THE EFFECT OF THE DOMAIN TOPOLOGY ON THE NUMBER OF MINIMAL NODAL SOLUTIONS OF AN ELLIPTIC EQUATION AT CRITICAL GROWTH IN A SYMMETRIC DOMAIN

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ABSTRACT. We consider the Dirichlet problem $\Delta u+\lambda u+\mid u\mid^{2^*-2} u = 0$ in Ω, $u = 0$ on $\partial\Omega$ where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 4$, and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. We show that if Ω is invariant under an orthogonal involution then, for $\lambda > 0$ sufficiently small, there is an effect of the equivariant topology of Ω on the number of solutions which change sign exactly once.

1. Introduction

Consider the problem

$$(\wp_\lambda) \qquad \begin{cases} \Delta u + \lambda u + \mid u\mid^{2^*-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$, and $\lambda \in \mathbb{R}$. We are interested in solutions to this problem which change sign exactly once, that is, solutions u such that $\Omega \setminus u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other.

Let us first recall some well known facts. If $\lambda = 0$ this problem does not have a least energy solution, and it does not have a nontrivial solution if Ω is strictly starshaped and $\lambda \leq 0$ [10]. In contrast with this situation Brézis and Nirenberg [2] showed that there is at least one positive solution of $(\wp_\lambda)$ if $N \geq 4$ and $0 < \lambda < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$. Furthermore, it was shown by Rey [11] for $N \geq 5$ and by Lazzo [9] for $N \geq 4$ that there is an effect of the domain topology on the number of low energy positive solutions of this problem, namely, they showed that there is a $0 < \overline{\lambda} < \lambda_1(\Omega)$ such that $(\wp_\lambda)$ has at least $cat(\Omega)$ positive solutions for all $0 < \lambda < \overline{\lambda}$, where $cat(\Omega)$ is the Lusternik-Schnirelmann category of Ω.

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The first result about sign changing solutions is due to Cerami, Solimini and Struwe [5] who showed the existence of a pair of least energy sign changing solutions if $N \geq 6$ and $0 < \lambda < \lambda_1(\Omega)$. These solutions change sign exactly once. Similar results were obtained by Zhang [17] and Tarantello [15].

Some multiplicity results for sign changing solutions are also known. Cerami, Solimini and Struwe [5] showed the existence of infinitely many radially symmetric solutions on a ball for $N \geq 7$ and $0 < \lambda < \lambda_1(\Omega)$. For domains with some special kind of symmetries Fortunato and Jannelli [8] showed the existence of solutions with arbitrarily large energy for $N \geq 4$ and $\lambda > 0$. However, these solutions change sign many times.

Here we shall obtain a multiplicity result for solutions which change sign exactly once. We shall consider symmetric domains and prove that, as for positive solutions, there is also an effect of the domain topology on the number of such solutions. More precisely, we consider the problem

$$(\wp_\lambda^\tau) \qquad \begin{cases} \Delta u + \lambda u + \mid u \mid^{2^*-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega \end{cases}$$

where τ is a nontrivial orthogonal involution, that is, an orthogonal linear transformation of $\mathbb{R}^N$ such that $\tau \neq I$ and $\tau^2 = I$, I being the identity of $\mathbb{R}^N$, and Ω is a bounded smooth domain in $\mathbb{R}^N$ which is τ-invariant, that is, $\tau x \in \Omega$ if $x \in \Omega$. This includes, for example, domains Ω which are symmetric with respect to the origin (that is, such that $x \in \Omega$ iff $-x \in \Omega$), as well as cylindrical or rotationally invariant domains as those considered by Fortunato and Jannelli. We shall prove the following.

Theorem 1. *If $N \geq 4$ then, for every $0 < \lambda < \lambda_1(\Omega)$, problem $(\wp_\lambda^\tau)$ has at least one pair of solutions which change sign exactly once.*

Theorem 2. *If $N \geq 4$ then there is a $0 < \lambda^* < \lambda_1(\Omega)$ such that, for each $0 < \lambda < \lambda^*$, problem $(\wp_\lambda^\tau)$ has at least $\tau\text{-}cat_\Omega(\Omega \setminus \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ is the set of fixed points of the involution τ, and τ-*cat* is the G_τ-equivariant Lusternik-Schnirelmann category for the group $G_\tau = \{I, \tau\}$. In many cases the equivariant category turns out to be larger than the nonequivariant one. For example, for the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ and $\tau = -I$, $\tau\text{-}cat(\mathbb{S}^{N-1}) = N$ whereas $cat(\mathbb{S}^{N-1}) = 2$. Thus, Theorem 2 provides many solutions for some domains like the following.

Corollary 3. *Let Ω be symmetric with respect to the origin and such that $0 \notin \Omega$. Assume further that there is an odd map $\varphi : \mathbb{S}^{N-1} \to \Omega$. Then if $N \geq 4$ there is a $0 < \lambda^* < \lambda_1(\Omega)$ such that, for each $0 < \lambda < \lambda^*$, problem $(\wp_\lambda)$ has at least N pairs of odd solutions which change sign exactly once.*

The solutions provided by Theorems 1 and 2 concentrate at symmetric points of the domain as $\lambda \to 0$. We shall show the following to hold.

Theorem 4. *Let $N \geq 4$, let (λ_k) be a sequence of positive numbers such that $\lambda_k \to 0$. The solutions u_k to the problem $(\wp^\tau_{\lambda_k})$ provided by Theorems 1 and 2 satisfy the following: There is a sequence of points (y_k) in Ω and a sequence of positive real numbers (ε_k) such that*

(i) $(\varepsilon_k)^{-1} dist(y_k, \partial\Omega) \to \infty$ *as* $k \to \infty$,

(ii) $y_k \neq \tau y_k$ *for all* $k \in \mathbb{N}$, *and* $\varepsilon_k^{-1} |y_k - \tau y_k| \to \infty$ *as* $k \to \infty$,

(iii) $u_k = a_N \left[\left(\frac{\varepsilon_k}{\varepsilon_k^2 + |x - y_k|^2} \right)^{\frac{N-2}{2}} - \left(\frac{\varepsilon_k}{\varepsilon_k^2 + |x - \tau y_k|^2} \right)^{\frac{N-2}{2}} \right] + o(1)$

where $a_N = [N(N-2)]^{(N-2)/4}$ *and* $o(1) \to 0$ *in* $D^{1,2}(\mathbb{R}^N)$ *as* $k \to \infty$.

This paper is organized as follows: In Section 2 we describe the variational setting for problem $(\wp^\tau_\lambda)$. In Section 3 we state a global compactness result for this problem, deduce some consequences of it, and prove Theorem 1. Section 4 is devoted to the proof of Theorems 2 and 4. Finally, in Section 5 we prove the global compactness result stated in Section 3 which provides a precise description of all Palais-Smale sequences for the variational problem associated to $(\wp^\tau_\lambda)$.

2. The variational problem

We assume throughout that $0 \leq \lambda < \lambda_1(\Omega)$. We write $p = 2^* = \frac{2N}{N-2}$, and denote

$$\|u\|^2_\lambda = \int_\Omega |\nabla u|^2 - \lambda u^2 \qquad \text{and} \qquad |u|^p_p = \int_\Omega |u|^p .$$

Observe that, since $0 \leq \lambda < \lambda_1(\Omega)$, $\|u\|_\lambda$ is a norm in the Sobolev space $H^1_0(\Omega)$ which is equivalent to the usual one.

The (classical) solutions of problem $(\wp_\lambda)$ are the critical points of the energy functional

$$E_\lambda(u) = \frac{1}{2} \|u\|^2_\lambda - \frac{1}{p} |u|^p_p$$

defined on $H_0^1(\Omega)$. The nontrivial critical points of E_λ lie in the Nehari manifold

$$\begin{aligned}\mathcal{N}_\lambda &= \{u \in H_0^1(\Omega) : u \neq 0,\ DE_\lambda(u)u = 0\} \\ &= \{u \in H_0^1(\Omega) : u \neq 0,\ \|u\|_\lambda^2 = |u|_p^p\}.\end{aligned}$$

This is a manifold of class C^1 which is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)$ [16, Lemma 4.1].

The involution τ of Ω induces an orthogonal involution of $H_0^1(\Omega)$, which we also denote by τ, as follows: For each $u \in H_0^1(\Omega)$ we define $\tau u \in H_0^1(\Omega)$ by

$$(\tau u)(x) = -u(\tau x).$$

The solutions of problem $(\wp_\lambda^\tau)$ are the critical points of E_λ which lie in the closed linear subspace

$$H_0^1(\Omega)^\tau = \{u \in H_0^1(\Omega) : \tau u = u\}$$

of $H_0^1(\Omega)$. Observe that $E_\lambda(\tau u) = E_\lambda(u)$ and that $\nabla E_\lambda(\tau u) = \tau \nabla E_\lambda(u)$. Thus $\tau \nabla E_\lambda(u) = \nabla E_\lambda(u)$ if $\tau u = u$. Therefore the nontrivial solutions of $(\wp_\lambda^\tau)$ are the critical points of the restriction of E_λ to the τ-invariant Nehari manifold

$$\mathcal{N}_\lambda^\tau = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap H_0^1(\Omega)^\tau.$$

Let

$$\mu_\lambda = \mu_\lambda(\Omega) = \inf_{\mathcal{N}_\lambda} E_\lambda \qquad \text{and} \qquad \mu_\lambda^\tau = \mu_\lambda^\tau(\Omega) = \inf_{\mathcal{N}_\lambda^\tau} E_\lambda.$$

If $\lambda = 0$ then $\mu_0(\Omega) = \frac{1}{N} S^{\frac{N}{2}}$ where S is the best Sobolev constant for the imbedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$. In particular, μ_0 is independent of Ω and, due to the maximum principle, it is not achieved by E_0 on $\mathcal{N}_0$ if $\Omega \neq \mathbb{R}^N$. On the other hand, if $N \geq 4$ Brézis and Nirenberg [2] showed that, for $0 < \lambda < \lambda_1(\Omega)$ and any bounded domain Ω,

$$\mu_\lambda(\Omega) < \mu_0 = \frac{1}{N} S^{\frac{N}{2}},$$

and that $\mu_\lambda(\Omega)$ is achieved by E_λ on $\mathcal{N}_\lambda$. Moreover,

$$\lim_{\lambda \to 0} \mu_\lambda = \frac{1}{N} S^{\frac{N}{2}}.$$

Proposition 5. *Let $N \geq 4$. For every $0 < \lambda < \lambda_1(\Omega)$ the following holds*

$$2\mu_\lambda \leq \mu_\lambda^\tau < \mu_0^\tau = \frac{2}{N} S^{\frac{N}{2}}.$$

Moreover $\mu_0^\tau = \frac{2}{N} S^{\frac{N}{2}}$ is not achieved by E_0 on $\mathcal{N}_0^\tau$.

Proof. Let $u^{\pm} = \pm\max\{\pm u, 0\}$. Observe that, if $u = \tau u$, then $\|u^+\|^2_\lambda = \|u^-\|^2_\lambda$ and $|u^+|^p_p = |u^-|^p_p$. So, if $u \in \mathcal{N}^\tau_\lambda$ then $u^+, u^- \in \mathcal{N}_\lambda$ and

$$E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \geq 2\mu_\lambda$$

This shows that $2\mu_\lambda \leq \mu^\tau_\lambda$ for every $0 \leq \lambda < \lambda_1(\Omega)$. To prove the second inequality, choose $x \in \Omega$ with $\tau x \neq x$, and $r > 0$ so that $B_r(x) \subset \Omega$ and $B_r(x) \cap B_r(\tau x) = \emptyset$, where $B_r(\xi)$ denotes the open ball in $\mathbb{R}^N$ with center ξ and radius r. For $0 < \lambda < \lambda_1(\Omega)$ let $u_{\lambda,r}$ be the positive ground state solution of the problem

$$(\wp_{\lambda,r}) \qquad \begin{cases} \Delta u + \lambda u + \mid u \mid^{2^*-2} u = 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases}$$

Then $\|u_{\lambda,r}\|^2_\lambda = |u_{\lambda,r}|^p_p$ and, since $u_{\lambda,r}$ is radially symmetric,

$$u = u_{\lambda,r}(\cdot - x) - u_{\lambda,r}(\cdot - \tau x) \in \mathcal{N}^\tau_\lambda.$$

Thus,

$$\mu^\tau_\lambda \leq E_\lambda(u) = 2E_\lambda(u_{\lambda,r}) < \frac{2}{N} S^{\frac{N}{2}}.$$

To show that $\mu^\tau_0 \leq \frac{2}{N} S^{\frac{N}{2}}$ we take a minimizing sequence for problem $(\wp_{0,r})$ consisting of positive functions, that is,

$$u_k \in H^1_0(B_r(0)), \quad \|u_k\|^2 = |u_k|^p_p, \quad u_k = u_k^+, \quad \lim_{k\to\infty} E_0(u_k) = \frac{1}{N} S^{\frac{N}{2}}.$$

Then $v_k = u_k(\cdot - x) - u_k(\cdot + x) \in \mathcal{N}^\tau_0$ and

$$\mu^\tau_0 \leq \lim_{k\to\infty} E_0(v_k) = \lim_{k\to\infty} 2E_0(u_k) = \frac{2}{N} S^{\frac{N}{2}}.$$

To prove the last assertion we argue by contradiction. If $u \in \mathcal{N}^\tau_0$ were such that $E_0(u) = \frac{2}{N} S^{\frac{N}{2}}$ then, since $u^+, u^- \in \mathcal{N}_0$ and $E_0(u) = E_0(u^+) + E_0(u^-)$, it would follow that $E_0(u^+) = E_0(u^-) = \frac{1}{N} S^{\frac{N}{2}}$. But $\frac{1}{N} S^{\frac{N}{2}}$ is not achieved by E_0 on $\mathcal{N}_0$. Therefore $\frac{2}{N} S^{\frac{N}{2}}$ is not achieved by E_0 on $\mathcal{N}^\tau_0$. ■

Let u be solution of problem $(\wp_\lambda)$. Then it is of class C^2. One says that u changes sign n times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $n+1$ connected components. If u is a solution of problem $(\wp^\tau_\lambda)$ then it changes sign an odd number of times.

Proposition 6. *If u is a solution of problem $(\wp^\tau_\lambda)$ which changes sign $2m-1$ times, then $E_\lambda(u) \geq m\mu^\tau_\lambda$.*

Proof. The set $\{x \in \Omega : u(x) > 0\}$ has m connected components $A_1, ..., A_m$. Let $u_i(x) = u(x)$ if $x \in A_i \cup \tau A_i$ and $u_i(x) = 0$ otherwise. Since u is a critical point of E_λ,

$$DE_\lambda(u)u_i = \int_\Omega (\nabla u \nabla u_i - \lambda u u_i - |u|^{p-2} u u_i) = \|u_i\|_\lambda^2 - |u_i|_p^p = 0.$$

Thus, $u_i \in \mathcal{N}_\lambda^\tau$ for all $i = 1, ..., m$, and

$$E_\lambda(u) = E_\lambda(u_1) + \cdots + E_\lambda(u_m) \geq m\mu_\lambda^\tau.$$

■

3. A compactness condition

We recall that a sequence (u_k) in $H_0^1(\Omega)$ such that

$$E_\lambda(u_k) \to c, \qquad \|DE_\lambda(u_k)\|_{H^{-1}(\Omega)} \to 0,$$

as $k \to \infty$ is called a Palais-Smale sequence (PS-sequence for short) for E_λ at the level c. Struwe has given a complete description of all PS-sequences for E_λ [12] [13, Theorem 3.1]. For PS-sequences in $H_0^1(\Omega)^\tau$ a more precise description may be given as follows.

Theorem 7. *Let (u_k) be a PS-sequence for E_λ such that $u_k \in H_0^1(\Omega)^\tau$. Then, after replacing (u_k) by a subsequence if necessary, there exist a solution u of problem $(\wp_\lambda^\tau)$, two numbers $m_1, m_2 \geq 0$ and, for each $1 \leq i \leq m = m_1 + m_2$, a sequence $(y_{i,k})$ in Ω, a sequence $(\varepsilon_{i,k})$ in $(0, \infty)$, and a solution $(\widetilde{u}_i)$ of the limiting problem*

$$(\wp_\infty) \qquad \begin{cases} \Delta u + |u|^{2^*-2} u = 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to 0 \end{cases}$$

such that

(i) $\varepsilon_{i,k}^{-1} dist(y_{i,k}, \partial\Omega) \to \infty$ as $k \to \infty$ for each $i = 1, ..., m$,

(ii) $\tau y_{i,k} \neq y_{i,k}$ and $\varepsilon_{i,k}^{-1} |y_{i,k} - \tau y_{i,k}| \to \infty$ as $k \to \infty$ for each $i = 1, ..., m_1$,

(iii) $\tau y_{i,k} = y_{i,k}$ and $\tau\widetilde{u}_i = \widetilde{u}_i$ for $i = m_1 + 1, ..., m$, for all $k \in \mathbb{N}$,

(iv)

$$\begin{aligned} u_k &= u + \sum_{i=1}^{m_1} \varepsilon_{i,k}^{\frac{2-N}{2}} \left[\widetilde{u}_i \left(\frac{\cdot \; - y_{i,k}}{\varepsilon_{i,k}} \right) + (\tau \widetilde{u}_i) \left(\frac{\cdot \; - \tau y_{i,k}}{\varepsilon_{i,k}} \right) \right] \\ &+ \sum_{i=m_1+1}^{m} \varepsilon_{i,k}^{\frac{2-N}{2}} \widetilde{u}_i \left(\frac{\cdot \; - y_{i,k}}{\varepsilon_{i,k}} \right) + o(1) \end{aligned}$$

where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^N)$ as $k \to \infty$, *(v)* $E_\lambda(u_k) \to E_\lambda(u) + 2\sum_{i=1}^{m_1} E_0(\widetilde{u}_i) + \sum_{i=m_1+1}^{m} E_0(\widetilde{u}_i)$ *as* $k \to \infty$.

The proof goes along the lines of [3, Theorem 1]. We postpone it to Section 5. Now we point out some consequences which are relevant to our purposes.

We say that E_λ satisfies the τ-Palais-Smale condition $(PS)_c^\tau$ at the level c if every sequence (u_k) such that

$$u_k \in H_0^1(\Omega)^\tau, \qquad E_\lambda(u_k) \to c \text{ and } \|DE_\lambda(u_k)\|_{H^{-1}(\Omega)} \to 0 \text{ as } k \to \infty$$

has a convergent subsequence. An immediate consequence of Theorem 7 is the following.

Corollary 8. *E_λ satisfies $(PS)_c^\tau$ at every $c < \frac{2}{N}S^{\frac{N}{2}}$.*

Proof. If $\widetilde{u}$ is a nontrivial solution of problem $(\wp_\infty)$ then $E_0(\widetilde{u}) \geq \frac{1}{N}S^{\frac{N}{2}}$. If, furthermore, $\tau\widetilde{u} = \widetilde{u}$ then $E_0(\widetilde{u}) \geq \frac{2}{N}S^{\frac{N}{2}}$. So, if (u_k) is a PS-sequence at the level c such that $u_k \in H_0^1(\Omega)^\tau$, both numbers m_1 and m_2 provided by Theorem 7 must be cero. Hence, up to a subsequence, $u_k \to u$. ■

Recall that, up to sign, the nontrivial least energy solutions of the limiting problem $(\wp_\infty)$ are the instantons

$$U_{\varepsilon,z}(x) = a_N \left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N-2}{2}},$$

$$a_N = [N(N-2)]^{\frac{N-2}{4}}, \ \varepsilon > 0, \ z \in \mathbb{R}^N.$$

cf. [1],[14]. They satisfy

$$\int_{\mathbb{R}^N} |\nabla U_{\varepsilon,z}|^2 = S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{2^*}.$$

Another consequence of Theorem 7 is the following.

Corollary 9. *If (u_k) is a PS-sequence for E_0 in $H_0^1(\Omega)^\tau$ such that $E_0(u_k) \to \frac{2}{N}S^{\frac{N}{2}}$ then there is a sequence of points (y_k) in Ω and a sequence of positive real numbers (ε_k) such that*

(i) $(\varepsilon_k)^{-1} dist(y_k, \partial\Omega) \to \infty$ *as* $k \to \infty$,

(ii) $y_k \neq \tau y_k$ *for all* $k \in N$, *and* $\varepsilon_k^{-1}|y_k - \tau y_k| \to \infty$ *as* $k \to \infty$,

(iii) $\|u_k - U_{\varepsilon_k,y_k} + U_{\varepsilon_k,\tau y_k}\| \to 0$ *in* $D^{1,2}(\mathbb{R}^N)$ *as* $k \to \infty$.

Proof. If there were a solution $\widetilde{u}$ of $(\wp_\infty)$ such that $\tau\widetilde{u} = \widetilde{u}$ and $E_0(\widetilde{u}) = \frac{2}{N}S^{\frac{N}{2}}$ then

$$\int_{\mathbb{R}^N} |\nabla \widetilde{u}^\pm|^2 = S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |\widetilde{u}^\pm|^{2^*}.$$

Thus, $\widetilde{u}^+$ and $\widetilde{u}^-$ would be solutions of $(\wp_\infty)$ which vanish in some open subset of $\mathbb{R}^N$. This is a contradiction. Therefore every solution $\widetilde{u}$ of $(\wp_\infty)$ such that $\tau\widetilde{u} = \widetilde{u}$ must satisfy $E_0(\widetilde{u}) > \frac{2}{N}S^{\frac{N}{2}}$. On the other hand, we have shown (Proposition 5) that problem $(\wp_0^\tau)$ does not have a nontrivial least energy solution in Ω. So Theorem 7 implies the result. ■

Proof of Theorem 1. Take a minimizing sequence (u_k) for E_λ on $\mathcal{N}_\lambda^\tau$. By Ekeland's variational principle [7] [16, Theorem 8.5] we may assume that it is a PS-sequence. Proposition 5 and Corollary 8 yield the existence of a minimum of E_λ on $\mathcal{N}_\lambda^\tau$, and Proposition 6 asserts that it changes sign exactly once. ■

4. The effect of the domain

We recall some facts about equivariant Lusternik-Schnirelmann theory. If G is a compact Lie group, then a G-space is a topological space X with a continuous G-action $G \times X \to X$, $(g,x) \mapsto gx$. A G-map is a continuous function $f : X \to Y$ between G-spaces X and Y which is compatible with the G-actions, that is, $f(gx) = gf(x)$ for all $x \in X$, $g \in G$. Two G-maps $f_0, f_1 : X \to Y$ are G -homotopic if there is a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(gx,t) = g\Theta(x,t)$ for all $x \in X$, $g \in G$, $t \in [0,1]$. A subset A of a X is G-invariant if $ga \in A$ for every $a \in A$, $g \in G$. The G-orbit of a point $x \in X$ is the set $Gx = \{gx : g \in G\}$. A detailed discussion on G-spaces may be found for example in [6].

In our applications G will be the group with two elements, acting as $G_\tau = \{I, \tau\}$ on Ω, and as $\mathbb{Z}/2 = \{1, -1\}$ by multiplication on the Nehari manifold $\mathcal{N}_\lambda^\tau$. The energy functional $E_\lambda : \mathcal{N}_\lambda^\tau \to \mathbb{R}$ is a $\mathbb{Z}/2$-map for this action, in other words, it is an even functional.

Definition 10. *The G-category of a G-map $f : X \to Y$ is the smallest number $k = G$ -cat(f) of open G-invariant subsets $X_1, ..., X_k$ of X which cover X and which have the property that, for each $i = 1, ..., k$, there is a point $y_i \in Y$ and a G-map $\alpha_i : X_i \to Gy_i \subset Y$ such that the restriction of f to X_i is G-homotopic to α_i . If no such covering exists we define G-cat$(f) = \infty$.*

If A is a G-subset of X and $\iota : A \hookrightarrow X$ is the inclusion map we write

$$G\text{-}cat_X(A) = G\text{-}cat(\iota) \qquad \text{and} \qquad G\text{-}cat(X) = G\text{-}cat_X(X).$$

The following properties can be easily verified.

Lemma 11. *a) If $f : X \to Y$ and $h : Y \to Z$ are G-maps then*

$$G\text{-}cat(h \circ f) \leq \min\{G\text{-}cat(f), G\text{-}cat(h)\}.$$

In particular, G-$cat(h \circ f) \leq G$-$cat(Y)$.

b) If $f_0, f_1 : X \to Y$ are G-homotopic then G-$cat(f_0) = G$- $cat(f_1)$.
■

Equivariant Lusternik-Schnirelmann category provides a lower bound for the number of critical G-orbits of a G-invariant functional. The following result is well known, see for example [4, Theorem 1.1], [13, Theorem 5.7].

Theorem 12. *Let $\phi : M \to \mathbb{R}$ be an even C^1-functional on a complete $C^{1,1}$-submanifold M of a Banach space which is symmetric with respect to the origin. Assume that ϕ is bounded below and satisfies the Palais-Smale condition $(PS)_c$ for every $c \leq d$. Then ϕ has at least $\mathbb{Z}/2$-$cat(\phi^d)$ antipodal pairs $\{u, -u\}$ of critical points with critical values $\phi(\pm u) \leq d$.*

Here ϕ^d stands, as usual, for the sublevel set

$$\phi^d = \{u \in M : \phi(u) \leq d\}$$

and the group $\mathbb{Z}/2 = \{1, -1\}$ acts by multiplication on V. There is a similar result for arbitrary group actions [4, Theorem 1.1].

Coming back to our problem, we assume from now on that $N \geq 4$ and $0 < \lambda < \lambda_1(\Omega)$. Given $r > 0$ let

$$\begin{aligned} \Omega_r^- &= \{x \in \Omega : dist(x, \partial\Omega \cup \Omega^\tau) \geq r\} \\ \Omega_r^+ &= \{x \in \mathbb{R}^N : dist(x, \Omega) \leq r\} \end{aligned}$$

where $\Omega^\tau = \{x \in \Omega : \tau x = x\}$. Fix $r > 0$ such that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \backslash \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are G_τ -homotopy equivalences where $G_\tau = \{I, \tau\}$. We shall start by proving the following.

Proposition 13. *Let $N \geq 4$. There is a $0 < \lambda^* < \lambda_1(\Omega)$ and, for each $0 < \lambda < \lambda^*$, there is a $\mu_\lambda^\tau < d_\lambda < \frac{2}{N} S^{\frac{N}{2}}$ and two maps*

$$\Omega_r^- \xrightarrow{\alpha_\lambda} \mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda} \xrightarrow{\beta_\lambda} \Omega_r^+$$

such that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$, $\beta_\lambda(-u) = \tau\beta_\lambda(u)$, and $\beta_\lambda \circ \alpha_\lambda$ is G_τ-homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$.

For the proof we need the following lemmas. Consider the baricenter map $\beta : H_0^1(\Omega)\backslash\{0\} \to \mathbb{R}^N$,

$$\beta(u) = \frac{\int_{\mathbb{R}^N} |u(x)|^p \, x dx}{\int_{\mathbb{R}^N} |u(x)|^p \, dx}.$$

Lemma 14. *Given $r > 0$ there exists $\kappa > 0$ such that, if $u \in \mathcal{N}_0^\tau$ and $E_0(u) \leq \frac{2}{N} S^{\frac{N}{2}} + 2\kappa$, then*

$$\beta(u^+) \in \Omega_r^+.$$

Proof. We argue by contradiction. Assume that for every $k \in \mathbb{N}$ there is a $u_k \in \mathcal{N}_0^\tau$ such that $\beta(u_k^+) \notin \Omega_r^+$ and $E_0(u_k) \leq \frac{2}{N} S^{\frac{N}{2}} + \frac{1}{2k}$. Then (u_k^+) is a minimizing sequence for E_0 in $\mathcal{N}_0$. By Ekeland's variational principle [7], [16, Theorem 8.5] we may assume that (u_k^+) is a PS-sequence. Thus, by Struwe's theorem [12] [13, Theorem 3.1], there are sequences (y_k) in Ω and (ε_k) in $(0, \infty)$ such that

$$\left\| u_k^+ - U_{\varepsilon_k, y_k} \right\| \to 0 \quad \text{in } D^{1,2}(\mathbb{R}^N) \;\; \text{as } k \to \infty.$$

Therefore

$$\left| \beta(u_k^+) - \beta(U_{\varepsilon_k, y_k}) \right| = \left| \beta(u_k^+) - y_k \right| \to 0 \;\; \text{as } k \to \infty,$$

contradicting our assumption that $\beta(u_k^+) \notin \Omega_r^+$. ■

Let $\rho_\lambda : H_0^1(\Omega) \backslash \{0\} \to \mathcal{N}_\lambda$ be the radial projection onto the Nehari manifold, that is,

$$\rho_\lambda(u) = \frac{\|u\|_\lambda^{(N-2)/2}}{|u|_p^{N/2}} u.$$

Lemma 15. *For every $\kappa > 0$ there is a $0 < \lambda^* < \lambda_1(\Omega)$ such that, for each $0 < \lambda < \lambda^*$, the following holds:*

(a) If $u \in \mathcal{N}_\lambda^\tau$ is such that $E_\lambda(u) \leq \mu_\lambda^\tau + \kappa$ then $\rho_0(u) \in \mathcal{N}_0^\tau$ is such that $E_0(\rho_0(u)) \leq \frac{2}{N} S^{\frac{N}{2}} + 2\kappa$.

(b) $\frac{2}{N} S^{\frac{N}{2}} \leq \mu_\lambda^\tau + \kappa$.

Proof. *(a)* If $u \in \mathcal{N}_\lambda$ then

$$\rho_0(u) = \frac{\|u\|^{(N-2)/2}}{|u|_p^{N/2}} u = \left(\frac{\|u\|}{\|u\|_\lambda} \right)^{\frac{N-2}{2}} u.$$

Let $\theta > 0$ be such that $(1+\theta)^{\frac{N}{2}}(\frac{2}{N}S^{\frac{N}{2}}+\kappa) \leq \frac{2}{N}S^{\frac{N}{2}}+2\kappa$. Then, if $\lambda \leq \left(\frac{\theta}{1+\theta}\right)\lambda_1(\Omega)$ and $E_\lambda(u) \leq \mu_\lambda^\tau + \kappa$,

$$\begin{aligned} E_0(\rho_0(u)) &= \frac{1}{N}\left(\frac{\|u\|}{\|u\|_\lambda}\right)^{N-2}\|u\|^2 \\ &= \frac{1}{N}\left(\frac{\|u\|}{\|u\|_\lambda}\right)^{N}\|u\|_\lambda^2 \\ &\leq \left(\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)-\lambda}\right)^{\frac{N}{2}}\frac{1}{N}\|u\|_\lambda^2 \\ &\leq (1+\theta)^{\frac{N}{2}}E_\lambda(u) \\ &\leq \frac{2}{N}S^{\frac{N}{2}}+2\kappa. \end{aligned}$$

(b) By Proposition 5, $2\mu_\lambda \leq \mu_\lambda^\tau < \frac{2}{N}S^{\frac{N}{2}}$. Since $\mu_\lambda \to \frac{1}{N}S^{\frac{N}{2}}$ as $\lambda \to 0$ there exists $0 < \lambda^* \leq \left(\frac{\theta}{1+\theta}\right)\lambda_1(\Omega)$ such that

$$\frac{2}{N}S^{\frac{N}{2}} - \mu_\lambda^\tau < \kappa$$

if $0 < \lambda < \lambda^*$. ■

We are ready to prove Proposition 13.

Proof of Proposition 13. For $\kappa > 0$ as in Lemma 14 choose λ^* as in Proposition 15. Fix $0 < \lambda < \lambda^*$ and let $u_{\lambda,r}$ be the positive ground state solution of the problem

$$(\wp_{\lambda,r}) \qquad \begin{cases} \Delta u + \lambda u + \mid u \mid^{2^*-2} u = 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases}$$

Choose d_λ such that $2E_\lambda(u_{\lambda,r}) < d_\lambda < \frac{2}{N}S^{\frac{N}{2}}$ and define

$$\begin{aligned} \alpha_\lambda &: \Omega_r^- \to \mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda}, \quad \alpha_\lambda(x) = u_{\lambda,r}(\ \cdot\ - x) - u_{\lambda,r}(\ \cdot\ - \tau x), \\ \beta_\lambda &: \mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda} \to \Omega_r^+, \quad \beta_\lambda(u) = \beta(\rho_0(u)^+) = \beta(u^+). \end{aligned}$$

These maps have obviously the desired properties. ■

Proof of Theorem 2. Let $0 < \lambda^* < \lambda_1(\Omega)$ and, for $0 < \lambda < \lambda^*$, let $\mu_\lambda^\tau < d_\lambda < \frac{2}{N}S^{\frac{N}{2}}$ be as in Proposition 13 above. Since $E_\lambda : \mathcal{N}_\lambda^\tau \to \mathbb{R}$ is even, bounded below and satisfies $(PS)_c^\tau$ for $c < \frac{2}{N}S^{\frac{N}{2}}$ it follows from Theorem 12 that E_λ has at least $\mathbb{Z}/2$-$cat(\mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda})$ pairs $\pm u$ of critical points in $\mathcal{N}_\lambda^\tau$ with $E_\lambda(\pm u) \leq d_\lambda$ where $\mathbb{Z}/2 = \{1, -1\}$ acts by multiplication on $\mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda}$. On the other hand, Lemma 11 and

Proposition 13 imply that

$$\begin{aligned} G_\tau\text{-}cat_\Omega(\Omega \setminus \Omega^\tau) &= G_\tau\text{-}cat(\Omega_r^- \hookrightarrow \Omega_r^+) \\ &= G_\tau\text{-}cat(\beta_\lambda \circ \alpha_\lambda) \leq \mathbb{Z}/2\text{-}cat(\mathcal{N}_\lambda^\tau \cap E_\lambda^{d_\lambda}). \end{aligned}$$

Taking λ^* even smaller if necessary we may assume that $d_\lambda < 2\mu_\lambda^\tau$. Thus, by Proposition 6, these solutions change sign exactly once. ■

Proof of Corollary 3. If $\{I, -I\}$-$cat(\Omega) = k$, then there exists an odd map $\Omega \to \mathbb{S}^{k-1}$. Indeed: Given an open covering $\{X_1, ..., X_k\}$ of Ω and odd maps $\alpha_i : X_i \to \{e_i, -e_i\}$, where $\{e_1, ..., e_k\}$ is the canonical orthonormal basis of $\mathbb{R}^k$, let $\{\pi_i : X_i \to [0, 1]\}$ be a partition of unity subordinated to the covering consisting of even functions. Then

$$\psi(x) = \frac{\sum_{i=1}^{k} \pi_i(x)\alpha_i(x)}{\left\| \sum_{i=1}^{k} \pi_i(x)\alpha_i(x) \right\|}$$

defines an odd map $\psi : \Omega \to \mathbb{S}^{k-1}$. Composing it with φ gives an odd map $\psi \circ \varphi : \mathbb{S}^{N-1} \to \mathbb{S}^{k-1}$ and the Borsuk-Ulam theorem implies that $N \geq k$. The result now follows from Theorem 2. ■

Proof of Theorem 4. The solutions u_k to problem $(\wp_{\lambda_k}^\tau)$ provided by Theorems 1 and 2 satisfy $u_k \in \mathcal{N}_{\lambda_k}^\tau$ and $\mu_{\lambda_k}^\tau \leq E_{\lambda_k}(u_k) < \frac{2}{N} S^{\frac{N}{2}}$. As in Lemma 15 above

$$\frac{2}{N} S^{\frac{N}{2}} \leq E_0(\rho_0(u_k)) \leq \left(\frac{\lambda_1(\Omega)}{\lambda_1(\Omega) - \lambda_k} \right)^{\frac{N}{2}} E_{\lambda_k}(u_k) \to \frac{2}{N} S^{\frac{N}{2}}$$

as $k \to \infty$, where ρ_0 is the radial projection onto $\mathcal{N}_0$. By Corollary 9 the sequence $(\rho_0(u_k))$ has the desired form. On the other hand,

$$\|\rho_0(u_k) - u_k\| = \left| \left(\frac{\|u_k\|}{\|u_k\|_{\lambda_k}} \right)^{\frac{N-2}{2}} - 1 \right| \|u_k\| \to 0$$

as $k \to \infty$. ■

5. Proof of Theorem 7

As in Struwe's global compactness result [12], [13], [16], Theorem 7 follows inductively from the following proposition (cf. [3]).

Proposition 16. *Let (u_k) be a PS-sequence for E_0 such that $u_k \in H_0^1(\Omega)^\tau$, $u_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)^\tau$ and $E_0(u_k) \to c > 0$. Then, replacing (u_k) by a subsequence if necessary, there exist a sequence (y_k) in Ω, a sequence ε_k in $(0,\infty)$, a solution $\widetilde{u}$ of the limiting problem*

$$(\wp_\infty) \qquad \begin{cases} \Delta u + \mid u \mid^{2^*-2} u = 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } \ |x| \to 0 \end{cases}$$

and a PS-sequence (v_k) for E_0 such that $v_k \in H_0^1(\Omega)^\tau$, $\varepsilon_k^{-1}\, dist(y_k, \partial\Omega) \to \infty$ as $k \to \infty$ and one of the following two assertions holds:

(I) $y_k \neq \tau y_k,\ \ \varepsilon_k^{-1}\,|y_k - \tau y_k| \to \infty$ *as* $k \to \infty$,
$u_k = v_k + \varepsilon_k^{\frac{2-N}{2}} \left[\widetilde{u}\left(\frac{\cdot\, - y_k}{\varepsilon_k}\right) - \widetilde{u}\left(\tau\left(\frac{\cdot\, - \tau y_k}{\varepsilon_k}\right)\right)\right] + o(1)$
in $D^{1,2}(\mathbb{R}^N)$,
and
$E_0(v_k) \to c - 2E_0(\widetilde{u})$ *as* $k \to \infty$.

(II) $y_k = \tau y_k,\ \ \widetilde{u} \circ \tau = -\widetilde{u}$,
$u_k = v_k + \varepsilon_k^{\frac{2-N}{2}} \widetilde{u}\left(\frac{\cdot\, - y_k}{\varepsilon_k}\right) + o(1)$ *in* $D^{1,2}(\mathbb{R}^N)$,
and
$E_0(v_k) \to c - E_0(\widetilde{u})$ *as* $k \to \infty$.

Proof. The proof will follow in several steps:

1) Since PS-sequences for E_0 are bounded in $H_0^1(\Omega)$,

$$\int_\Omega |u_k|^{2^*}\, dx = NE_0(u_k) - \frac{N}{2} DE_0(u_k)u_k \to Nc > 0.$$

Let $\delta = \min\{\frac{Nc}{2}, (\frac{S}{2})^{\frac{N}{2}}\}$ where S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$. Let $B(x,r)$ denote the closed ball in $\mathbb{R}^N$ with center x and radius r. The Levy concentration function

$$\Phi_k(r) := \sup_{x\in\mathbb{R}^N} \int_{B(x,r)} |u_k|^{2^*}$$

satisfies that $\Phi_k(0) = 0$ and $\Phi_k(\infty) > \delta$ for k large enough. Hence we may choose $\xi_k \in \Omega$ and $\varepsilon_k > 0$ such that

$$\sup_{x\in\mathbb{R}^N} \int_{B(x,\varepsilon_k)} |u_k|^{2^*} = \int_{B(\xi_k,\varepsilon_k)} |u_k|^{2^*} = \delta, \tag{$*$}$$

Observe that, since Ω is bounded, the sequence (ε_k) is bounded.

2) Let ξ_k^τ be the orthogonal projection of ξ_k onto the fixed point set $\{x \in \mathbb{R}^N : \tau x = x\}$. We distinguish two cases and define y_k as follows:

(I) $\quad y_k = \xi_k \quad$ if $\ (\varepsilon_k^{-1}\,|\xi_k - \xi_k^\tau|)$ is unbounded,

(II) $y_k = \xi_k^\tau$ if $(\varepsilon_k^{-1}|\xi_k - \xi_k^\tau|)$ is bounded.

Taking a subsequence if necessary we may assume that, in case *(I)*, $y_k \neq \tau y_k$ for all k. Let $\widetilde{u}_k \in D^{1,2}(\mathbb{R}^N)$ be given by

$$\widetilde{u}_k(z) := \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k z + y_k).$$

Notice that in case *(II)* $\widetilde{u}_k \circ \tau = -\widetilde{u}_k$. Since

$$\int |\nabla \widetilde{u}_k|^2 = \int |\nabla u_k|^2 \quad \text{and} \quad \int |\widetilde{u}_k|^{2^*} = \int |u_k|^{2^*},$$

up to a subsequence, $\widetilde{u}_k \rightharpoonup \widetilde{u}$ weakly in $D^{1,2}(\mathbb{R}^N)$, $\widetilde{u}_k \to \widetilde{u}$ a.e. on $\mathbb{R}^N$ and $\widetilde{u}_k \to \widetilde{u}$ in $L^2_{loc}(\mathbb{R}^N)$. If $\widetilde{u} \equiv 0$ then, for every $z \in \mathbb{R}^N$ and every $h \in C_c^\infty(B(z,1))$,

$$\begin{aligned}
&S\left(\int |h\widetilde{u}_k|^{2^*}\right)^{\frac{2}{2^*}} \\
&\quad\leq \int |\nabla(h\widetilde{u}_k)|^2 = \int \nabla\widetilde{u}_k \cdot \nabla(h^2\widetilde{u}_k) + \int |\nabla h|^2 \widetilde{u}_k^2 \\
&\quad= \int h^2 |\widetilde{u}_k|^{2^*} - DE_0(u_k)\left(\left[h^2\left(\frac{\cdot - y_k}{\varepsilon_k}\right)\right] u_k\right) + o(1) \\
&\quad\leq \left(\int_{B(z,1)} |\widetilde{u}_k|^{2^*}\right)^{\frac{2}{N}} \left(\int |h\widetilde{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1) \\
&\quad\leq \delta^{\frac{2}{N}} \left(\int |h\widetilde{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1) \\
&\quad\leq \frac{S}{2}\left(\int |h\widetilde{u}_k|^{2^*}\right)^{\frac{2}{2^*}} + o(1),
\end{aligned}$$

where the first inequality is Sobolev's inequality, the second one follows from the fact that (u_k) is a PS-sequence and from Hölder's inequality, the third one uses $(*)$ and the fourth one our definition of δ. It follows that $\widetilde{u}_k \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$. On the other hand, since $\varepsilon_k^{-1}|\xi_k - y_k| < C < \infty$ for all k,

$$0 < \delta = \int_{B(\xi_k,\varepsilon_k)} |u_k|^{2^*} \leq \int_{B(y_k,\varepsilon_k(C+1))} |u_k|^{2^*} = \int_{B(0,C+1)} |\widetilde{u}_k|^{2^*}.$$

This is a contradiction. Therefore, $\widetilde{u} \not\equiv 0$.

3) Since Ω is bounded and $u_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, up to a subsequence, $y_k \to y \in \overline{\Omega}$ and $\varepsilon_k \to 0$.

If $(\varepsilon_k^{-1} dist(y_k, \partial\Omega))$ were bounded then $\widetilde{u}$ would be a solution of $-\Delta u = |u|^{2^*-2} u$ in a half space and, by Pohozaev's identity [10], $\widetilde{u} \equiv 0$.

This is a contradiction. Therefore,

$$\varepsilon_k^{-1} dist(y_k, \partial\Omega) \to \infty$$

and $\widetilde{u}$ is a nontrivial solution of the limiting problem $(\wp_\infty)$ in $\mathbb{R}^N$. Moreover, since $\varepsilon_k^{-1} |\xi_k - y_k| < C < \infty$ for all k, it follows that $y_k \in \Omega$.

4) We define $v_k \in H_0^1(\Omega)^\tau$ as follows: Let $\varphi \in C^\infty(\mathbb{R}^N)$ be radially symmetric and such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B(0,1)$ and $\varphi \equiv 0$ outside of $B(0,2)$. Let

$$4\rho_k = \begin{cases} \min\{dist(y_k, \partial\Omega), |y_k - \tau y_k|\} & \text{in case } \textit{(I)} \\ dist(y_k, \partial\Omega) & \text{in case } \textit{(II)} \end{cases}.$$

Thus, $\varepsilon_k^{-1}\rho_k \to \infty$. In case *(I)* we take

$$w_k = \varepsilon_k^{\frac{2-N}{2}} \left[\widetilde{u}(\varepsilon_k^{-1}(\cdot - y_k))\ \varphi(\rho_k^{-1}(\cdot - y_k)) - \widetilde{u}(\varepsilon_k^{-1}\tau(\cdot - \tau y_k))\ \varphi(\rho_k^{-1}(\cdot - \tau y_k))\right]$$

and in case *(II)*

$$w_k = \varepsilon_k^{\frac{2-N}{2}} \widetilde{u}(\varepsilon_k^{-1}(\cdot - y_k))\ \varphi(\rho_k^{-1}(\cdot - y_k)).$$

In both cases $w_k \circ \tau = -w_k$. We define

$$v_k = u_k - w_k \in H_0^1(\Omega)^\tau.$$

As in [13], [16], [3] one verifies that v_k has the desired properties. ■

Proof of Theorem 7. Since PS-sequences for E_λ are bounded in $H_0^1(\Omega)$,

$$\|u_k\|_\lambda^2 = NE_\lambda(u_k) - \frac{N-2}{2} DE_\lambda(u_k)u_k \to Nc.$$

Therefore $c \geq 0$. We may assume that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)^\tau$ and $u_k \to u$ a.e. in Ω. It is easy to see that $DE_\lambda(u) = 0$ and that $u_k^1 := u_k - u$ is a PS-sequence for E_0 such that $u_k^1 \in H_0^1(\Omega)^\tau$, $u_k^1 \rightharpoonup 0$ weakly in $H_0^1(\Omega)^\tau$ and $E_0(u_k^1) = E_\lambda(u_k) - E_\lambda(u) = c - E_\lambda(u) + o(1)$. The result now follows inductively from Proposition 16 above. ■

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