

MINIMAL NODAL SOLUTIONS OF A SCHRÖDINGER EQUATION WITH CRITICAL NONLINEARITY AND SYMMETRIC POTENTIAL

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Abstract. We study the nonlinear Schrödinger equation

$$-\Delta u + \lambda a(x)u = \mu u + u^{2^*-1}, \quad u \in \mathbb{R}^N,$$

with critical exponent $2^* = 2N/(N-2)$, $N \geq 4$, where $a \geq 0$ has a potential well and is invariant under an orthogonal involution of \mathbb{R}^N . Using variational methods we establish existence and multiplicity of solutions which change sign exactly once. These solutions localize near the potential well for μ small and λ large.

1. INTRODUCTION AND STATEMENT OF RESULTS

In recent years much attention has been paid to the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi + a(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N \quad (1.1)$$

where \hbar is the Planck constant. When looking for stationary waves of the form $\psi(t, x) = e^{-i\mu\hbar t} \hbar^{2/(p-2)} u(x)$ with $\mu \in \mathbb{R}$, and writing $\lambda = \hbar^{-2}$, one is led to considering the elliptic equation

$$-\Delta u + \lambda a(x)u = \mu u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

The goal of this paper is to investigate the number of solutions u of equation (1.2) which change sign exactly once; that is, each of the sets $\{x \in \mathbb{R}^N : u(x) > 0\}$ and $\{x \in \mathbb{R}^N : u(x) < 0\}$ is nonempty and connected. We assume that $a(x) \geq 0$ has a nonempty potential well $\Omega = \text{int } a^{-1}(0)$ and that p is the critical Sobolev exponent $2^* = \frac{2N}{N-2}$.

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Recently Bartsch and Wang [1] studied equation (1.2) for $a(x)$ having a potential well, $\mu = -1$, and $2 < p < 2^*$. They observed that, as $\lambda \rightarrow \infty$, positive low-energy solutions concentrate at a positive solution of the Dirichlet problem $-\Delta u + u = u^{p-1}$ in Ω , $u = 0$ on $\partial\Omega$. Motivated by a result of Benci and Cerami [2] concerning the effect of the domain topology on the Dirichlet problem near the critical exponent, they showed that there is an effect of the topology of the potential well on the number of positive solutions of equation (1.2) for p sufficiently close to the critical exponent and λ large enough.

In [6] we investigated the critical exponent case. For $p = 2^*$ positive low-energy solutions of equation (1.2) concentrate at positive solutions of the Dirichlet problem

$$-\Delta u = \mu u + |u|^{2^*-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

as $\lambda \rightarrow \infty$. It is known that for $N \geq 4$ there is an effect of the domain topology on the number of positive solutions of (3) for μ small enough [10], [9]. Motivated by this result we showed in [6] that there is an effect of the topology of the potential well Ω on the number of positive solutions of equation (1.2) for $p = 2^*$, $0 < \mu$ small enough and λ large enough.

Recently Castro and the first author showed [5] that there is also an effect of the domain topology on the number of solutions of the Dirichlet problem (1.3) which change sign exactly once, whenever the domain Ω has certain symmetries. Here we shall show that a similar result holds true for the Schrödinger equation (1.2).

More precisely, we consider the problem

$$(S_{\lambda,\mu}^\tau) \quad \begin{cases} -\Delta u + \lambda a(x)u = \mu u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \\ u(\tau x) = -u(x) & \text{for all } x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 4$, $2^* = \frac{2N}{N-2}$, $\lambda > 0$, and $\mu \in \mathbb{R}$, $\tau : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a nontrivial orthogonal involution, that is, an orthogonal linear function such that $\tau \neq I$ and $\tau^2 = I$ (where I denotes the identity of \mathbb{R}^N), and $a(x)$ satisfies the following assumptions:

- (A1) $a : \mathbb{R}^N \rightarrow \mathbb{R}$ is Hölder continuous, $a \geq 0$, $\Omega := \text{int } a^{-1}(0)$ is a nonempty bounded set with smooth boundary, and $\overline{\Omega} = a^{-1}(0)$.
- (A2) There exists $M_0 > 0$ such that

$$\mathcal{L}\{x \in \mathbb{R}^N : a(x) \leq M_0\} < \infty,$$

where \mathcal{L} denotes Lebesgue measure in \mathbb{R}^N .

- (A3) a is τ -invariant; that is, $a(\tau x) = a(x)$ for every $x \in \mathbb{R}^N$.

Observe that the solutions of $(S_{\lambda,\mu}^\tau)$ appear in pairs; that is, $-u$ is a solution if u is a solution. We shall prove the following results.

Theorem 1. *Assume (A1), (A2), and (A3) hold and $N \geq 4$. Then, for every $0 < \mu < \mu_1(\Omega)$, there exists $\lambda(\mu) > 0$ such that, for each $\lambda \geq \lambda(\mu)$, problem $(S_{\lambda,\mu}^\tau)$ has at least one pair of solutions which change sign exactly once.*

Theorem 2. *Assume (A1), (A2), and (A3) hold and $N \geq 4$. Then there exist $0 < \mu^* < \mu_1(\Omega)$ and for each $0 < \mu \leq \mu^*$ a number $\Lambda(\mu) > 0$ such that, for every $\lambda \geq \Lambda(\mu)$, problem $(S_{\lambda,\mu}^\tau)$ has at least $\tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ is the set of fixed points of the involution, and $\tau\text{-cat}$ is the G_τ -equivariant Lusternik-Schnirelmann category for the group $G_\tau = \{I, \tau\}$. We would like to point out that in many cases the equivariant category turns out to be larger than the classical (nonequivariant) one. For example, for the unit sphere \mathbb{S}^{N-1} in \mathbb{R}^N and the antipodal involution $\tau = -I$, $\tau\text{-cat}(\mathbb{S}^{N-1}) = N$ whereas $\text{cat}(\mathbb{S}^{N-1}) = 2$. Thus, Theorem 2 provides many solutions for some potentials like the following.

Corollary 3. *Assume (A1) and (A2) hold and $N \geq 4$. Assume further that $a(x)$ is even, $a(0) \neq 0$, and that there is an odd map $\varphi : \mathbb{S}^{N-1} \rightarrow \Omega$. Then there exist $0 < \mu^* < \mu_1(\Omega)$ and for each $0 < \mu \leq \mu^*$ a number $\Lambda(\mu) > 0$ such that, for every $\lambda \geq \Lambda(\mu)$, the problem*

$$\begin{cases} -\Delta u + \lambda a(x)u = \mu u + |u|^{2^*-2} u & \text{in } \mathbb{R}^N \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

has at least N pairs of odd solutions which change sign exactly once.

The solutions given by Theorems 1 and 2 have the following concentration behavior.

Theorem 4. *Let $\mu \in (0, \mu_1(\Omega))$ and let $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Let u_n be a solution of problem $(S_{\lambda_n,\mu}^\tau)$ given by Theorem 1, or by Theorem 2 if $\mu \leq \mu^*$. Then the sequence (u_n) converges in $H^1(\mathbb{R}^N)$ to a nontrivial solution of the Dirichlet problem*

$$(D_\mu^\tau) \quad \begin{cases} -\Delta u = \mu u + |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(\tau x) = -u(x) & \text{for all } x \in \Omega \end{cases}$$

which changes sign exactly once.

This paper is organized as follows. In Section 2 we describe the variational problem associated to $(S_{\lambda,\mu}^\tau)$. In Section 3 we establish a compactness result and prove Theorem 1. Theorems 2 and 4 will be proved in Section 4.

2. THE VARIATIONAL PROBLEM

We assume throughout that $N \geq 4$ and that assumptions (A1), (A2) and (A3) hold. We denote by $\mu_1(\Omega)$ the first eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$. We write $|\cdot|_q$ for the norm in L^q , (\cdot, \cdot) for the inner product in L^2 and $\|\cdot\|_{H^1}$ for the norm in $H^1(\mathbb{R}^N)$. Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 < \infty \right\}$$

be the Hilbert space endowed with the norm

$$\|u\| = \left(\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} a(x)u^2 \right)^{1/2}.$$

Write

$$\|u\|_\lambda^2 = \|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} a(x)u^2 \quad \text{and} \quad (A_\lambda u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda uv)$$

for $u, v \in E$. Then $\|u\|_\lambda$ is equivalent to the norm $\|u\|$ in E , and it was shown in [6, Lemma 5] that the following holds.

Lemma 5. *For each $\mu \in (0, \mu_1(\Omega))$, there are $\alpha_\mu > 0$, $\lambda(\mu) \geq \mu/M_0$ such that*

$$\alpha_\mu \|u\|_\lambda^2 \leq ((A_\lambda - \mu)u, u)$$

for every $u \in E$, $\lambda \geq \lambda(\mu)$.

Consider the functional

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda a u^2 - \mu u^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \\ &= \frac{1}{2} ((A_\lambda - \mu)u, u) - \frac{1}{2^*} |u|_{2^*}^{2^*}. \end{aligned}$$

$I_{\lambda,\mu} \in C^1(E, \mathbb{R})$, and the critical points of $I_{\lambda,\mu}$ are the solutions of

$$-\Delta u + \lambda a(x)u = \mu u + |u|^{2^*-2} u, \quad u \in H^1(\mathbb{R}^N).$$

The nontrivial critical points lie on the Nehari manifold

$$\begin{aligned} \mathcal{N}_{\lambda,\mu} &= \{u \in E : u \neq 0, DI_{\lambda,\mu}(u)u = 0\} \\ &= \{u \in E : u \neq 0, ((A_\lambda - \mu)u, u) = |u|_{2^*}^{2^*}\}. \end{aligned}$$

This is a manifold of class $C^{1,1}$ which is radially diffeomorphic to the unit sphere in E for $\lambda \geq \lambda(\mu)$, $0 < \mu < \mu_1(\Omega)$. Notice that

$$I_{\lambda,\mu}(u) = \frac{1}{N}((A_\lambda - \mu)u, u) = \frac{1}{N}|u|_{2^*}^{2^*} \quad \text{if } u \in \mathcal{N}_{\lambda,\mu}.$$

The orthogonal involution $\tau : \mathbb{R}^N \rightarrow \mathbb{R}^N$ induces an involution on E , which we denote again by $\tau : E \rightarrow E$, as follows: $(\tau u)(x) := -u(\tau x)$. It satisfies $((A_\lambda - \mu)(\tau u), \tau v) = ((A_\lambda - \mu)u, v)$ and $|\tau u|_{2^*}^{2^*} = |u|_{2^*}^{2^*}$ for all $u, v \in E$. Thus, $I_{\lambda,\mu}$ is τ -invariant, that is, $I_{\lambda,\mu}(\tau u) = I_{\lambda,\mu}(u)$, and $\nabla I_{\lambda,\mu}(\tau u) = \tau \nabla I_{\lambda,\mu}(u)$. In particular, $\tau \nabla I_{\lambda,\mu}(u) = \nabla I_{\lambda,\mu}(u)$ if $\tau u = u$. It follows that the solutions of problem $(S_{\lambda,\mu}^\tau)$ are the critical points of the restriction

$$I_{\lambda,\mu} : \mathcal{N}_{\lambda,\mu}^\tau \rightarrow \mathbb{R}$$

of $I_{\lambda,\mu}$ to the τ -Nehari manifold

$$\mathcal{N}_{\lambda,\mu}^\tau = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap E^\tau,$$

where $E^\tau = \{u \in E : \tau u = u\}$. Set

$$c_{\lambda,\mu} = \inf_{\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu} \quad \text{and} \quad c_{\lambda,\mu}^\tau = \inf_{\mathcal{N}_{\lambda,\mu}^\tau} I_{\lambda,\mu}.$$

Similarly, we consider the functional

$$I_{\mu,\Omega}(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \mu u^2) - \frac{1}{2^*} \int_\Omega |u|^{2^*} = \frac{1}{2} (|\nabla u|_2^2 - \mu |u|_2^2) - \frac{1}{2^*} |u|_{2^*}^{2^*}$$

on $H_0^1(\Omega)^\tau = \{u \in H_0^1(\Omega) : \tau u = u\}$ associated to problem (D_μ^τ) . The nontrivial solutions of (D_μ^τ) are the critical points of $I_{\mu,\Omega}$ on the τ -invariant Nehari manifold

$$\mathcal{N}_{\mu,\Omega}^\tau = \{u \in H_0^1(\Omega)^\tau \setminus \{0\} : |\nabla u|_2^2 - \mu |u|_2^2 = |u|_{2^*}^{2^*}\}.$$

Set

$$c(\mu, \Omega) = \inf_{u \in \mathcal{N}_{\mu,\Omega}} I_{\mu,\Omega}(u) \quad \text{and} \quad c(\mu, \Omega)^\tau = \inf_{u \in \mathcal{N}_{\mu,\Omega}^\tau} I_{\mu,\Omega}(u).$$

As usual, let S denote the best Sobolev constant for the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$; that is,

$$S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^{2^*}}.$$

Proposition 6. *For each $\mu \in (0, \mu_1(\Omega))$ there is a $\lambda(\mu)$ such that the following inequalities hold for every $\lambda \geq \lambda(\mu)$:*

$$0 < \frac{1}{N} (\alpha_\mu S)^{\frac{N}{2}} \leq c_{\lambda,\mu} < c(\mu, \Omega) < \frac{1}{N} S^{\frac{N}{2}}$$

and

$$2c_{\lambda,\mu} \leq c_{\lambda,\mu}^\tau < c(\mu, \Omega)^\tau < \frac{2}{N} S^{\frac{N}{2}}.$$

Furthermore,

$$\lim_{\lambda \rightarrow \infty} c_{\lambda,\mu} = c(\mu, \Omega).$$

Proof. The first row of inequalities were proved in [6, Lemma 9]. We turn to the second row. Let $u^\pm = \pm \max\{\pm u, 0\}$. Observe that, if $u = \tau u$, then $((A_\lambda - \mu)u^+, u^+) = ((A_\lambda - \mu)u^-, u^-)$ and $|u^+|_{2^*}^{2^*} = |u^-|_{2^*}^{2^*}$. So, if $u \in \mathcal{N}_{\lambda,\mu}^\tau$ then $u^+, u^- \in \mathcal{N}_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u^+) + I_{\lambda,\mu}(u^-) \geq 2c_{\lambda,\mu}.$$

This shows that $2c_{\lambda,\mu} \leq c_{\lambda,\mu}^\tau$. Since $\mathcal{N}_{\mu,\Omega}^\tau \subset \mathcal{N}_{\lambda,\mu}^\tau$ and $I_{\lambda,\mu}(u) = I_{\mu,\Omega}(u)$ for every $u \in \mathcal{N}_{\mu,\Omega}^\tau$, it follows that $c_{\lambda,\mu}^\tau \leq c(\mu, \Omega)^\tau$. It was shown in [5] that $c(\mu, \Omega)^\tau < \frac{2}{N} S^{\frac{N}{2}}$ and that $c(\mu, \Omega)^\tau$ is achieved by $I_{\mu,\Omega}$ at some $\tilde{u} \in \mathcal{N}_{\mu,\Omega}^\tau$ if $\mu \in (0, \mu_1(\Omega))$. Therefore $c_{\lambda,\mu}^\tau < c(\mu, \Omega)^\tau$; otherwise, $c_{\lambda,\mu}^\tau$ would also be achieved at $\tilde{u} \neq 0$ which vanishes outside Ω , and hence \tilde{u} would solve $-\Delta u = \mu u + |u|^{2^*-2}u$ in \mathbb{R}^N , contradicting the Pohozaev identity [12, Corollary B.4]. The last assertion was proved in [6, Corollary 10]. \square

One says that a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ changes sign n times if the set $\{x \in \mathbb{R}^N : u(x) \neq 0\}$ has $n + 1$ connected components. If u is a solution of problem $(S_{\lambda,\mu}^\tau)$ then it is of class C^2 and τ induces a bijection between the connected components of $\{x \in \mathbb{R}^N : u(x) > 0\}$ and those of $\{x \in \mathbb{R}^N : u(x) < 0\}$, so u changes sign an odd number of times.

Proposition 7. *If u is a solution of problem $(S_{\lambda,\mu}^\tau)$ which changes sign $2m - 1$ times, then $I_{\lambda,\mu}(u) \geq mc_{\lambda,\mu}^\tau$.*

Proof. The set $\{x \in \mathbb{R}^N : u(x) > 0\}$ has m connected components X_1, \dots, X_m . Let $u_i(x) = u(x)$ if $x \in X_i \cup \tau X_i$ and $u_i(x) = 0$ otherwise. Since u is a critical point of $I_{\lambda,\mu}$,

$$DI_{\lambda,\mu}(u)u_i = ((A_\lambda - \mu)u, u_i) - \int |u|^{2^*-2}uu_i = ((A_\lambda - \mu)u_i, u_i) - |u_i|_{2^*}^{2^*} = 0.$$

Thus, $u_i \in \mathcal{N}_{\lambda,\mu}^\tau$ for all $i = 1, \dots, m$, and

$$I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u_1) + \dots + I_{\lambda,\mu}(u_m) \geq mc_{\lambda,\mu}^\tau.$$

3. A COMPACTNESS CONDITION

A sequence (u_n) in E^τ is called a τ -Palais-Smale sequence for $I_{\lambda,\mu}$ at the level c if

$$I_{\lambda,\mu}(u) \rightarrow c \quad \text{and} \quad DI_{\lambda,\mu}(u_n) \rightarrow 0.$$

$I_{\lambda,\mu}$ is said to satisfy the τ -Palais-Smale condition $(PS)_c^\tau$ if every such sequence has a convergent subsequence.

Proposition 8. *If $\mu \in (0, \mu_1(\Omega))$ and $\lambda \geq \lambda(\mu)$, then $I_{\lambda,\mu}$ satisfies $(PS)_c^\tau$ at every $c < \frac{2}{N}S^{\frac{N}{2}}$.*

Proof. Let $(u_n) \subset E^\tau$ be a τ -Palais-Smale sequence for $I_{\lambda,\mu}$ at the level c . It follows easily from Lemma 5 that (u_n) is bounded in E . So we may assume that $u_n \rightharpoonup u$ weakly in E^τ , $u_n \rightarrow u$ in L^2_{loc} and $u_n(x) \rightarrow u(x)$ for almost every $x \in \mathbb{R}^N$. A standard argument shows that $u \in E^\tau$ is a weak solution of

$$-\Delta u + \lambda a(x)u = \mu u + |u|^{2^*-2}u.$$

One can easily see that

$$((A_\lambda - \mu)u_n, u_n) = Nc + o(1) = |u_n|_{2^*}^{2^*}.$$

Setting $w_n = u_n - u \in E^\tau$ and applying the Brezis-Lieb lemma [3] we obtain

$$((A_\lambda - \mu)w_n, w_n) = b + o(1) = |w_n|_{2^*}^{2^*}$$

with $b \leq Nc < 2S^{\frac{N}{2}}$. Let $F = \{x \in \mathbb{R}^N : a(x) \leq M_0\}$. Since $w_n \rightarrow 0$ in L^2_{loc} and $\mathcal{L}(F) < \infty$, it follows that

$$\int_F w_n^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Recall that $\lambda(\mu) \geq \mu/M_0$. Thus,

$$\begin{aligned} |\nabla w_n|_2^2 &\leq |\nabla w_n|_2^2 + \int_{\mathbb{R}^N \setminus F} (\lambda a - \mu)w_n^2 \\ &\leq ((A_\lambda - \mu)w_n, w_n) + \mu \int_F w_n^2 = b + o(1). \end{aligned}$$

Since $w_n \in E^\tau$, we have $|\nabla w_n|_2^2 = 2|\nabla w_n^+|_2^2$. It follows that

$$S|w_n^+|_{2^*}^2 \leq |\nabla w_n^+|_2^2 + o(1) \leq b/2 + o(1).$$

Passing to the limit yields $S(b/2)^{2/2^*} \leq b/2$. Since $b < 2S^{\frac{N}{2}}$ it follows that $b = 0$. Lemma 5 implies that $\|w_n\| \rightarrow 0$ in E ; that is, $u_n \rightarrow u$ strongly in E^τ . \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $\lambda(\mu)$ be as in Lemma 5, let $\lambda \geq \lambda(\mu)$ and let (u_n) be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^\tau$. By Ekeland’s variational principle [8], [12, Theorem 8.5], we may assume that (u_n) is a τ -Palais-Smale sequence. It follows from Propositions 6, 8, and 7 that a subsequence converges to a nontrivial solution of $(S_{\lambda,\mu}^\tau)$ which changes sign exactly once.

4. THE EFFECT OF THE POTENTIAL WELL

We start by recalling some facts about equivariant Lusternik-Schnirelmann theory. An involution on a topological space X is a continuous function $\tau_X : X \rightarrow X$ such that τ_X^2 is the identity map of X . A subset A of X is called a τ_X -invariant subset if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y respectively, then an equivariant map is a continuous function $f : X \rightarrow Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1 : X \rightarrow Y$ are equivariantly homotopic if there is a homotopy $\Theta : X \times [0, 1] \rightarrow Y$ such that $\Theta(x, 0) = f_0(x)$, $\Theta(x, 1) = f_1(x)$ and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x, t))$ for every $x \in X, t \in [0, 1]$.

Definition 9. The equivariant category (τ_X, τ_Y) - $cat(f)$ of an equivariant map $f : X \rightarrow Y$ is the smallest number k of open invariant subsets X_1, \dots, X_k of X which cover X and which have the property that, for each $i = 1, \dots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i : X_i \times [0, 1] \rightarrow Y$ such that $\Theta(x, 0) = x$, $\Theta(x, 1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta(\tau_X(x), t) = \tau_Y(\Theta(x, t))$ for every $x \in X, t \in [0, 1]$. If no such covering exists we define (τ_X, τ_Y) - $cat(f) = \infty$.

If $\tau_X = \tau$, A is a τ -invariant subset of X , and $\iota : A \hookrightarrow X$ is the inclusion map we write

$$\tau\text{-cat}_X(A) = \tau\text{-cat}(\iota) \quad \text{and} \quad \tau\text{-cat}(X) = \tau\text{-cat}_X(X).$$

In the literature $\tau\text{-cat}(X)$ is usually called $\mathbb{Z}/2\text{-cat}(X)$. Here it is more convenient to specify the involution in the notation.

The following properties can be easily verified.

Lemma 10. a) *If $f : X \rightarrow Y$ and $h : Y \rightarrow Z$ are equivariant maps then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \leq \tau_Y\text{-cat}(Y).$$

b) *If $f_0, f_1 : X \rightarrow Y$ are equivariantly homotopic, then (τ_X, τ_Y) - $cat(f_0) = (\tau_X, \tau_Y)$ - $cat(f_1)$.*

We denote by $\tau_a : V \rightarrow V$ the antipodal involution $\tau_a(u) = -u$ on the vector space V . A τ_a -invariant subset of V is usually called a symmetric subset. Equivariant Lusternik-Schnirelmann category provides a lower bound for the

number of pairs $\{u, -u\}$ of critical points of an even functional. The following result is well known; see for example [7, Theorem 1.1], [11, Theorem 5.7].

Theorem 11. *Let $I : M \rightarrow \mathbb{R}$ be an even C^1 -functional on a complete symmetric $C^{1,1}$ -submanifold M of some Banach space V . Assume that I is bounded below and satisfies the Palais-Smale condition $(PS)_c$ for all $c \leq d$. Then I has at least τ_a -cat(I^d) antipodal pairs $\{u, -u\}$ of critical points with critical values $I(\pm u) \leq d$.*

Here I^d stands, as usual, for the sublevel set $I^d = \{u \in M : I(u) \leq d\}$. A similar result holds for arbitrary group actions; see for example [7, Theorem 1.1].

Now we come back to our problem. Given $\rho > 0$ let

$$\begin{aligned} \Omega_\rho^- &= \{x \in \Omega : \text{dist}(x, \partial\Omega \cup \Omega^\tau) \geq \rho\} \\ \Omega_{2\rho}^+ &= \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq 2\rho\}, \end{aligned}$$

where $\Omega^\tau = \{x \in \Omega : \tau x = x\}$. Fix $\rho > 0$ such that the inclusion maps $\Omega_\rho^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_\rho^+$ are equivariant homotopy equivalences for the given orthogonal involution τ on Ω .

Let $u_{\mu,\rho}$ be the positive ground-state solution of the Dirichlet problem

$$\begin{cases} -\Delta u = \mu u + |u|^{2^*-2} u & \text{in } B_\rho(0) \\ u = 0 & \text{on } \partial B_\rho(0) \end{cases}$$

on the ball $B_\rho(0) = \{x \in \mathbb{R}^N : |x| < \rho\}$ which Brezis and Nirenberg have shown to exist if $0 < \mu < \mu_1(\Omega)$ [4]. Thus $I_{\mu,\Omega}(u_{\mu,\rho}) = c(\mu, B_\rho(0))$. We write

$$c(\mu, \rho) := c(\mu, B_\rho(0)) < \frac{1}{N} S^{\frac{N}{2}}$$

and define $\iota_\mu : \Omega_\rho^- \rightarrow \mathcal{N}_{\mu,\Omega}^\tau$, $\iota_\mu(x) = u_{\mu,\rho}(\cdot - x) - u_{\mu,\rho}(\cdot - \tau x)$. Since $u_{\mu,\rho}$ is radially symmetric and $|x - \tau x| \geq 2\rho$ for $x \in \Omega_\rho^-$ it follows that ι_μ is well defined and $I_{\mu,\Omega}(\iota_\mu(x)) \leq 2c(\mu, \rho)$.

As in [1], we choose $R > 0$ such that $\overline{\Omega} \subset B_R(0)$ and set

$$\xi(t) = \begin{cases} 1 & 0 \leq t \leq R, \\ R/t & R \leq t. \end{cases}$$

Define

$$\beta_0(u) = \frac{\int_{\mathbb{R}^N} |u|^{2^*} \xi(|x|) x \, dx}{\int_{\mathbb{R}^N} |u|^{2^*} \, dx} \quad \text{for } u \in L^{2^*}(\mathbb{R}^N) \setminus \{0\}.$$

We have proved [6, Lemma 12] that the following holds.

Lemma 12. *There exist $\mu^* = \mu^*(\rho) \in (0, \mu_1(\Omega))$ and for each $0 < \mu \leq \mu^*$ a number $\Lambda(\mu) \geq \lambda(\mu)$ with the following properties:*

- i) $c(\mu, \rho) < 2c_{\lambda, \mu}$ for all $\lambda \geq \Lambda(\mu)$, and*
- ii) $\beta_0(u) \in \Omega_{2\rho}^+$ for all $\lambda \geq \Lambda(\mu)$ and all $u \in \mathcal{N}_{\lambda, \mu}$ with $I_{\lambda, \mu}(u) \leq c(\mu, \rho)$.*

We apply this lemma to prove Theorem 2.

Proof of Theorem 2. Since $I_{\lambda, \mu} : \mathcal{N}_{\lambda, \mu}^\tau \rightarrow \mathbb{R}$ is even, bounded below and satisfies $(PS)_c^\tau$ for all $c < \frac{2}{N}S^{\frac{N}{2}}$ it follows from Theorem 11 that $I_{\lambda, \mu}$ has at least $\tau_a\text{-cat}(\mathcal{N}_{\lambda, \mu}^\tau \cap I_{\lambda, \mu}^{2c(\mu, \rho)})$ pairs $\pm u$ of critical points with $I_{\lambda, \mu}(\pm u) \leq 2c(\mu, \rho)$ where $\tau_a(u) = -u$. Let $\mu^* \in (0, \mu_1(\Omega))$ and, for $0 < \mu \leq \mu^*$, let $\Lambda(\mu) \geq \lambda(\mu)$ be as in Lemma 12. We wish to show that for every $\lambda \geq \Lambda(\mu)$

$$\tau_a\text{-cat}(\mathcal{N}_{\lambda, \mu}^\tau \cap I_{\lambda, \mu}^{2c(\mu, \rho)}) \geq \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau). \tag{4.1}$$

Recall that if $u \in \mathcal{N}_{\lambda, \mu}^\tau$ then $u^\pm \in \mathcal{N}_{\lambda, \mu}$, $I_{\lambda, \mu}(u^+) = I_{\lambda, \mu}(u^-)$ and $I_{\lambda, \mu}(u) = 2I_{\lambda, \mu}(u^\pm)$. Thus, if $\lambda \geq \Lambda(\mu)$, $u \in \mathcal{N}_{\lambda, \mu}^\tau$ and $I_{\lambda, \mu}(u) \leq 2c(\mu, \rho)$ Lemma 12 implies that $\beta_0(u^+) \in \Omega_{2\rho}^+$. We define

$$\beta_{\lambda, \mu} : \mathcal{N}_{\lambda, \mu}^\tau \cap I_{\lambda, \mu}^{2c(\mu, \rho)} \rightarrow \Omega_{2\rho}^+, \quad \beta_{\lambda, \mu}(u) = \beta_0(u^+).$$

The maps $\Omega_\rho^- \xrightarrow{\iota_\mu} \mathcal{N}_{\lambda, \mu}^\tau \cap I_{\lambda, \mu}^{2c(\mu, \rho)} \xrightarrow{\beta_{\lambda, \mu}} \Omega_{2\rho}^+$ satisfy $\iota_\mu(\tau x) = -\iota_\mu(x)$, $\beta_{\lambda, \mu}(-u) = \tau\beta_{\lambda, \mu}(u)$, and $\beta_{\lambda, \mu} \circ \iota_\mu$ is the inclusion map $\Omega_\rho^- \hookrightarrow \Omega_{2\rho}^+$. Furthermore, ρ was chosen so that the inclusion maps $\Omega_\rho^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_\rho^+$ are equivariant homotopy equivalences. Hence the inequality (4.1) follows from Lemma 10. Notice that, if $\lambda \geq \Lambda(\mu)$, Lemma 13 and Proposition 6 imply that the solutions satisfy $I_{\lambda, \mu}(u) \leq 2c(\mu, \rho) < 2c_{\lambda, \mu}^\tau$ so, by Proposition 7, they change sign exactly once. \square

Proof of Corollary 3. We apply Theorem 2 with $\tau = -I$. As in [5, Corollary 3] our assumptions imply that $\{I, -I\}\text{-cat}(\Omega) \geq N$. We repeat here the argument for the reader's convenience. If $\{I, -I\}\text{-cat}(\Omega) = k$, then one can define an odd map $\psi : \Omega \rightarrow \mathbb{S}^{k-1}$ as follows: Given an open covering $\{X_1, \dots, X_k\}$ of Ω and odd maps $\alpha_i : X_i \rightarrow \{e_i, -e_i\}$, where $\{e_1, \dots, e_k\}$ is the canonical orthonormal basis of \mathbb{R}^k , let $\{\pi_i : X_i \rightarrow [0, 1]\}$ be a partition of unity subordinated to the covering consisting of even functions. Define

$$\psi(x) = \frac{\sum_{i=1}^k \pi_i(x) \alpha_i(x)}{\left\| \sum_{i=1}^k \pi_i(x) \alpha_i(x) \right\|}.$$

Composing ψ with φ gives an odd map $\psi \circ \varphi : \mathbb{S}^{N-1} \rightarrow \mathbb{S}^{k-1}$, and the Borsuk-Ulam theorem implies that $N \leq k$. \square

For the proof of Theorem 4 we shall need the following lemma, which was proved in [6, Lemma 4].

Lemma 13. *Let $\lambda_n \geq 1$ and $u_n \in E$ be such that $\lambda_n \rightarrow \infty$ and $\|u_n\|_{\lambda_n}^2 < K$. Then there is a $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E and $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$.*

Proof of Theorem 4. Let $\mu \in (0, \mu_1(\Omega))$, let $\lambda_n \rightarrow \infty$, and let $u_n \in \mathcal{N}_{\lambda_n, \mu}^\tau$ be a solution of $(S_{\lambda_n, \mu}^\tau)$ given by Theorem 1, or by Theorem 2 if $\mu \leq \mu^*$. Then

$$I_{\lambda_n, \mu}(u_n) = \frac{1}{N}((A_{\lambda_n} - \mu)u_n, u_n) = \frac{1}{N}|u_n|_{2^*}^{2^*} \rightarrow c \leq 2c(\mu, \rho) < \frac{2}{N}S^{\frac{N}{2}}.$$

Lemmas 5 and 13 imply that there is a $u \in H_0^1(\Omega)^\tau$ such that a subsequence $u_n \rightharpoonup u$ weakly in E and $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$. Since u_n is a solution of $(S_{\lambda_n, \mu}^\tau)$,

$$\int_{\mathbb{R}^N} \nabla u_n \nabla v + \lambda_n a u_n v - \mu u_n v = \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v \quad \text{for all } v \in E^\tau.$$

If $v \in H_0^1(\Omega)^\tau$, then $\int \lambda_n a u_n v = 0$ for all n , so letting $n \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}^N} \nabla u \nabla v - \mu u v = \int_{\mathbb{R}^N} |u|^{2^*-2} u v \quad \text{for all } v \in H_0^1(\Omega)^\tau;$$

that is, u is a solution of the Dirichlet problem (D_μ^τ) . Let $w_n = u_n - u \in E^\tau$. Since $u_n \in \mathcal{N}_{\lambda, \mu}$ and $|\nabla u|_2^2 - \mu|u|_2^2 = |u|_{2^*}^{2^*}$, applying the Brezis-Lieb lemma [3] one easily obtains

$$((A_{\lambda_n} - \mu)w_n, w_n) = b + o(1) = |w_n|_{2^*}^{2^*},$$

with $b \leq Nc < 2S^{\frac{N}{2}}$, and since $w_n \rightarrow 0$ in $L^2(\mathbb{R}^N)$,

$$|\nabla w_n|_2^2 \leq ((A_{\lambda_n} - \mu)w_n, w_n) + \mu|w_n|_2^2 = b + o(1).$$

Observe that for $w_n \in E^\tau$, $|\nabla w_n|_2^2 = 2|\nabla w_n^+|_2^2$. The above inequality thus yields

$$S|w_n^+|_{2^*}^{2^*} \leq |\nabla w_n^+|_2^2 + o(1) \leq b/2 + o(1).$$

Passing to the limit we obtain $S(b/2)^{2/2^*} \leq b/2$. Since $b < 2S^{\frac{N}{2}}$ it follows that $b = 0$. By Lemma 5, for n large enough

$$\alpha_\mu \|w_n\|^2 \leq ((A_{\lambda_n} - \mu)w_n, w_n) = o(1).$$

Therefore, $u_n \rightarrow u$ in E , and thus in $H^1(\mathbb{R}^N)$, and $I_{\lambda_n, \mu}(u_n) = \frac{1}{N}|u_n|_{2^*}^{2^*} \rightarrow \frac{1}{N}|u|_{2^*}^{2^*} = I_{\mu, \Omega}(u)$. Since the solutions v given by Theorems 1 and 2 satisfy

$$0 < \frac{2}{N}(\alpha_\mu S)^{\frac{N}{2}} \leq c_{\lambda, \mu}^\tau \leq I_{\lambda, \mu}(v) \leq 2c(\mu, \rho) < 2c_{\lambda, \mu}^\tau < 2c(\mu, \Omega)^\tau$$

it follows that $0 < I_{\mu,\Omega}(u) < 2c(\mu, \Omega)^\tau$. Hence u is nontrivial and changes sign exactly once [5, Proposition 6]. \square

Finally, we state the following consequence of Theorems 1 and 4.

Corollary 14. *For each $\mu \in (0, \mu_1(\Omega))$, $\lim_{\lambda \rightarrow \infty} c_{\lambda,\mu}^\tau = c(\mu, \Omega)^\tau$.*

Proof. By Proposition 6 it follows that $c_{\lambda,\mu}^\tau \rightarrow c \leq c(\mu, \Omega)^\tau$. By Theorem 1, $c_{\lambda,\mu}^\tau = I_{\lambda,\mu}(u_\lambda)$ for some $u_\lambda \in \mathcal{N}_{\lambda,\mu}^\tau$, and, by Theorem 4, (u_λ) converges to a nontrivial solution $u \in \mathcal{N}_{\mu,\Omega}^\tau$ of the Dirichlet problem (D_μ^τ) . Thus, $I_{\mu,\Omega}(u) = c \geq c(\mu, \Omega)^\tau$. \square

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