2^*} := \int_\Omega f(x)\,|u|^{2^*}.$$

Properties (a_1) and (f_1) guarantee that $\|u\|_a$ and $|u|_{f,2^*}$ are equivalent to the usual norms,

$$\|u\|^2 := \int_\Omega |\nabla u|^2, \qquad |u|_{2^*}^{2^*} := \int_\Omega |u|^{2^*},$$

of $H_0^1(\Omega)$ and $L^{2^*}(\Omega)$ respectively. They are G-invariant with respect to the action defined in (3.1). Therefore, the functional

$$E_{a,f}(u) := \frac{1}{2}\|u\|_a^2 - \frac{1}{2^*}|u|_{f,2^*}^{2^*}$$

is G-invariant and, by the principle of symmetric criticality [16], the critical points of its restriction to $H_0^1(\Omega)^\tau$ are the solutions of problem $(\wp_{a,f}^\tau)$. The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}_{a,f}^\tau := \{u \in H_0^1(\Omega)^\tau : u \neq 0, \ \ \|u\|_a^2 = |u|_{f,2^*}^{2^*}\}$$

which is of class C^2 and radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\tau$. Hence, the nontrivial solutions of problem $(\wp_{a,f}^\tau)$ are precisely the critical points of the restriction of $E_{a,f}$ to the Nehari manifold $\mathcal{N}_{a,f}^\tau$. The radial projection is given by

$$\pi_{a,f} : H_0^1(\Omega)^\tau \smallsetminus \{0\} \to \mathcal{N}_{a,f}^\tau, \qquad \pi_{a,f}(u) = \left(\frac{\|u\|_a^2}{|u|_{f,2^*}^{2^*}} \right)^{\frac{N-2}{4}} u. \tag{3.2}$$

Note that

$$E_{a,f}(\pi_{a,f}(u)) = \frac{1}{N}\left(\frac{\|u\|_a^2}{|u|_{f,2^*}^2} \right)^{\frac{N}{2}} \qquad \forall u \in H_0^1(\Omega)^\tau \smallsetminus \{0\}.$$

Set $\mathcal{N}_{a,f}^\Gamma := \mathcal{N}_{a,f}^\tau \cap H_0^1(\Omega)^\Gamma$, and define

$$\mu^\Gamma(a,f) := \inf_{\mathcal{N}_{a,f}^\Gamma} E_{a,f}, \qquad \mu^\tau(a,f) := \inf_{\mathcal{N}_{a,f}^\tau} E_{a,f}.$$

It is easy to see that $\mu^\tau(a,f) \geq \mu^\Gamma(a,f) > 0$. Following Benci and Cerami [3] one proves the following.

Proposition 1. *If* $u \in \mathcal{N}_{a,f}^{\Gamma}$ *is a critical point of* $E_{a,f}$ *such that* $E_{a,f}(u) < 2\mu^{\Gamma}(a,f)$, *then* $u \geq 0$ *or* $u \leq 0$.

Proof. Since $u \in \mathcal{N}_{a,f}^{\Gamma}$ is a critical point of $E_{a,f}$ we have that

$$0 = E_{a,f}'(u)u^{\pm} = \int_{\Omega} \left(\nabla u \nabla u^{\pm} + a(x)uu^{\pm} - |u|^{2^*-2} uu^{\pm} \right) = \left\| u^{\pm} \right\|_a^2 - \left| u^{\pm} \right|_{f,2^*}^{2^*},$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Therefore, if $u^+ \neq 0$ and $u^- \neq 0$, then $u^{\pm} \in \mathcal{N}_{a,f}^{\Gamma}$ and, consequently,

$$E_{a,f}(u) = E_{a,f}(u^+) + E_{a,f}(u^-) \geq 2\mu^{\Gamma}(a,f).$$

This is a contradiction. We conclude that, either $u^+ = 0$ or $u^- = 0$. □

Recall the definition of a $(\Gamma, 2)$-nodal function given in the Introduction (Definition 1). The following holds.

Proposition 2. *Let* $\tau : G \to \mathbb{Z}/2$ *be an epimorphism. If* $u \in \mathcal{N}_{a,f}^{\tau}$ *is a critical point of* $E_{a,f}$ *such that* $E_{a,f}(u) < 2\mu^{\tau}(a,f)$ *then* u *is* $(\Gamma, 2)$*-nodal.*

Proof. Assume there are two disjoint open Γ-invariant sets U_1 and U_2 such that $\{x \in \Omega : u(x) > 0\} = U_1 \cup U_2$. Fix $g \in G$ such that $\tau(g) = -1$, and set

$$u_i(x) := \begin{cases} u(x) & \text{if } x \in U_i \cup g(U_i) \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_i \in H_0^1(\Omega)^{\tau}$, $i = 1, 2$. Since u is a critical point of $E_{a,f}$, we have

$$0 = E_{a,f}'(u)u_i = \int_{\Omega} \left(\nabla u \nabla u_i + a(x)uu_i - |u|^{2^*-2} uu_i \right) = \|u_i\|_a^2 - |u_i|_{f,2^*}^{2^*}.$$

Therefore, if $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$, then $u_1, u_2 \in \mathcal{N}_{a,f}^{\tau}$ and

$$E_{a,f}(u) = E_{a,f}(u_1) + E_{a,f}(u_2) \geq 2\mu^{\tau}(a,f),$$

contradicting our assumption. We conclude that $\{x \in \Omega : u(x) > 0\}$ and $\{x \in \Omega : u(x) < 0\}$ are Γ-connected. □

As usual, we denote by λ_1 be the first Dirichlet eigenvalue of $-\Delta$ in Ω.

Lemma 1. *For every* $-\lambda_1 < a_0 \leq \min\{0, a(x) : x \in \overline{\Omega}\}$ *the following holds:*

$$E_{0,f}(\pi_{0,f}(u)) \leq \left(\frac{\lambda_1}{\lambda_1 + a_0} \right)^{\frac{N}{2}} E_{a,f}(u) \qquad \forall u \in H_0^1(\Omega)^{\tau} \smallsetminus \{0\}.$$

In particular,

$$\mu^{\tau}(0,f) \leq \left(\frac{\lambda_1}{\lambda_1 + a_0} \right)^{\frac{N}{2}} \mu^{\tau}(a,f).$$

Proof. Let $u \in H_0^1(\Omega)$. Since $-\lambda_1 < a_0 \leq \min\{0, a(x) : x \in \overline{\Omega}\}$, we have that

$$\|u\|_a^2 \geq \int (|\nabla u|^2 + a_0 u^2) \geq \left(\frac{\lambda_1 + a_0}{\lambda_1} \right) \|u\|^2.$$

Therefore, if $u \neq 0$,

$$E_{0,f}(\pi_{0,f}(u)) = \frac{1}{N} \left(\frac{\|u\|^N}{|u|_{f,2^*}^N} \right) \leq \left(\frac{\lambda_1}{\lambda_1 + a_0} \right)^{\frac{N}{2}} \frac{1}{N} \left(\frac{\|u\|_a^N}{|u|_{f,2^*}^N} \right) = \left(\frac{\lambda_1}{\lambda_1 + a_0} \right)^{\frac{N}{2}} E_{a,f}(u),$$

as claimed. $\square$

We write $G_y := \{g \in G : gy = y\}$ for the isotropy subgroup of y. Recall that the G-orbit Gy is G-homeomorphic to the homogeneous space G/G_y. Let M be the set defined in (2.1) and ℓ_f^Γ be the number defined in (2.3). We have the following estimates.

Lemma 2. *(a) If $\Omega \cap M \neq \emptyset$, then $\mu^\Gamma(0,f) \leq \frac{1}{N}\ell_f^\Gamma$.*
(b) If there exists $y \in \Omega \cap M$ with $\Gamma y \neq Gy$, then $\mu^\tau(0,f) \leq \frac{2}{N}\ell_f^\Gamma$.

Proof. If $\ell_f^\Gamma = \infty$ there is nothing to prove. So assume $\ell_f^\Gamma < \infty$ and let $y \in \Omega \cap M$. If τ is an epimorphism, we also assume that $\Gamma y \neq Gy$. Let $s > 0$ be such that $|y - gy| > 2s$ for every $g \in G$ with $y \neq gy$, and set $f_s := \min\{f(x) : \operatorname{dist}(x, Gy) \leq s\}$. Fix $-\lambda_1 < \lambda < 0$ and let $u_{\lambda,s}$ be a positive least energy solution to the problem

$$-\Delta u + \lambda u = |u|^{2^*-2} u \ \text{ in } B_s(0), \qquad u = 0 \ \text{ on } \partial B_s(0),$$

where $B_s(0) := \{x \in \mathbb{R}^N : |x| < s\}$, which Brezis and Nirenberg showed to exist [4]. Define

$$u_\lambda(x) := \sum_{[g]\in G/G_y} f_s^{\frac{2-N}{4}} \tau(g) u_{\lambda,s}(x - gy).$$

Our assumptions on y guarantee that u_λ is well defined. Since $\operatorname{supp}(u_\lambda) \subset \{x \in \mathbb{R}^N : \operatorname{dist}(x, Gy) \leq s\}$, one has that

$$\|u_\lambda\|_\lambda^2 = |u_\lambda|_{f_s,2^*}^{2^*} \leq |u_\lambda|_{f,2^*}^{2^*}.$$

Therefore, using the previous lemma, we obtain

$$\begin{aligned}
\mu^\tau(0,f) &\leq E_{0,f}(\pi_{0,f}(u_\lambda)) \leq \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} E_{\lambda,f}(u_\lambda) \\
&\leq \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} E_{\lambda,f_s}(u_\lambda) = \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} \frac{1}{N}\|u_\lambda\|_\lambda^2 \\
&= \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} \left(\frac{\#Gy}{f_s^{\frac{N-2}{2}}}\right) \frac{1}{N}\|u_{\lambda,s}\|_\lambda^2 \\
&< \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} \left(\frac{\#Gy}{f_s^{\frac{N-2}{2}}}\right) \frac{1}{N} S^{N/2}.
\end{aligned}$$

Letting $s \to 0$ yields

$$\mu^\tau(0,f) \leq \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} \left(\frac{\#Gy}{f(y)^{\frac{N-2}{2}}}\right) \frac{1}{N} S^{N/2} = \left(\frac{\lambda_1}{\lambda_1+\lambda}\right)^{\frac{N}{2}} \frac{\#(G/\Gamma)}{N}\ell_f^\Gamma,$$

and letting $\lambda \to 0$ we obtain $\mu^\tau(0,f) \leq \frac{\#(G/\Gamma)}{N}\ell_f^\Gamma$, as claimed. $\square$

4. Estimates for $\mu^\Gamma(a,f)$ and $\mu^\tau(a,f)$

We assume throughout this section that $(a_1), (a_2), (f_1), (f_2)$ hold. We also assume that Ω has a finite orbit. Hence, all Γ-orbits in M are finite, where M is the set defined in (2.1).

For $\varepsilon > 0$, $y \in \mathbb{R}^N$, consider the Aubin-Talenti instantons [1, 21]

$$U_{\varepsilon,y}(x) = a_N \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2}{2}}, \quad a_N := [N(N-2)]^{\frac{N-2}{4}}, \tag{4.1}$$

They are the positive solutions to the limit problem

$$(\wp_\infty) \qquad -\Delta u = |u|^{2^*-2} u, \quad u \in D^{1,2}(\mathbb{R}^N)$$

and satisfy $\int_{\mathbb{R}^N} |\nabla U_{\varepsilon,y}|^2 = S^{N/2} = \int_{\mathbb{R}^N} |U_{\varepsilon,y}|^{2^*}$. Assumption (a_2) allows us to choose $s > 0$ such that

$$\max_{B_s(M)} a < 0, \tag{4.2}$$

where $B_s(M) := \{y \in \mathbb{R}^N : \operatorname{dist}(y, M) \le s\}$, as defined in (2.4). For such an s we consider

$$M_s^- := \{y \in M : \ \operatorname{dist}(y, \partial\Omega) \ge s\},$$
$$\rho_s^\Gamma := \inf\{r, \frac{s}{2}, \frac{|\gamma y - y|}{4} : y \in M, \ \gamma \in \Gamma, \ \gamma y \ne y\},$$

where $r > 0$ is the constant appearing in assumption (f_2). Since we are assuming all orbits in M to be finite, our definition of M yields that $\rho_s^\Gamma > 0$. Fix $0 < \rho \le \rho_s^\Gamma$ and a radially symmetric cut-off function $\varphi \in C^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 2$. For $z \in \mathbb{R}^N$, set $\varphi_z(x) := \varphi\left(\rho^{-1}(x-z)\right)$.
For each $y \in M_s^-$ and $\varepsilon > 0$, consider the multibump function

$$w_{\varepsilon,y}^\Gamma(x) := \sum_{[\gamma] \in \Gamma/\Gamma_y} f(y)^{\frac{2-N}{4}} \varphi_{\gamma y}(x) U_{\varepsilon,\gamma y}(x). \tag{4.3}$$

Note that $w_{\varepsilon,y}^\Gamma$ is Γ-invariant and $\operatorname{supp}(w_{\varepsilon,y}^\Gamma) \subset \overline{\Omega}$. Thus, $w_{\varepsilon,y}^\Gamma \in H_0^1(\Omega)^\Gamma$. Let ℓ_f^Γ be the number defined in (2.3).

Lemma 3. *For $y \in M_s^-$ and $\varepsilon > 0$ small enough, the function $w_{\varepsilon,y}^\Gamma$ satisfies*

$$\left\| w_{\varepsilon,y}^\Gamma \right\|^2 = \ell_f^\Gamma + O(\varepsilon^{N-2}), \tag{4.4}$$

$$\left| w_{\varepsilon,y}^\Gamma \right|_{f,2^*}^{2^*} = \ell_f^\Gamma + O(\varepsilon^N), \tag{4.5}$$

$$\int_\Omega a(x) \left(w_{\varepsilon,y}^\Gamma \right)^2 \le \begin{cases} -c\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N \ge 5, \\ -c\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \end{cases} \tag{4.6}$$

for some positive constant c.

Proof. Estimates (4.4) and (4.6) follow immediately from the well known Brezis-Nirenberg estimates [4] using (4.2). We prove (4.5). Let $y \in M_s^-$ and $\varepsilon > 0$. Then,

$$\begin{aligned} \left| w_{\varepsilon,y}^\Gamma \right|_{f,2^*}^{2^*} &= \frac{\#\Gamma y}{f(y)^{\frac{N-2}{2}}} \int \frac{f(x) \left| \varphi_y U_{\varepsilon,y} \right|^{2^*}}{f(y)} dx \\ &= \frac{\#\Gamma y}{f(y)^{\frac{N-2}{2}}} \left(\int \left| U_{\varepsilon,y} \right|^{2^*} + \int \frac{f(x)\varphi_y^{2^*} - f(y)}{f(y)} \left| U_{\varepsilon,y} \right|^{2^*} dx \right) \\ &= \left(\min_{x \in \overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} \right) \left(S^{\frac{N}{2}} + \varepsilon^N a_N^{2^*} \int \frac{f(x)\varphi_y^{2^*} - f(y)}{f(y)(\varepsilon^2 + |x-y|^2)^N} dx \right). \end{aligned}$$

Assumption (f_2) yields

$$\int \frac{f(x)\varphi_y^{2^*} - f(y)}{f(y)(\varepsilon^2 + |x-y|^2)^N} dx \leq c \int_{|x|\leq\rho} |x|^{\nu-2N}\, dx + c\int_{|x|\geq\rho} |x|^{-2N}\, dx < \infty.$$

This finishes the proof. □

As before, let $\pi_{a,f}$ denote the radial projection onto the Nehari manifold (3.2).

Proposition 3. *Given $s > 0$ such that $\max_{B_s(M)} a < 0$, there exists $\varepsilon_s > 0$ with the property that, for each $\varepsilon \in (0, \varepsilon_s)$, there exists θ_ε which satisfies*

$$E_{a,f}(\pi_{a,f}(w^\Gamma_{\varepsilon,y})) \leq \theta_\varepsilon < \frac{1}{N}\ell^\Gamma_f$$

for every $y \in M^-_s$. Hence, if $M^-_s \neq \emptyset$, then $\mu^\Gamma(a,f) < \frac{1}{N}\ell^\Gamma_f$.

Proof. Estimates (4.4) and (4.5) in Lemma 3 imply that there are constants $c_1, c_2 > 0$ such that

$$\begin{aligned} E_{a,f}(\pi_{a,f}(w^\Gamma_{\varepsilon,y})) &= \frac{1}{N}\left(\frac{\left\|w^\Gamma_{\varepsilon,y}\right\|^2_a}{\left|w^\Gamma_{\varepsilon,y}\right|^2_{f,2^*}}\right)^{\frac{N}{2}} \\ &\leq \frac{1}{N}\left[(\ell^\Gamma_f)^{\frac{2}{N}} + c_1\varepsilon^{N-2} + c_2\int\limits_\Omega a(x)\left(w^\Gamma_{\varepsilon,y}\right)^2\right]^{\frac{N}{2}} =: \theta_\varepsilon, \end{aligned}$$

for all $y \in M^-_s$ and ε small enough. Estimate (4.6) yields

$$c_1\varepsilon^{N-2} + c_2\int\limits_\Omega a(x)\left(w^\Gamma_{\varepsilon,y}\right)^2 < 0$$

for all $y \in M^-_s$ and ε small enough. Hence,

$$\mu^\Gamma(a,f) \leq E_{a,f}(\pi_{a,f}(w^\Gamma_{\varepsilon,y})) \leq \theta_\varepsilon < \frac{1}{N}\ell^\Gamma_f,$$

as claimed. □

Let $\tau : G \to \mathbb{Z}/2$ be an epimorphism with $\Gamma = \ker\tau$, and assume that Ω, a and f are G-invariant. We shall now obtain some estimates for $\mu^\tau(a,f)$. Set $\Omega^\tau := \{y \in \Omega : \Gamma y = Gy\}$ and define

$$M^-_{\tau,s} := \{y \in M : \operatorname{dist}(y, \partial\Omega \cup \Omega^\tau) \geq s\},$$

$$\rho^\tau_s := \inf\{\rho^\Gamma_s, \frac{|gy-y|}{4} : y \in M^-_{\tau,s},\ g \in G,\ gy \neq y\},$$

with s and ρ^Γ_s as before. Then $\rho^\tau_s > 0$, and we may choose ρ in the definition (4.3) of $w^\Gamma_{\varepsilon,y}$ to satisfy $0 < \rho \leq \rho^\tau_s$. Let $g_\tau \in G$ be such that $\tau(g_\tau) = -1$. For each $y \in M^-_{\tau,s}$, $\varepsilon > 0$, we define

$$w^\tau_{\varepsilon,y} := w^\Gamma_{\varepsilon,y} - w^\Gamma_{\varepsilon,g_\tau y}.$$

Since $w^\Gamma_{\varepsilon,y}$ depends only on Γy and not on y itself, $w^\tau_{\varepsilon,y}$ does not depend on our choice of g_τ. Moreover, our choice of ρ guarantees that the supports of $w^\Gamma_{\varepsilon,y}$ and $w^\Gamma_{\varepsilon,g_\tau y}$ have disjoint interiors. Hence, $w^\tau_{\varepsilon,y} \in H^1_0(\Omega)^\tau \smallsetminus \{0\}$ and

$$\left(w^\tau_{\varepsilon,y}\right)^+ = w^\Gamma_{\varepsilon,y}, \qquad \left(w^\tau_{\varepsilon,y}\right)^- = -w^\Gamma_{\varepsilon,g_\tau y}. \tag{4.7}$$

An immediate consequence of Proposition 3 is the following.

Corollary 1. *Given* $s > 0$ *such that* $\max_{B_s(M)} a < 0$, *there exists* $\varepsilon_s > 0$ *such that, for each* $\varepsilon \in (0, \varepsilon_s)$, *there exists* θ_ε *which satisfies*

$$E_{a,f}(\pi_{a,f}(w^\tau_{\varepsilon,y})) \leq 2\theta_\varepsilon < \frac{2}{N}\ell^\Gamma_f,$$

for every $y \in M^-_{\tau,s}$. *Hence, if* $M^-_{\tau,s} \neq \emptyset$, *then* $\mu^\tau(a,f) < \frac{2}{N}\ell^\Gamma_f$.

Proof. It follows from (4.7) that $\left\| w^\tau_{\varepsilon,y} \right\|^2_a = 2\left\| w^\Gamma_{\varepsilon,y} \right\|^2_a$ and $\left| w^\tau_{\varepsilon,y} \right|^{2^*}_{f,2^*} = 2\left| w^\Gamma_{\varepsilon,y} \right|^{2^*}_{f,2^*}$. Hence,

$$E_{a,f}(\pi_{a,f}(w^\tau_{\varepsilon,y})) = 2E_{a,f}(\pi_{a,f}(w^\Gamma_{\varepsilon,y}))$$

and our claim follows with θ_ε as in Proposition 3. □

For every $s > 0$ such that $\max_{B_s(M)} a < 0$, and every $\varepsilon > 0$, we have maps

$$\alpha^\Gamma_s : M^-_s/\Gamma \to \mathcal{N}^\Gamma_{a,f}, \qquad \alpha^\Gamma_s(\Gamma y) = \pi_{a,f}(w^\Gamma_{\varepsilon,y}), \tag{4.8}$$

$$\alpha^\tau_s : M^-_{\tau,s}/\Gamma \to \mathcal{N}^\tau_{a,f}, \qquad \alpha^\tau_s(\Gamma y) = \pi_{a,f}(w^\tau_{\varepsilon,y}), \tag{4.9}$$

defined on the Γ-orbit spaces

$$M^-_s/\Gamma = \{\Gamma y : y \in M^-_s\}, \qquad M^-_{\tau,s}/\Gamma = \{\Gamma y : y \in M^-_{\tau,s}\}$$

of M^-_s and $M^-_{\tau,s}$ respectively.

If $\tau : G \to \mathbb{Z}/2$ is an epimorphism, then $G/\Gamma \cong \mathbb{Z}/2$. The action of G on $\mathbb{R}^N$ induces an action of G/Γ on its Γ-orbit space $\mathbb{R}^N/\Gamma$ in the obvious way. The fixed point set of this action is the set $\{y \in \mathbb{R}^N : \Gamma y = Gy\}/\Gamma$. Since $M^-_{\tau,s}$ does not intersect this set, we have that $G/\Gamma \cong \mathbb{Z}/2$ acts freely on $M^-_{\tau,s}/\Gamma$. The map α^τ_s is $\mathbb{Z}/2$-equivariant, that is,

$$\alpha^\tau_s(\Gamma(gy)) = \tau(g)\alpha^\tau_s(\Gamma y), \qquad \forall y \in M^-_{\tau,s},\ g \in G. \tag{4.10}$$

5. A COMPACTNESS RESULT

Let $\tau : G \to \mathbb{Z}/2$ be a homomorphism with $\Gamma = \ker \tau$. Assume that Ω, a and f are G-invariant and that (a_1) and (f_1) hold. A sequence (u_n) in $H^1_0(\Omega)$ which satisfies

$$u_n \in H^1_0(\Omega)^\tau, \qquad E_{a,f}(u_n) \to c, \qquad \nabla E_{a,f}(u_n) \to 0.$$

will be called a τ-*equivariant PS-sequence* for $E_{a,f}$ at c. If $\tau \equiv 1$ we shall call it a Γ-*invariant PS-sequence.* And we shall say that $E_{a,f}$ satisfies $(PS)^\tau_c$ if every τ-equivariant PS-sequence for $E_{a,f}$ at c has a convergent subsequence. If $\tau \equiv 1$ we say that it satisfies $(PS)^\Gamma_c$.

A complete description of all PS-sequences for $E_{\lambda,1}$ was given by Struwe [19, 20]. Γ-invariant PS-sequences have been described in [7]. We now describe the τ-equivariant ones. Let

$$E_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}, \qquad E_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}$$

be the energy functional for problem $(\wp_\infty)$. The following holds.

Theorem 8. *Let* (u_n) *be a* τ*-equivariant PS-sequence for* $E_{a,f}$ *at* c*. Then, up to a subsequence, there exists a solution* u *of problem* $(\wp_{a,f}^{\tau})$*, and, for some integer* $m \geq 0$*, there exist* m *closed subgroups* $G_1, ..., G_m$ *of finite index in* G*,* m *sequences* $(y_{1,n}), ..., (y_{m,n})$ *in* Ω*,* m *sequences* $(\varepsilon_{1,n}), ..., (\varepsilon_{m,n})$ *in* $(0,\infty)$*, and* m *nontrivial solutions* $\widehat{u}_1, ..., \widehat{u}_m$ *of the limit problem* $(\wp_{\infty})$*, with the following properties:*
(i) $G_{y_{i,n}} = G_i$ *for all* $n \geq 1$*, and* $y_{i,n} \to y_i$ *as* $n \to \infty$*, for each* $i = 1, ..., m$*,*
(ii) $\varepsilon_{i,n}^{-1}\operatorname{dist}(y_{i,n}, \partial\Omega) \to \infty$ *and* $\varepsilon_{i,n}^{-1}|gy_{i,n} - g'y_{i,n}| \to \infty$ *as* $n \to \infty$ *for all* $[g] \neq [g'] \in G/G_i$*,* $i = 1, ..., m$*,*
(iii) $\widehat{u}_i(gz) = \tau(g)\widehat{u}_i(z)$ *for all* $z \in \mathbb{R}^N$*,* $g \in G_i$*,* $i = 1, ..., m$*,*
(iv) $||u_n - u - \sum_{i=1}^{m} \sum_{[g]\in G/G_i} f(y_i)^{\frac{2-N}{4}} \varepsilon_{i,n}^{\frac{2-N}{2}} \tau(g)\widehat{u}_i(g^{-1}\varepsilon_{i,n}^{-1}(\cdot - gy_{i,n}))|| \to 0$ *in* $D^{1,2}(\mathbb{R}^N)$ *as* $n \to \infty$*,*
(v) $E_{a,f}(u) + \sum_{i=1}^{m} \left(\frac{\#(G/G_i)}{f(y_i)^{\frac{N-2}{2}}} \right) E_{\infty}(\widehat{u}_i) = c.$

The proof is completely analogous to the one given in [7] for $\tau \equiv 1$. Here we shall only need the following consequence of this theorem. Recall the definitions of M and ℓ_f^{Γ} given in (2.1) and (2.3), and let U be the instanton $U(x) = a_N(1 + |x|^2)^{(2-N)/2}$.

Corollary 2. *Let* (u_n) *be a* τ*-equivariant PS-sequence for* $E_{a,f}$ *at* c*. For some subsequence of* (u_n) *the following holds:*
(a) If $c < \frac{\#(G/\Gamma)}{N}\ell_f^{\Gamma}$ *then* (u_n) *converges in* $H_0^1(\Omega)^{\tau}$*.*
(b) If $c = \frac{\#(G/\Gamma)}{N}\ell_f^{\Gamma}$ *then, either* (u_n) *converges in* $H_0^1(\Omega)^{\tau}$*, or there exist sequences* (y_n) *in* Ω *and* (ε_n) *in* $(0,\infty)$*, and a point* $y_0 \in M$*, with the following properties:*
  (b.1) $y_n \to y_0$ *as* $n \to \infty$*, and* $G_{y_n} = \Gamma_{y_n} = \Gamma_{y_0}$ *for all* $n \geq 1$*,*
  (b.2) $\varepsilon_n^{-1}\operatorname{dist}(y_n, \partial\Omega) \to \infty$*,* $\varepsilon_n^{-1}|gy_n - g'y_n| \to \infty$ *if* $[g] \neq [g'] \in G/\Gamma_{y_0}$*,*
  (b.3) $||u_n - \sum_{[g]\in G/\Gamma_{y_0}} f(y_0)^{\frac{2-N}{4}}\tau(g)\varepsilon_n^{\frac{2-N}{2}}\widehat{u}(\varepsilon_n^{-1}(\cdot - gy_n))|| \to 0$*, with* $\widehat{u} = \pm U$*.*

Proof. Assume that no subsequence of (u_n) converges and that

$$c \leq \#(G/\Gamma)\frac{1}{N}\ell_f^{\Gamma}. \tag{5.1}$$

Then Theorem 8 implies that there exist a closed subgroup G_0 of finite order in G, sequences (y_n) in Ω and (ε_n) in $(0,\infty)$, and a nontrivial solution $\widehat{u}$ of problem $(\wp_{\infty})$ such that $y_n \to y_0$ in $\overline{\Omega}$, $G_{y_n} = G_0$, $\varepsilon_n^{-1}dist(y_n, \partial\Omega) \to \infty$, $\varepsilon_n^{-1}|gy_n - g'y_n| \to \infty$ if $[g] \neq [g'] \in G/G_0$, $\widehat{u}$ is $\tau|_{G_0}$-equivariant, and

$$c \geq \frac{\#Gy_n}{f(y_0)^{\frac{N-2}{2}}}E_{\infty}(\widehat{u}). \tag{5.2}$$

If $\tau|_{G_0}\colon G_0 \to \mathbb{Z}/2$ is an epimorphism then $\widehat{u}$ is sign changing. Since $\widehat{u} \neq 0$, this implies that $E_{\infty}(\widehat{u}) > \frac{2}{N}S^{N/2}$, and (5.2) yields

$$c \geq \frac{\#Gy_0}{f(y_0)^{\frac{N-2}{2}}}E_{\infty}(\widehat{u}) \geq \frac{\#\Gamma y_0}{f(y_0)^{\frac{N-2}{2}}}E_{\infty}(\widehat{u}) > \frac{2}{N}\ell_f^{\Gamma},$$

contradicting (5.1). Consequently, $G_0 = G_{y_n} \subset \Gamma$. It follows from (5.2) that

$$c \geq \#(G/\Gamma)\left(\frac{\#\Gamma y_n}{f(y_0)^{\frac{N-2}{2}}}\right)E_{\infty}(\widehat{u}) \geq \#(G/\Gamma)\left(\frac{\#\Gamma y_0}{f(y_0)^{\frac{N-2}{2}}}\right)\frac{1}{N}S^{N/2} \geq \#(G/\Gamma)\frac{1}{N}\ell_f^{\Gamma}.$$

Assumption (5.1) implies that $y_0 \in M$, $\Gamma_{y_n} = \Gamma_{y_0} = G_0$, and $E_\infty(\widehat{u}) = \frac{1}{N}S^{\frac{N}{2}}$. Replacing the sequence (ε_n) by a positive multiple of it, if necessary, property *(iv)* of Theorem 8 gives

$$u_n = \sum_{[g]\in G/G_0} f(y_0)^{\frac{2-N}{4}}\tau(g)\varepsilon_n^{\frac{2-N}{2}}\widehat{u}(\varepsilon_n^{-1}(\cdot - gy_n)) + o(1)$$

with $\widehat{u} = \pm U$, as claimed. □

Corollary 2*(a)* implies, in particular, that $\mu^\tau(a,f)$ is achieved if $\mu^\tau(a,f) < \frac{\#(G/\Gamma)}{N}\ell_f^\Gamma$. This was proved by P.L. Lions [15] for $\tau \equiv 1$, and by Hebey and Vaugon [13] when τ is an epimorphism. We use this fact to prove Theorems 2 and 4.

Proof of Theorem 2. Proposition 3 gives $\mu^\Gamma(a,f) < \frac{1}{N}\ell_f^\Gamma$. Hence there exists $u \in \mathcal{N}_{a,f}^\Gamma$ with $E_{a,f}(u) = \mu^\Gamma(a,f)$. Proposition 1 asserts that either $u \geq 0$ or $-u \geq 0$. □

Proof of Theorem 4. Since $Gx \neq \Gamma x$ for some $x \in \Omega \cap M$, the set $M_{\tau,s}^- \neq \emptyset$ for $s > 0$ small enough. Corollary 1 gives $\mu^\tau(a,f) < \frac{2}{N}\ell_f^\Gamma$. Hence there exists $u \in \mathcal{N}_{a,f}^\tau$ with $E_{a,f}(u) = \mu^\tau(a,f)$. Proposition 2 asserts that u is $(\Gamma,2)$-nodal. □

6. A local baryorbit map

Throughout this section we shall assume that condition (A_f^Γ) holds.

For every $y \in \mathbb{R}^N$, $\gamma \in \Gamma$, the isotropy subgroups satisfy $\Gamma_{\gamma y} = \gamma\Gamma_y\gamma^{-1}$. Therefore, the set of isotropy subgroups of a Γ-invariant subset consists of complete conjugacy classes. We choose subgroups $\Gamma_1, ..., \Gamma_m$ of Γ, one in each conjugacy class of an isotropy subgroup of M. Set

$$M_{\Gamma_i} := \{y \in M : \Gamma_y = \Gamma_i\}.$$

Then M is the union of closed disjoint subsets $M = \Gamma M_{\Gamma_1} \cup \Gamma M_{\Gamma_2} \cup \cdots \cup \Gamma M_{\Gamma_m}$, where $\Gamma M_{\Gamma_i} := \{\gamma y : \gamma \in \Gamma,\, y \in M_{\Gamma_i}\}$, and f is constant on each ΓM_{Γ_i}.

To simplify notation we write

$$V^i = (\mathbb{R}^N)^{\Gamma_i} := \{z \in \mathbb{R}^N : \gamma z = z\ \forall \gamma \in \Gamma_i\}$$

for the subspace of Γ_i-fixed points of $\mathbb{R}^N$. Fix $\delta_0 > 0$ such that

$$|y - \gamma y| \geqq \delta_0 \quad \text{if } \gamma y \neq y \in M, \tag{6.1}$$

$$\mathrm{dist}(\Gamma M_{\Gamma_i}, \Gamma M_{\Gamma_j}) \geqq \delta_0 \quad \text{if } i \neq j \in \{1, ..., m\}, \tag{6.2}$$

and such that the isotropy subgroup of each point in $(M_{\Gamma_i})_{\delta_0} := \{z \in V^i : \mathrm{dist}(z, M_{\Gamma_i}) \leq \delta_0\}$ is precisely Γ_i. Define

$$W_{\varepsilon,z} := \sum_{[g]\in\Gamma/\Gamma_i} f_i^{\frac{2-N}{4}} U_{\varepsilon,gz} \quad \text{if } z \in (M_{\Gamma_i})_{\delta_0},$$

where $U_{\varepsilon,y}$ is the instanton (4.1) and $f_i := f(\Gamma M_{\Gamma_i}) \in \mathbb{R}$. Finally, for each $\delta \in (0,\delta_0)$, set $(M_{\Gamma_i})_\delta := \{z \in V^i : \mathrm{dist}(z, M_{\Gamma_i}) \leq \delta\}$, $M_\delta := (M_{\Gamma_1})_\delta \cup \cdots \cup (M_{\Gamma_m})_\delta$,

$$\Theta_\delta := \{\pm W_{\varepsilon,z} : \varepsilon \in (0,\delta],\ z \in M_\delta\}, \qquad \Theta_* := \Theta_{\delta_0}.$$

As usual, we set $E_{0,f}^\eta := \{u \in H_0^1(\Omega) : E_{0,f}(u) \leq \eta\}$. The following proposition will be proved in the Appendix.

Proposition 4. *Assume that condition* (A_f^Γ) *holds. Let* $\delta \in (0, \delta_0)$. *There exists* $\eta > \frac{1}{N}\ell_f^\Gamma$ *with the following properties: For each* $u \in \mathcal{N}_{0,f}^\Gamma \cap E_{0,f}^\eta$ *the inequality*

$$\inf_{W \in \Theta_*} \|u - W\| < \sqrt{\frac{1}{2}\ell_f^\Gamma},$$

holds, and there exist precisely one $\nu \in \{-1, 1\}$, *one* $\varepsilon \in (0, \delta_0)$ *and one* Γ-*orbit* Γz, $z \in M_{\delta_0}$, *such that*

$$\|u - \nu W_{\varepsilon,z}\| = \inf_{W \in \Theta_*} \|u - W\|.$$

Moreover, $\varepsilon \in (0, \delta)$ *and* $z \in M_\delta$.

Fix $\delta \in (0, \delta_0)$ and choose $\eta > \frac{1}{N}\ell_f^\Gamma$ as in Proposition 4. Set $B_\delta(M) := \{z \in \mathbb{R}^N : \operatorname{dist}(z, M) \leq \delta\}$.

Definition 2. *The* Γ-*bariorbit map* $\beta^\Gamma : \mathcal{N}_{0,f}^\Gamma \cap E_{0,f}^\eta \to B_\delta(M)/\Gamma$ *is defined as follows:*

$$\beta^\Gamma(u) = \Gamma z \overset{\text{def}}{\iff} \|u \pm W_{\varepsilon,z}\| = \min_{W \in \Theta_*} \|u - W\|.$$

This map is continuous and $\mathbb{Z}/2$-invariant, that is, $\beta^\Gamma(u) = \beta^\Gamma(-u)$.

If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ and $u \in \mathcal{N}_{0,f}^\tau$, then $u^+(gx) = -u^-(x)$ for every $g \in \tau^{-1}(-1)$. Hence $\|u^+\| = \|u^-\|$, $|u^+|_{f,2^*} = |u^-|_{f,2^*}$, $u^+, u^- \in \mathcal{N}_{0,f}^\Gamma$, and $E_{0,f}(u) = 2E_{0,f}(u^\pm)$. The following holds.

Lemma 4. $\beta^\Gamma(u^+) \neq \beta^\Gamma(u^-)$ *for every* $u \in \mathcal{N}_{0,f}^\tau \cap E_{0,f}^{2\eta}$.

Proof. For every $g \in \tau^{-1}(-1)$ and $u \in \mathcal{N}_{0,f}^\tau$, one has that

$$\left\|u^+ - \nu W_{\varepsilon,z}\right\| = \min_{W \in \Theta_*} \left\|u^+ - W\right\| \iff \left\|u^- + \nu W_{\varepsilon,gz}\right\| = \min_{W \in \Theta_*} \left\|u^- - W\right\|,$$

that is, $\beta^\Gamma(u^+) = \Gamma z$ if and only if $\beta^\Gamma(u^-) = \Gamma(gz)$ for each $g \in \tau^{-1}(-1)$. Assume that $\beta^\Gamma(u^+) = \beta^\Gamma(u^-)$. Then $W_{\varepsilon,gz} = W_{\varepsilon,z}$. By Proposition 4, we have that

$$\begin{aligned} \|u\| &= \left\|u^+ - \nu W_{\varepsilon,z} + u^- + \nu W_{\varepsilon,z}\right\| \leq \left\|u^+ - \nu W_{\varepsilon,z}\right\| + \left\|u^- + \nu W_{\varepsilon,z}\right\| \\ &= \min_{W \in \Theta_*} \left\|u^+ - W\right\| + \min_{W \in \Theta_*} \left\|u^- - W\right\| < (2\ell_f^\Gamma)^{1/2}. \end{aligned}$$

Since (A_f^Γ) holds, $E_{0,f}(u^\pm) \geq \frac{1}{N}\ell_f^\Gamma$. Hence,

$$\frac{2}{N}\ell_f^\Gamma \leq E_{0,f}(u^+) + E_{0,f}(u^-) = E_{0,f}(u) = \frac{1}{N}\|u\|^2 < \frac{2}{N}\ell_f^\Gamma.$$

This is a contradiction. We conclude that $\beta^\Gamma(u^+) \neq \beta^\Gamma(u^-)$. ☐

Set $B_\delta(M)^\tau := \{z \in B_\delta(M) : Gz = \Gamma z\}$.

Definition 3. *The* τ-*bariorbit map is defined as follows:*

$$\beta^\tau : \mathcal{N}_{0,f}^\tau \cap E_{0,f}^{2\eta} \to \left(B_\delta(M) \smallsetminus B_\delta(M)^\tau\right)/\Gamma, \qquad \beta^\tau(u) := \beta^\Gamma(u^+).$$

The previous lemma asserts that this map is well defined. It is also continuous and $\mathbb{Z}/2$-equivariant, that is,

$$\beta^\tau(u) = \Gamma z \iff \beta^\tau(-u) = \Gamma(gz) \text{ for any } g \in \tau^{-1}(-1).$$

7. Multiplicity of solutions

We start by recalling the notion of equivariant Lusternik-Schnirelmann category. An involution on a topological space X is a map $\varrho_X : X \to X$ such that $\varrho_X \circ \varrho_X = id_X$. Providing X with an involution amounts to defining an action of $\mathbb{Z}/2$ on X. We shall consider the trivial action given by the identity $\varrho_X = id_X$, the action of $G/\Gamma \cong \mathbb{Z}/2$ on the orbit space $\mathbb{R}^N/\Gamma$ where $G \subset O(N)$ and Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, and the antipodal action $\varrho(u) = -u$ on $\mathcal{N}^\tau_{a,f}$. A map $f : X \to Y$ is called $\mathbb{Z}/2$-equivariant (or a $\mathbb{Z}/2$-map) if $\varrho_Y \circ f = f \circ \varrho_X$, and two $\mathbb{Z}/2$-maps $f_0, f_1 : X \to Y$ are said to be $\mathbb{Z}/2$-homotopic if there exists a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\varrho_X x, t) = \varrho_Y \Theta(x,t)$ for every $x \in X$, $t \in [0,1]$. A subset A of X is $\mathbb{Z}/2$-invariant if $\varrho_X a \in A$ for every $a \in A$.

Definition 4. *The $\mathbb{Z}/2$-category of a $\mathbb{Z}/2$-map $f : X \to Y$ is the smallest integer $k =: \mathbb{Z}/2\text{-}\mathrm{cat}(f)$ for which there exists a cover of X by k open $\mathbb{Z}/2$-invariant subsets $X_1, ..., X_k$ such that the restriction $f\mid_{X_i}: X_i \to Y$ is $\mathbb{Z}/2$-homotopic to the composition $\kappa_i \circ \alpha_i$ of a $\mathbb{Z}/2$-map $\alpha_i : X_i \to \{y_i, \varrho_Y y_i\}$, $y_i \in Y$, and the inclusion $\kappa_i : \{y_i, \varrho_Y y_i\} \hookrightarrow Y$. If no such covering exists, we define $\mathbb{Z}/2\text{-}\mathrm{cat}(f) := \infty$.*

If A is a $\mathbb{Z}/2$-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion, we write

$$\mathbb{Z}/2\text{-}cat_X(A) := \mathbb{Z}/2\text{-}cat(\iota) \qquad \text{and} \qquad \mathbb{Z}/2\text{-}cat(X) := \mathbb{Z}/2\text{-}cat_X(X).$$

Note that, if $\varrho_X = id_X$ then

$$\mathbb{Z}/2\text{-}cat_X(A) =: cat_X(A), \qquad \mathbb{Z}/2\text{-}cat(X) := cat(X)$$

is the usual Lusternik-Schnirelmann category [22, Definición 5.4].

Lemma 5. *a) For any $\mathbb{Z}/2$-maps $f : X \to Y$ and $h : Y \to Z$ one has*

$$\mathbb{Z}/2\text{-}\mathrm{cat}(h \circ f) \leq \min\{\mathbb{Z}/2\text{-}\mathrm{cat}(f), \mathbb{Z}/2\text{-}\mathrm{cat}(h)\}.$$

In particular, $\mathbb{Z}/2\text{-}\mathrm{cat}(f) \leq \mathbb{Z}/2\text{-}\mathrm{cat}(Y)$.
b) If $f_0, f_1 : X \to Y$ are $\mathbb{Z}/2$-homotopic, then $\mathbb{Z}/2\text{-}\mathrm{cat}(f_0) = \mathbb{Z}/2\text{-}\mathrm{cat}(f_1)$.

The proof is straightforward, cf. [8, 2.2].

Proof of Theorem 5. Corollary 2 asserts that $E_{a,f}$ satisfies $(PS)^\tau_\theta$ for every $\theta < \frac{2}{N}\ell^\Gamma_f$. Thus, by Lusternik-Schnirelmann theory, $E_{a,f}$ has at least $\mathbb{Z}/2\text{-}\mathrm{cat}(\mathcal{N}^\tau_{a,f} \cap E^\theta_{a,f})$ pairs $\pm u$ of critical points in $\mathcal{N}^\tau_{a,f} \cap E^\theta_{a,f}$ (cf., for example, [8]). We now estimate this category for an appropriate choice of θ.
Without loss of generality we assume that $\delta \in (0, \delta_0)$ and that it satisfies $\max_{B_\delta(M)} a < 0$. We choose $\eta > \frac{1}{N}\ell^\Gamma_f$ as in Proposition 4. We also assume that $\delta' < 2\ell^\Gamma_f$. Let $a_0 \in (-\lambda_1, 0)$ be given by

$$\left(\frac{\lambda_1}{\lambda_1 + a_0}\right)^{\frac{N}{2}} = \min\left\{2, N\eta/\ell^\Gamma_f, 2\ell^\Gamma_f/(2\ell^\Gamma_f - \delta')\right\}. \tag{7.1}$$

If $\min_{\overline{\Omega}} a \geq a_0$ then, for every $\theta < \frac{2}{N}\ell^\Gamma_f$, Lema 1 yields

$$E_{0,f}(\pi_{0,f}(u)) \leq \left(\frac{\lambda_1}{\lambda_1 + a_0}\right)^{\frac{N}{2}} E_{a,f}(u) < \left(\frac{\lambda_1}{\lambda_1 + a_0}\right)^{\frac{N}{2}} \frac{2}{N}\ell^\Gamma_f \leq 2\eta \quad \forall u \in \mathcal{N}^\tau_{a,f} \cap E^\theta_{a,f},$$

where $\pi_{0,f} : \mathcal{N}^{\tau}_{a,f} \cap E^{\theta}_{a,f} \to \mathcal{N}^{\tau}_{0,f}$ is the restriction of the radial projection (3.2). Consequently, we may compose it with the τ-bariorbit map $\beta^{\tau} : \mathcal{N}^{\tau}_{0,f} \cap E^{2\eta}_{0,f} \to (B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma$ of Definition 3 to obtain a $\mathbb{Z}/2$-map

$$\beta^{\tau} \circ \pi_{0,f} : \mathcal{N}^{\tau}_{a,f} \cap E^{\theta}_{a,f} \to (B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma.$$

By Corollary 1 we may choose $\varepsilon > 0$ small enough and $\theta := 2\theta_{\varepsilon} < \frac{2}{N}\ell^{\Gamma}_{f}$ with the property that $E_{a,f}(\pi_{a,f}(w^{\tau}_{\varepsilon,y})) \leq \theta$ for every $y \in M^{-}_{\tau,\delta}$. Therefore, the map

$$\alpha^{\tau}_{\delta} : M^{-}_{\tau,\delta}/\Gamma \to \mathcal{N}^{\tau}_{a,f} \cap E^{\theta}_{a,f}, \qquad \alpha^{\tau}_{\delta}(\Gamma y) := \pi_{a,f}(w^{\tau}_{\varepsilon,y}),$$

is well defined and it is $\mathbb{Z}/2$-equivariant, cf. (4.10). Moreover, $\beta^{\tau}(\pi_{0,f}(\alpha^{\tau}_{\delta}(\Gamma y))) = \Gamma y$ for every $y \in M^{-}_{\tau,\delta}$. Hence Lemma 5 yields

$$\mathbb{Z}/2\text{-cat}(\mathcal{N}^{\tau}_{a,f} \cap E^{\theta}_{a,f}) \geq \mathbb{Z}/2\text{-cat}_{(B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma}(M^{-}_{\tau,\delta}/\Gamma).$$

Since the action of $G/\Gamma \cong \mathbb{Z}/2$ on $(B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma$ is free, one has that

$$\mathbb{Z}/2\text{-cat}_{(B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/\Gamma}(M^{-}_{\tau,\delta}/\Gamma) = \text{cat}_{(B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/G}(M^{-}_{\tau,\delta}/G).$$

We conclude that problem $(\wp^{\Gamma}_{a,f})$ has at least

$$\text{cat}_{(B_{\delta}(M) \setminus B_{\delta}(M)^{\tau})/G}(M^{-}_{\tau,\delta}/G)$$

pairs $\pm u$ of τ-equivariant solutions which satisfy $E_{a,f}(u) \leq \theta$. Now, Lemma 1 and (7.1) yield

$$\theta < \frac{2}{N}\ell^{\Gamma}_{f} = \mu^{\tau}(0,f) \leq \left(\frac{\lambda_1}{\lambda_1 + a_0}\right)^{\frac{N}{2}} \mu^{\tau}(a,f) \leq 2\mu^{\tau}(a,f).$$

Thus, Proposition 2 asserts that these solutions are $(\Gamma, 2)$-nodal. They also yield

$$\frac{1}{N}\left(2\ell^{\Gamma}_{f} - \delta'\right) \leq \left(\frac{\lambda_1 + a_0}{\lambda_1}\right)^{\frac{N}{2}} \frac{2}{N}\ell^{\Gamma}_{f} \leq \mu^{\tau}(a,f) \leq E_{a,f}(u) = \frac{1}{N}\left\|u\right\|^2_a.$$

This concludes the proof. □

Proof of Theorem 3. The proof of Theorem 3 is completely analogous to that of Theorem 5. One uses the Γ-barycenter map β^{Γ} of Definition 2 instead of β^{τ}, and the map α^{Γ}_{s} defined in (4.8) instead of α^{τ}_{s}. □

Proof of Theorem 6. Theorem 3 asserts the existence of $a_1 \in (-\lambda_1, 0)$ such that problem $(\wp^{\Gamma}_{a,f})$ has at least $\text{cat}_{B_{\delta}(M)/\Gamma}(M^{-}_{\delta}/\Gamma)$ positive solutions u with $\ell^{\Gamma}_{f} - \delta' \leq \left\|u\right\|^2_a < \ell^{\Gamma}_{f}$ if $\min_{\overline{\Omega}} a \geq a_1$.

Observe that $\frac{1}{N}\ell^{\Gamma}_{f} < \mu^{\widetilde{\Gamma}}(0,f)$. Indeed, if $\mu^{\widetilde{\Gamma}}(0,f)$ is not achieved, then $\mu^{\widetilde{\Gamma}}(0,f) = \frac{1}{N}\ell^{\widetilde{\Gamma}}_{f}$, and assumption (2.5) guarantees that $\ell^{\Gamma}_{f} < \ell^{\widetilde{\Gamma}}_{f}$. On the other hand, if $u \in \mathcal{N}^{\widetilde{\Gamma}}_{0,f} \subset \mathcal{N}^{\Gamma}_{0,f}$ satisfies $E_{0,f}(u) = \mu^{\widetilde{\Gamma}}(0,f)$, then assumption $(\mathrm{A}^{\Gamma}_{f})$ implies that $\mu^{\widetilde{\Gamma}}(0,f) > \frac{1}{N}\ell^{\Gamma}_{f}$.

Let $a_2 \in (-\lambda_1, 0)$ be given by

$$\left(\frac{\lambda_1 + a_2}{\lambda_1}\right)^{\frac{N}{2}} \mu^{\widetilde{\Gamma}}(0,f) = \frac{1}{N}\ell^{\Gamma}_{f}.$$

Lemma 1 asserts that $\mu^{\widetilde{\Gamma}}(a,f) \geq \frac{1}{N}\ell_f^{\Gamma}$ if $\min_{\overline{\Omega}} a \geq a_2$. Set $a_0 := \max\{a_1, a_2\}$. Then, since $\mu^{\widetilde{\Gamma}}(a,f)$ is the smallest energy of a $\widetilde{\Gamma}$-invariant solution, the solutions provided by Theorem 3 are not $\widetilde{\Gamma}$-invariant if $\min_{\overline{\Omega}} a \geq a_0$. □

Proof of Theorem 1. *(a)* Lazzo's theorem [14] (or Theorem 3 with $\Gamma = \{1\}$) yields $\operatorname{cat}(\Omega)$ positive solutions of $(\wp_\lambda)$ for λ close to 0. If $\Gamma = \mathbb{Z}/2$ then $\Omega \cap M = \{0\}$, and Theorem 2 yields at least one positive even solution for $\lambda \in (-\lambda_1, 0)$. Let $\tau : \mathbb{Z}/2 \to \mathbb{Z}/2$ be the identity homomorphism. Then $\Gamma = \{1\}$, $M = \overline{\Omega}$, $M_{\tau,\delta}^{-} = \{y \in \overline{\Omega} : \operatorname{dist}(y, \partial\Omega \cup \{0\}) \geq \delta\}$, and $B_\delta(M) \smallsetminus B_\delta(M)^\tau = B_\delta(\overline{\Omega}) \smallsetminus \{0\}$. So, for δ small enough, the inclusions $M_{\tau,\delta}^{-} \hookrightarrow \Omega \smallsetminus \{0\} \hookrightarrow B_\delta(\overline{\Omega}) \smallsetminus \{0\}$ are $\mathbb{Z}/2$-homotopy equivalences. Theorem 5 yields at least $\mathbb{Z}/2$-$\operatorname{cat}(\Omega \smallsetminus \{0\})$ pairs of odd 2-nodal solutions for λ close to 0. Since Ω contains a small sphere centered at the origin, one has that $\mathbb{Z}/2$-$\operatorname{cat}(\Omega \smallsetminus \{0\}) \geq N$.
(b) For λ close to 0 Theorem 6, with $\Gamma = \{1\}$, $\widetilde{\Gamma} = \mathbb{Z}/2$, and δ small enough, yields at least $\operatorname{cat}(\Omega)$ positive solutions which are not even. If $\Gamma = \mathbb{Z}/2$ then $M = \overline{\Omega}$, and Theorem 2 yields at least one positive even solution for $\lambda \in (-\lambda_1, 0)$. Let $\tau : \mathbb{Z}/2 \to \mathbb{Z}/2$ be the identity homomorphism. Then Theorem 5 yields at least $\mathbb{Z}/2$-$\operatorname{cat}(\Omega)$ pairs of odd 2-nodal solutions for λ close to 0. Finally observe that, since Ω is symmetric with respect to the origin and $0 \notin \Omega$, one has that $\operatorname{cat}(\Omega) \geq 2$. And since, in addition, Ω admits an odd map $\mathbb{S}^{k-1} \to \Omega$, one has that $\mathbb{Z}/2$-$\operatorname{cat}(\Omega) \geq k$. □

8. A NONEXISTENCE RESULT

Let $\varphi : \mathbb{S}^{N-1} \to [1,2]$ be a Γ-invariant C^∞-function and let $S_\varphi := \{\varphi(z)z : z \in \mathbb{S}^{N-1}\}$. For $\kappa > 0$ set $A_{\varphi,\kappa} := \{z + tn_z : z \in S_\varphi,\ 0 < t < \kappa\}$, where n_z is the outward unit normal to S_φ at z. Ben Ayed, El Mehdi and Hammami [2] showed that, for every $C > 0$ and all $\kappa > 0$ small enough, problem

$$-\Delta u = |u|^{2^*-2}u \ \text{ in } A_{\varphi,\kappa}, \qquad u = 0 \ \text{ on } \partial A_{\varphi,\kappa} \tag{8.1}$$

has no positive solution with $\|u\|^2 \leq C$. We use this result to prove Theorem 7.

Proof of Theorem 7. For $C := (\min_{x\in\overline{\Omega}} \#\Gamma x) S^{N/2}$ let $\kappa > 0$ be such that problem (8.1) has no positive solution with $\|u\|^2 \leq C$. Set

$$\mathcal{N}^\Gamma(\Omega) := \{u \in H_0^1(\Omega)^\Gamma : u \neq 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\}, \qquad \mu^\Gamma(\Omega) := \inf_{\mathcal{N}^\Gamma(\Omega)} E_{0,f}.$$

Let $\Omega \subset A_{\varphi,\kappa}$ and assume, by contradiction, that $E_{0,1}$ has a nontrivial critical point $u \in \mathcal{N}^\Gamma(\Omega)$ with $\|u\|^2 \leq C$. Using property (2.6) we obtain

$$\mu^\Gamma(A_{\varphi,\kappa}) \leq \mu^\Gamma(\Omega) \leq E_{0,1}(u) \leq \frac{1}{N}C = \frac{1}{N}(\min_{x\in\overline{A}_{\varphi,\kappa}} \#\Gamma x) S^{N/2}. \tag{8.2}$$

Proposition 1 and our choice of κ imply that $\mu^\Gamma(A_{\varphi,\kappa})$ is not achieved. Hence, Corollary 2 asserts that equalities in (8.2) most hold, in particular, $E_{0,1}(u) = \mu^\Gamma(A_{\varphi,\kappa})$. This is a contradiction. □

Finally, we apply this result to the Brezis-Nirenberg problem $(\wp_\lambda)$.

Theorem 9. *Let $\varphi : \mathbb{S}^{N-1} \to [1,2]$ be an even C^∞-function, $N \geq 4$. Then there exists $\kappa > 0$ with the following property: If $\Omega \subset A_{\varphi,\kappa}$ is symmetric with respect to the origen and admits an odd map $\mathbb{S}^{k-1} \to \Omega$, then there exists $\lambda^* \in (-\lambda_1, 0)$ such that, for every $\lambda \in (\lambda^*, 0)$, problem $(\wp_\lambda)$ has at least $\operatorname{cat}(\Omega) \geq 2$ positive solutions which are not even functions, $\mathbb{Z}/2\text{-}\operatorname{cat}(\Omega) \geq k$ even positive solutions, and $\mathbb{Z}/2\text{-}\operatorname{cat}(\Omega) \geq k$ pairs $\pm u$ of odd 2-nodal solutions.*

Proof. Let $\Gamma = \mathbb{Z}/2$. Since $\min_{x\in\overline{\Omega}} \#\Gamma x = \min_{z\in\mathbb{S}^{N-1}} \#\Gamma z = 2$, Theorem 7 asserts that (A_1^Γ) holds. If δ is small enough, Theorem 3 yields at least $\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma) = \operatorname{cat}(\Omega/\Gamma) = \mathbb{Z}/2\text{-}cat(\Omega)$ even positive solutions, for λ close enough to 0. Since $0 \notin \Omega$ and Ω admits an odd map $\mathbb{S}^{k-1} \to \Omega$, one has that $\mathbb{Z}/2\text{-}cat(\Omega) \geq k$. The other solutions are given by Theorem 1. $\square$

Appendix A. The proof of Proposition 4

We split the proof of this proposition into three lemmas.

Lemma 6. *(a)* $\|W_{\varepsilon,z}\|^2 \to \ell_f^\Gamma$ *as* $\varepsilon \to 0$.
(b) $\|W_{\varepsilon,z} + W_{\varepsilon',z'}\|^2 \geq 2\ell_f^\Gamma$ *for all* $z,\ z' \in M_{\delta_0}$, $\varepsilon, \varepsilon' > 0$.
(c) If $\varepsilon'_n \to 0$ *and either* $\varepsilon_n \geq \delta > 0$ *or* $\operatorname{dist}(\Gamma z_n, \Gamma z'_n) \geq \delta$, *then* $\left\|W_{\varepsilon_n,z_n} - W_{\varepsilon'_n,z'_n}\right\|^2 + o_n(1) \geq 2\ell_f^\Gamma$, *where* $o_n(1) \to 0$ *as* $n \to \infty$.
(d) If $\left\|W_{\varepsilon_n,z_n} - W_{\varepsilon'_n,z'_n}\right\| \to 0$, $\varepsilon_n \to 0$ *and* $\varepsilon'_n \to 0$, *then* $\left|\varepsilon_n(\varepsilon_n^{'})^{-1} - 1\right| \to 0$ *and* $(\varepsilon_n\varepsilon'_n)^{-1}\operatorname{dist}(\Gamma z_n, \Gamma z'_n)^2 \to 0$.

Proof. Observe that $U_{\varepsilon,z} = \varepsilon^{-\frac{N-2}{2}} U(\frac{x-z}{\varepsilon})$ with $U(x) = a_N(1+|x|^2)^{(2-N)/2}$. Since U is a solution of $(\wp_\infty)$ one has that

$$
\begin{aligned}
\langle U_{\varepsilon,z}, U_{\varepsilon',z'}\rangle &= (\varepsilon\varepsilon')^{-\frac{N}{2}} \int_{\mathbb{R}^N} \nabla U(\frac{x-z}{\varepsilon}) \nabla U(\frac{x-z'}{\varepsilon'}) dx \\
&= (\frac{\varepsilon}{\varepsilon'})^{\frac{N}{2}} \int_{\mathbb{R}^N} \nabla U(y) \nabla U(\frac{\varepsilon}{\varepsilon'} y - \frac{z'-z}{\varepsilon'}) dy \\
&= (\frac{\varepsilon}{\varepsilon'})^{\frac{N}{2}} \int_{\mathbb{R}^N} U(y)^{2^*-1} \frac{\varepsilon'}{\varepsilon} U(\frac{\varepsilon}{\varepsilon'} y - \frac{z'-z}{\varepsilon'}) dy \\
&= \int_{\mathbb{R}^N} U(y)^{2^*-1} (\frac{\varepsilon}{\varepsilon'})^{\frac{N-2}{2}} U(\frac{\varepsilon}{\varepsilon'}(y - \frac{z'-z}{\varepsilon})) dy \\
&= \left\langle U, U_{\varepsilon'\varepsilon^{-1},(z'-z)\varepsilon^{-1}} \right\rangle .
\end{aligned}
\tag{A.1}
$$

So, for $z_1 \in (M_{\Gamma_i})_{\delta_0}$, $z_2 \in (M_{\Gamma_j})_{\delta_0}$, we obtain

$$
\begin{aligned}
\langle W_{\varepsilon,z}, W_{\varepsilon',z'}\rangle &= \sum_{[\gamma]\in\Gamma/\Gamma_i} \sum_{[\gamma']\in\Gamma/\Gamma_j} f_i^{\frac{2-N}{4}} f_j^{\frac{2-N}{4}} \langle U_{\varepsilon,\gamma z}, U_{\varepsilon',\gamma' z'}\rangle \\
&= \sum_{[\gamma]\in\Gamma/\Gamma_i} \sum_{[\gamma']\in\Gamma/\Gamma_j} f_i^{\frac{2-N}{4}} f_j^{\frac{2-N}{4}} \left\langle U, U_{\varepsilon'\varepsilon^{-1},(\gamma' z'-\gamma z)\varepsilon^{-1}} \right\rangle . \qquad \text{(A.2)}
\end{aligned}
$$

In particular, $\langle W_{\varepsilon,z}, W_{\varepsilon',z'}\rangle \geq 0$ and

$$
\|W_{\varepsilon,z}\|^2 = \sum_{[\gamma]\in\Gamma/\Gamma_i} f_i^{\frac{2-N}{2}} \|U\|^2 + \sum_{[\gamma]\neq[\gamma']} f_i^{\frac{2-N}{2}} \left\langle U, U_{1,(\gamma' z-\gamma z)\varepsilon^{-1}} \right\rangle \geq \ell_f^\Gamma. \tag{A.3}
$$

Hence $\|W_{\varepsilon,z} + W_{\varepsilon',z'}\|^2 = \|W_{\varepsilon,z}\|^2 + \|W_{\varepsilon',z'}\|^2 + 2\langle W_{\varepsilon,z}, W_{\varepsilon,z'}\rangle \geq 2\ell_f^\Gamma$, as asserted in *(b)*. The equality in (A.3), together with (6.1), yields *(a)*.

If either $\varepsilon_n \geq \delta > 0$ and $\varepsilon_n' \to 0$, or $\varepsilon_n \to 0$ and $\operatorname{dist}(\Gamma z_n, \Gamma z_n') \geq \delta$, the identity (A.2) yields $\left\langle W_{\varepsilon_n, z_n}, W_{\varepsilon_n', z_n'} \right\rangle \to 0$. Since

$$\left\| W_{\varepsilon_n, z_n} - W_{\varepsilon_n', z_n'} \right\|^2 + 2\left\langle W_{\varepsilon_n, z_n}, W_{\varepsilon_n', z_n'} \right\rangle = \left\| W_{\varepsilon_n, z_n} \right\|^2 + \left\| W_{\varepsilon_n', z_n'} \right\|^2 \geq 2\ell_f^\Gamma,$$

we obtain *(c).*

If $\max\{\varepsilon_n, \varepsilon_n', \left\| W_{\varepsilon_n, z_n} - W_{\varepsilon_n', z_n'} \right\|\} \to 0$ then, applying *(a)* to $\left\| W_{\varepsilon_n, z_n} - W_{\varepsilon_n', z_n'} \right\|^2 = \left\| W_{\varepsilon_n, z_n} \right\|^2 + \left\| W_{\varepsilon_n', z_n'} \right\|^2 - 2\left\langle W_{\varepsilon_n, z_n}, W_{\varepsilon_n', z_n'} \right\rangle$, we get that $\left\langle W_{\varepsilon_n, z_n}, W_{\varepsilon_n', z_n'} \right\rangle \to \ell_f^\Gamma$. It follows from (A.2) that $i = j$, $\operatorname{dist}(\Gamma z_n, \Gamma z_n') = |z_n - \gamma z_n'|$ for some $[\gamma] \in \Gamma/\Gamma_i$,

$$\left\langle U, U_{\varepsilon_n'\varepsilon_n^{-1}, (\gamma z_n' - z_n)\varepsilon_n^{-1}} \right\rangle \to S^{N/2}, \quad \text{and} \quad \left\langle U, U_{\varepsilon_n'\varepsilon_n^{-1}, (\gamma' z_n' - z_n)\varepsilon_n^{-1}} \right\rangle \to 0 \text{ if } [\gamma'] \neq [\gamma].$$

Hence, $\varepsilon_n'\varepsilon_n^{-1} \to 1$ and $(\gamma z_n' - z_n)\varepsilon_n^{-1} \to 0$. This proves *(d).* □

Lemma 7. *Given $\delta \in (0, \delta_0)$, $\rho \in (0, (\ell_f^\Gamma)^{1/2})$ and $R > 0$, there exists $\eta > \frac{1}{N}\ell_f^\Gamma$ such that, for every $u \in \mathcal{N}_{0,f}^\Gamma \cap E_{0,f}^\eta$, the following holds:*

(i) $\inf_{W \in \Theta_*} \|u - W\| < \rho$ *and this infimum is achieved.*

(ii) If $W_0 \in \Theta_*$ satisfies $\|u - W_0\| = \inf_{W \in \Theta_*} \|u - W\|$, then $W_0 \in \Theta_\delta$.

(iii) If $\nu_j W_{\varepsilon_j, z_j} \in \Theta_\delta$ satisfy $\left\| u - \nu_j W_{\varepsilon_j, z_j} \right\| = \inf_{W \in \Theta_*} \|u - W\|$, $j = 1, 2$, then $z_1, z_2 \in (M_{\Gamma_i})_\delta$ for the same $i \in \{1, ..., m\}$ and one has that

$$\nu_1 = \nu_2, \quad \left| \varepsilon_1 \varepsilon_2^{-1} - 1 \right| < R \quad \text{and} \quad (\varepsilon_1\varepsilon_2)^{-1} \operatorname{dist}(\Gamma z_1, \Gamma z_2)^2 < R. \tag{A.4}$$

Proof. *(i)* Observe that, since $\langle u, U_{\varepsilon, z} \rangle \to 0$ as $\varepsilon \to 0$, there exists $\varepsilon_0 \in (0, \delta_0)$ such that

$$\|u \pm W_{\varepsilon, z}\|^2 \geq \|u\|^2 \geq \ell_f^\Gamma \quad \forall \varepsilon \in (0, \varepsilon_0],\ z \in M_{\delta_0}.$$

Since $[\varepsilon_0, \delta_0] \times M_{\delta_0}$ is compact, we have that $\inf_{W \in \Theta_*} \|u - W\|$ must be achieved if $\inf_{W \in \Theta_*} \|u - W\| < \rho < (\ell_f^\Gamma)^{1/2}$. Assume, by contradiction, there is a sequence (u_n) in $\mathcal{N}_{0,f}^\Gamma$ such that $E_{0,f}(u) \leq \frac{1}{N}\ell_f^\Gamma + \frac{1}{n}$ and

$$\inf_{W \in \Theta_*} \|u_n - W\| \geq \rho. \tag{A.5}$$

By Ekeland's variational principle we may assume that (u_n) is a Γ-invariant Palais-Smale sequence. Since condition (A_f^Γ) holds, (u_n) does not contain a convergent subsequence. Thus, by Corollary 2, there exists $W_n \in \Theta_{\delta_n}$ with $\|u_n - W_n\| \to 0$ and $\delta_n \to 0$, contradicting (A.5). Therefore, there exists $\eta > \frac{1}{N}\ell_f^\Gamma$ such that *(i)* holds.

(ii) Assume there is a sequence (u_n) in $\mathcal{N}_{0,f}^\Gamma$ with $E_{0,f}(u) \leq \frac{1}{N}\ell_f^\Gamma + \frac{1}{n}$ such that $\|u_n - W_n'\| = \inf_{W \in \Theta_*} \|u_n - W\|$ for some $W_n' \in \Theta_* \smallsetminus \Theta_\delta$. Arguing as in *(i)* there is also a sequence $W_n \in \Theta_{\delta_n}$ with $\|u_n - W_n\| \to 0$ and $\delta_n \to 0$. Since $\|u_n - W_n'\| \leq \|u_n - W_n\|$ we obtain that $\|W_n - W_n'\| \to 0$, contradicting assertion *(c)* of Lemma 6. Therefore *(ii)* holds for some $\eta > \frac{1}{N}\ell_f^\Gamma$.

(iii) Without loss of generality we may assume that $R < 1$ and that δ and ρ are small enough to ensure that $\nu_1 = \nu_2$, $\left| \varepsilon_1\varepsilon_2^{-1} - 1 \right| < R$ and $(\varepsilon_1\varepsilon_2)^{-1}\operatorname{dist}(\Gamma z_1, \Gamma z_2)^2 < R$ if $\|\nu_1 W_{\varepsilon_1, z_1} - \nu_2 W_{\varepsilon_2, z_2}\| < 2\rho$ and $\varepsilon_1, \varepsilon_2 \in (0, \delta]$, see Lemma 6*(d).* For these data we choose η satisfying *(i)* and *(ii).* If $u \in \mathcal{N}_{0,f}^\Gamma \cap E_{0,f}^\eta$ and $\nu_j W_{\varepsilon_j, z_j} \in \Theta_\delta$ satisfy $\left\| u - \nu_j W_{\varepsilon_j, z_j} \right\| = \inf_{W \in \Theta_\delta} \|u - W\|$, then $\varepsilon_1, \varepsilon_2 \in (0, \delta]$ and $\|\nu_1 W_{\varepsilon_1, z_1} - \nu_2 W_{\varepsilon_2, z_2}\| < 2\rho$. Therefore, properties (A.4) hold. In particular, $\operatorname{dist}(\Gamma z_1, \Gamma z_2)^2 < R\varepsilon_1\varepsilon_2 < \delta^2$. Hence (6.2) guarantees that $z_1, z_2 \in (M_{\Gamma_i})_\delta$ for the same $i \in \{1, ..., m\}$. □

Lemma 8. *There exist* $\delta \in (0,\delta_0)$, $\rho \in (0,(\ell_f^{\Gamma})^{1/2})$ *and* $R > 0$ *such that, if* $\nu_s W_{\varepsilon_s,z_s} \in \Theta_\delta$, $s = 1,2$, *satisfy* $\|u - \nu_s W_{\varepsilon_s,z_s}\| = \inf_{W\in\Theta_\delta}\|u - W\| < \rho$, $z_1, z_2 \in (M_{\Gamma_i})_\delta$ *for the same* $i \in \{1,...,m\}$, $\left|\varepsilon_1\varepsilon_2^{-1} - 1\right| \leq R$, *and* $(\varepsilon_1\varepsilon_2)^{-1}\left|z_1 - z_2\right|^2 \leq R$, *then* $\varepsilon_1 = \varepsilon_2$ *and* $z_1 = z_2$.

Proof. Consider the function $\chi_u : (0,\delta_0) \times (M_{\Gamma_i})_{\delta_0} \to \mathbb{R}$ given by

$$\chi_u(\varepsilon,z) := \|u - W_{\varepsilon,z}\|^2 .$$

Set $\zeta := (\varepsilon,z)$ and $W_\zeta := W_{\varepsilon,z}$, and let $h = (h_0,h_1,...,h_d) \in \mathbb{R}\times V^i$ where $d := \dim V^i$. The second derivative of χ_u is given by

$$\frac{1}{2}\chi_u''(\zeta)(h,h) = \|D_\zeta W_\zeta(\cdot)h\|^2 - \left\langle u - W_\zeta, D_\zeta^2 W_\zeta(\cdot)(h,h)\right\rangle .$$

A straightforward computation shows that

$$\langle \partial_j U_{\varepsilon,\gamma z}(\cdot)h_j, \partial_k U_{\varepsilon,\gamma' z}(\cdot)h_k\rangle = \varepsilon^{-2}\int \nabla V_j(y)\cdot\nabla V_k(y - \frac{\gamma' z - \gamma z}{\varepsilon})h_j h_k dy,$$

where $\partial_0 := \partial_\varepsilon$, $\partial_j := \partial_{z_j}$ are the partial derivatives with respect to ε and z_j,

$$V_0(y) := \nabla U(y)\cdot y + \frac{N-2}{2}U(y), \qquad V_j(y) = \frac{\partial U}{\partial y_j}(y), \quad j = 1,...,d.$$

Therefore,

$$\begin{aligned}
\|D_\zeta W_\zeta(\cdot)h\|^2 &= f_i^{\frac{2-N}{2}} \sum_{j,k=0}^{d} \sum_{[\gamma],[\gamma']\in\Gamma/\Gamma_i} \langle \partial_j U_{\varepsilon,\gamma z}(\cdot)h_j, \partial_k U_{\varepsilon,\gamma' z}(\cdot)h_k\rangle \\
&= \varepsilon^{-2}\frac{\#\left(\Gamma/\Gamma_i\right)}{f_i^{\frac{N-2}{2}}}\left(\sum_{j,k=0}^{d} a_{jk}h_j h_k + o_\varepsilon(1)\right) \\
&= \varepsilon^{-2}c_i\left(Ah\cdot h + o_\varepsilon(1)\right),
\end{aligned}$$

where $a_{jk} = \langle V_j, V_k\rangle$, $A = (a_{jk})$ and $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Similarly,

$$\begin{aligned}
\left\langle u - W_\zeta, D_\zeta^2 W_\zeta(\cdot)(h,h)\right\rangle &\leq \|u - W_\zeta\|\left\|D_\zeta^2 W_\zeta(\cdot)(h,h)\right\| \\
&= \varepsilon^{-2}c_i\left(\|u - W_\zeta\|\widetilde{A}h\cdot h + o_\varepsilon(1)\right).
\end{aligned}$$

Hence there exist $\delta \in (0,\delta_0)$, $\rho > 0$ and $c > 0$ such that

$$\begin{aligned}
\frac{1}{2}\chi_u''(\zeta)(h,h) &\geq \varepsilon^{-2}c_i\left(Ah\cdot h - \|u - W_\zeta\|\widetilde{A}h\cdot h + o_\varepsilon(1)\right) \\
&\geq c\varepsilon^{-2}\left|h\right|^2 \qquad \text{if } \|u - W_\zeta\| < \rho \text{ and } \varepsilon < \delta.
\end{aligned}$$

Assume that $\zeta_1 = (\varepsilon_1,z_1)$, $\zeta_2 = (\varepsilon_2,z_2) \in (0,\delta)\times(M_{\Gamma_i})_\delta$ are minima of χ_u and that $\|u - \nu_s W_{\varepsilon_s,z_s}\| < \rho$, $s = 1,2$. Taylor's formula yields

$$\chi_u(\zeta_1 + h) - \chi_u(\zeta_1) = \frac{1}{2}\chi_u''(\zeta_1)(h,h) + r(h) \geq c\varepsilon_1^{-2}\left|h\right|^2 + r(h). \tag{A.6}$$

Let $h := \zeta_2 - \zeta_1$. If $h \neq 0$ we obtain from (A.6) that

$$0 \geq c + \frac{r(h)}{\varepsilon_1^{-2}\left|h\right|^2}. \tag{A.7}$$

A straightforward computation shows that

$$\frac{r(h)}{\varepsilon_1^{-2}\left|h\right|^2} \to 0 \quad \text{as} \ \ \varepsilon_1\varepsilon_2^{-1} \to 1 \text{ and } (\varepsilon_1\varepsilon_2)^{-1}\left|z_1 - z_2\right|^2 \to 0.$$

Hence there exists $R > 0$ such that

$$\frac{\left|r(h)\right|}{\varepsilon_1^{-2}\left|h\right|^2} < c \quad \text{if} \ \ \left|\varepsilon_1\varepsilon_2^{-1} - 1\right| < R \ \text{ and } \ (\varepsilon_1\varepsilon_2)^{-1}\left|z_1 - z_2\right|^2 < R,$$

contradicting (A.7). We conclude that $h = 0$, that is, $(\varepsilon_1, z_1) = (\varepsilon_2, z_2)$ as claimed. □

Proof of Proposition 4. Let $\delta \in (0, \delta_0)$, $\rho \in (0, (\ell_f^\Gamma)^{1/2})$ and $R > 0$ be as in Lemma 8. We assume that this δ is smaller that the given one. For these data let $\eta > \frac{1}{N}\ell_f^\Gamma$ be as in Lemma 7. This η has the desired properties. □

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