Sign changing solutions of semilinear elliptic problems in exterior domains

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Abstract. We prove the existence of a sign changing solution to the semilinear elliptic problem $-\Delta u + u =| u |^{p-2} u$, $u \in H_0^1(\Omega)$, in an exterior domain Ω having finite symmetries.

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1. Introduction

Consider the problem

$$(\wp_\Omega) \qquad \begin{cases} -\Delta u + u =| u |^{p-2} u \\ u \in H_0^1(\Omega) \end{cases}$$

where $\Omega \subset \mathbb{R}^N \smallsetminus \{0\}$ is an unbounded smooth domain such that $\mathbb{R}^N \smallsetminus \Omega$ is bounded, $N \geq 3$, and $2 < p < 2^* := \frac{2N}{N-2}$.

This problem has a variational structure, but it is well known that the usual variational methods cannot be applied to it in a straightforward manner, due to the lack of compactness caused by the unboundedness of Ω. This fact which, roughly speaking, originates from the invariance of $\mathbb{R}^N$ under the action of the noncompact group of translations, makes problems like $(\wp_\Omega)$ interesting and challenging. Indeed, in the last two decades, there has been a considerable amount of research on this subject.

The main difficulty, which is technically expressed in the fact that the variational functional $J : H_0^1(\Omega) \to \mathbb{R}$,

$$J(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) - \frac{1}{p}\int_\Omega |u|^p ,$$

associated to the problem does not satisfy the basic Palais-Smale condition, has been deeply investigated and the behaviour of Palais-Smale sequences has been fully described (see [4], [2]), allowing to conclude that the only obstructions to compactness are the solutions of the "problem at infinity" related to $(\wp_\Omega)$, that is, the infinitely many solutions of problem $(\wp_{\mathbb{R}^N})$.

As a consequence, the question of existence of a solution to $(\wp_\Omega)$ was successfully approached in [4], under some restriction on the size of the complement of Ω, and completely settled in [2], where it was shown that $(\wp_\Omega)$ has at least one positive solution, without any restriction on the complement of Ω (other than it being bounded).

Concerning the existence of more than one solution, less progress has been made. If Ω is the complement of a ball or, more generally, if Ω is invariant under a group of symmetries whose orbits are all infinite, then compactness is restored, and standard variational arguments, together with the symmetric criticality principle, yield the existence of infinitely many solutions to problem $(\wp_\Omega)$ (see, for example, [9, Theorem 1.28]).

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It is also worth mentioning, that many multiplicity results for positive solutions of the singularly perturbed problem $-\epsilon^2\Delta u+u=\mid u\mid^{p-2} u$, ϵ small, have been obtained, and that the existence of infinitely many solutions for the non autonomous problem $-\Delta u+a(x)u=\mid u\mid^{p-2} u$, under appropriate growth assumptions on a, has been recently proved (see, for example, [6] for a survey on these results).

As far as we know, the question of existence of sign changing solutions for the autonomous problem $(\wp_\Omega)$ is still largely open. The results we present here are a contribution to this question.

In what follows we assume that Ω is invariant under the action of some (possibly finite) closed subgroup G of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$. Recall that a function $u:\Omega\to\mathbb{R}$ is *G-invariant* if $u(gx)=u(x)$ for every $g\in G$, $x\in\Omega$. We denote by $Gx:=\{gx:g\in G\}$ the G-orbit of x, by $\#Gx$ its cardinality, and by

$$\ell=\ell(G):=\min\{\#Gx:x\in\mathbb{R}^N\smallsetminus\{0\}\}.$$

A G-orbit of $\mathbb{R}^N\smallsetminus\{0\}$ of cardinality precisely ℓ will be called a *minimal G-orbit*.

As usual, we put $B_R(0):=\{x\in\mathbb{R}^N:|x|\leq R\}$. Our first result concerns exterior domains with small complement.

Theorem 1. *Assume that $\mathbb{R}^N\smallsetminus\{0\}$ contains a minimal G-orbit Gx with*

$$|x|>\min\{|y-x|:y\in Gx,\ y\neq x\}. \tag{1}$$

Then there exists $R_0>0$ such that, if Ω is G-invariant and $\mathbb{R}^N\smallsetminus\Omega\subset B_{R_0}(0)$, problem $(\wp_\Omega)$ has at least one G-invariant sign changing solution.

The condition on the size of the complement can be dropped by making an additional assumption on the symmetries. Namely, assume now that G is the kernel of a homomorphism $\phi:\Gamma\to\mathbb{Z}/2:=\{1,-1\}$, where Γ is a closed subgroup of $O(N)$. Assume also that Ω is Γ-invariant. We look for solutions of problem $(\wp_\Omega)$ which satisfy

$$u(\gamma x)=\phi(\gamma)u(x)\quad\text{ for all } x\in\Omega,\ \gamma\in\Gamma. \tag{2}$$

Note that, if $\phi\equiv 1$ is the trivial homomorphism, then $\Gamma=G$ and u is simply G-invariant. If ϕ is an epimorphism, then u is not only G-invariant, but it has the additional property that $u(\gamma x)=-u(x)$ for all $x\in\Omega$, $\gamma\in\phi^{-1}(-1)$. In particular, it is sign changing, provided it is nontrivial. We shall prove the following result.

Theorem 2. *Assume there is a minimal G-orbit Gx of $\mathbb{R}^N\smallsetminus\{0\}$ such that*

$$\operatorname{dist}(\gamma x,Gx)>\min\{|x-y|:y\in Gx,y\neq x\}\quad\text{for } \gamma\in\phi^{-1}(-1). \tag{3}$$

Then, if Ω is Γ-invariant, problem $(\wp_\Omega)$ has at least one sign changing solution which satisfies (2).

Note that assumption (3) implies that $\Gamma x\neq Gx$. An example which ilustrates assumptions (1) and (3) is the following: If N is even, we identify $\mathbb{R}^N$ with $\mathbb{C}^{N/2}$ and consider the group of rotations $G=\{e^{2\pi ik/n}:k=0,...,n-1\}$ acting by multiplication on each complex coordinate. If $n\geq 7$, then assumption (1) holds for every $x\in\mathbb{R}^N\smallsetminus\{0\}$. Moreover, if Γ is the group generated by G and the reflection $\gamma(z_1,z_2,z_3,...,z_{N/2})=(z_2,z_1,z_3,...,z_{N/2})$,

$z_i \in \mathbb{C}$, and if $\phi : \Gamma \to \{1,-1\}$ is the determinant homomorphism and $n \geq 5$, then assumption (3) holds for $x = (1,0,...,0)$.

The proof of Theorem 2, which is given in section 5, rests on a quite natural argument: The Nehari manifold of the subspace of $H_0^1(\Omega)$ which consists of those functions satisfying (2) is a natural constraint for this problem. It consists of sign changing functions, and the functional J is bounded from below on it. A careful estimate of J on some test functions, combined with the representation theorem of symmetric Palais-Smale sequences allows to conclude that J achieves its infimum on this Nehari manifold.

The proof of Theorem 1 requires more subtle tools, which are developed in sections 2, 3 and 4. In this case we look at the restriction of the functional J to the Nehari manifold of G-invariant functions. J is bounded from below on this manifold and the infimum is also achieved, but this time by a positive function. We develop a variational principle for sign changing solutions, which combined with a compactness argument and careful estimates on test functions gives the existence of the desired sign changing solution.

2. The variational setting

Given $u \in H_0^1(\Omega)$ we denote by

$$\|u\| := \left(\int_\Omega \left(|\nabla u|^2 + u^2\right)\right)^{1/2} \qquad \text{and} \qquad |u|_p := \left(\int_\Omega |u|^p\right)^{1/p}.$$

The action of G on Ω induces an orthogonal action on $H_0^1(\Omega)$ as follows: For $g \in G$ and $u \in H_0^1(\Omega)$ define $gu \in H_0^1(\Omega)$ by

$$(gu)(x) := u(g^{-1}x).$$

The fixed point space of this action is the space

$$\begin{aligned} H_0^1(\Omega)^G \quad &:= \{u \in H_0^1(\Omega) : gu = u \ \ \forall g \in G\} \\ &= \{u \in H_0^1(\Omega) : u(gx) = u(x) \ \ \forall g \in G, \ x \in \Omega\}. \end{aligned}$$

The functional $J : H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p$$

satisfies $J(gu) = J(u)$ for all $g \in G$ and $u \in H_0^1(\Omega)$. So, by the principle of symmetric criticality [9, Theorem 1.28], the G-invariant solutions of problem $(\wp_\Omega)$ are the critical points of the restriction of J to the space $H_0^1(\Omega)^G$. The nontrivial ones lie on the *Nehari manifold*

$$\begin{aligned} \mathcal{N}^G \quad &:= \{u \in H_0^1(\Omega)^G : u \neq 0, \ J'(u)u = 0\} \\ &= \{u \in H_0^1(\Omega)^G : u \neq 0, \ \|u\|^2 = |u|_p^p\}, \end{aligned}$$

which is of class C^2. We denote by

$$\pi : H_0^1(\Omega)^G \smallsetminus \{0\} \to \mathcal{N}^G, \qquad \pi(u) := \left(\frac{\|u\|^2}{|u|_p^p}\right)^{\frac{1}{p-2}} u,$$

the radial projection onto the Nehari manifold. Note that

$$J(\pi(u)) = \frac{p-2}{2p}\left(\frac{\|u\|^2}{|u|_p^2}\right)^{\frac{p}{p-2}}.$$

Set

$$c_0^G := \inf_{\mathcal{N}^G} J.$$

Let ω be the unique positive radially symmetric solution to the problem $(\wp_{\mathbb{R}^N})$. It is well known that

$$\lim_{|x|\to\infty} |D^\nu \omega(x)|\, |x|^{\frac{N-1}{2}} \exp|x| = a_\nu > 0, \qquad \nu = 1, 2, \tag{4}$$

(see [8, 5]). Let

$$c_\infty := J(\omega) = \frac{p-2}{2p}\, \|\omega\|^2 = \frac{p-2}{2p}\, |\omega|_p^p .$$

Observe that $c_0^G \geq c_\infty$. One has the following representation lemma for G-invariant Palais-Smale sequences.

Lemma 3. *Let (u_n) be a sequence in $H_0^1(\Omega)^G$ such that $J(u_n) \to c < 2\ell c_\infty$ and $\nabla J(u_n) \to 0$. Then (u_n) contains a subsequence, still denoted by (u_n), for which one of the following holds: Either (u_n) converges in $H_0^1(\Omega)^G$, or there exists a G-invariant solution u_0 of $(\wp_\Omega)$ and a sequence of minimal G-orbits Gx_n in Ω such that $|x_n| \to \infty$,*

$$\left\| u_n - u_0 - \sum_{y\in Gx_n} \omega(\cdot - y) \right\| \to 0,$$

and $c = J(u_0) + \ell c_\infty$.

Proof. This follows from the representation result for Palais-Smale sequences given in [4], taking the symmetries into account. ■

We need the following numerical lemma.

Lemma 4. *(i) If $p \geq 2$, and $a_1, ..., a_m \geq 0$, then*

$$\left|\sum_{i=1}^m a_i\right|^p \geq \sum_{i=1}^m a_i^p + (p-1) \sum_{i\neq j} a_i^{p-1} a_j. \tag{5}$$

(ii) If $p \geq 2$, and $a, b \geq 0$, then

$$|a-b|^p \geq a^p + b^p - pab(a^{p-2} + b^{p-2}).$$

Proof. *(i)* Let $a, b, c \geq 0$. Then

$$\frac{d}{dt}\frac{d}{ds}(sa + tb + c)^p = p(p-1)(sa + tb + c)^{p-2} ab.$$

Integrating we obtain

$$\begin{aligned}\int_0^1\int_0^1 \frac{d}{dt}\left[\frac{d}{ds}(sa+tb+c)^p\right] dtds &= \int_0^1 \left[\frac{d}{ds}(sa+b+c)^p - \frac{d}{ds}(sa+c)^p\right] ds \\ &= (a+b+c)^p - (b+c)^p - (a+c)^p + c^p,\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 (sa+tb+c)^{p-2}ab\ dsdt &= \iint\limits_{t\leq s}(sa+tb+c)^{p-2}ab\ dsdt + \iint\limits_{s\leq t}(sa+tb+c)^{p-2}ab\ dsdt\\ &\geq \int_0^1\int_0^s (sa)^{p-2}ab\ dtds + \int_0^1\int_0^t (tb)^{p-2}ab\ dsdt\\ &= \int_0^1 (sa)^{p-1}b\ ds + \int_0^1 (tb)^{p-1}a\ dt\\ &= \frac{1}{p}\left(a^{p-1}b+b^{p-1}a\right).\end{aligned}$$

Hence,

$$(a+b+c)^p \geq (b+c)^p + (a+c)^p - c^p + (p-1)\left(a^{p-1}b+b^{p-1}a\right).$$

If $m=2$ the above inequality with $c=0$ yields (5) (see also [7]). If $m\geq 3$ we argue by induction to obtain

$$\begin{aligned}\left|\sum_{i=1}^m a_i\right|^p &= \left(a_1+a_2+\sum_{i=3}^m a_i\right)^p\\ &\geq \left(a_2+\sum_{i=3}^m a_i\right)^p + \left(a_1+\sum_{i=3}^m a_i\right)^p - \left(\sum_{i=3}^m a_i\right)^p + (p-1)\left(a_1^{p-1}a_2+a_2^{p-1}a_1\right)\\ &\geq \sum_{i\neq 1} a_i^p + \sum_{i\neq 2} a_i^p - \sum_{i\neq 1,2} a_i^p\\ &\quad + (p-1)\left[\sum_{\substack{i,j\neq 1\\ i\neq j}} a_i^{p-1}a_j + \sum_{\substack{i,j\neq 2\\ i\neq j}} a_i^{p-1}a_j - \sum_{\substack{i,j\neq 1,2\\ i\neq j}} a_i^{p-1}a_j + a_1^{p-1}a_2 + a_2^{p-1}a_1\right]\\ &= \sum_{i=1}^m a_i^p + (p-1)\sum_{i\neq j} a_i^{p-1}a_j.\end{aligned}$$

The proof of *(ii)* is straightforward. ■

The following lemma was proved in [1, Proposition 1.2].

Lemma 5. *Let $f\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $h\in C(\mathbb{R}^N)$ be radially symmetric functions and let $a,b\geq 0$, $c\in\mathbb{R}$ be such that*

$$\lim_{|x|\to\infty} f(x)\,|x|^b\exp(a\,|x|) = c \qquad \text{and} \qquad \int |h(x)|\,(1+|x|^b)\exp(a\,|x|)dx < \infty.$$

Then

$$\lim_{|y|\to\infty}\left(\int f(x+y)h(x)dx\right)|y|^b\exp(a\,|y|) = c\int h(x)\exp(-ax_1)dx. \quad ■$$

Let $\psi:\mathbb{R}^N\to[0,1]$ be a radially symmetric smooth cut-off function such that $\psi(x)=0$ if $|x|\leq R$ and $\psi(x)=1$ if $|x|\geq 2R$, where R satisfies $\mathbb{R}^N\smallsetminus\Omega\subset B_R(0)$. Let $Gx=\{y_1,...,y_m\}$ be a G-orbit in $\mathbb{R}^N\smallsetminus\{0\}$, and consider the function

$$\sigma_\rho(x) := \sum_{i=1}^m \omega(x-\rho y_i)\in H^1(\mathbb{R}^N)^G.$$

Then $\psi\sigma_\rho \in H_0^1(\Omega)^G$ and the following holds.

Lemma 6. *If $m \geq 2$, there exists $\rho_0 > 0$ such that*

$$J(\pi(\psi\sigma_\rho)) = \frac{p-2}{2p}\left(\frac{\|\psi\sigma_\rho\|^2}{|\psi\sigma_\rho|_p^2}\right)^{\frac{p}{p-2}} < mc_\infty \qquad \forall \rho \geq \rho_0.$$

In particular, if $\ell := \min\{\#Gx : x \in \mathbb{R}^N \smallsetminus \{0\}\} \geq 2$, then $c_0^G < \ell c_\infty$.

Proof. First note that, for every $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}
\|\psi u\|^2 &= \int \left(|\psi\nabla u + u\nabla\psi|^2 + |\psi u|^2\right) \\
&= \int \psi^2\left(|\nabla u|^2 + |u|^2\right) + \int |\nabla\psi|^2 u^2 + \frac{1}{2}\int \nabla(\psi^2)\nabla\left(u^2\right) \\
&= \int \psi^2\left(|\nabla u|^2 + |u|^2\right) + \int \left[|\nabla\psi|^2 - \frac{1}{2}\Delta(\psi^2)\right]u^2 \\
&\leq \|u\|^2 - \int (\psi\Delta\psi)\, u^2, \\
|\psi u|_p^p &= |u|_p^p + \int (\psi^p - 1)\,|u|^p\,.
\end{aligned}$$

Without loss of generality we assume that $|y_i| = 1$. For $\rho > 0$, $i, j \in \{1, ..., m\}$, $i \neq j$, we set

$$\begin{aligned}
\varepsilon_{\rho,i,j} &:= \int \omega(x-\rho y_i)^{p-1}\omega(x-\rho y_j)dx \\
&= \int \left[\nabla\omega(x-\rho y_i)\nabla\omega(x-\rho y_j) + \omega(x-\rho y_i)\omega(x-\rho y_j)\right]dx, \\
\varepsilon_\rho &:= \sum_{i\neq j}\varepsilon_{\rho,i,j}.
\end{aligned}$$

Lemma 5 together with (4) yield

$$\lim_{\rho\to\infty}\varepsilon_{\rho,i,j}\left(|d_{ij}\rho|^{\frac{N-1}{2}}\exp(d_{ij}\rho)\right) = \gamma_0 > 0$$

where $d_{i,j} := |y_i - y_j|$, and

$$\lim_{\rho\to\infty}\left(\int h(x)\omega(x-\rho y_i)^q dx\right)|q\rho|^{\frac{N-1}{2}}\exp(q\rho) = 0 \quad \text{ if } h \in C_c(\mathbb{R}^N) \text{ and } q > 1.$$

Since $d_{i,j} \leq 2 < p$ and $\psi\Delta\psi$, $\psi^p - 1 \in C_c(\mathbb{R}^N)$, using Lemma 4 we obtain

$$\begin{aligned}
\|\psi\sigma_\rho\|^2 &\leq \|\sigma_\rho\|^2 - \int (\psi\Delta\psi)\,\sigma_\rho^2 = m\,\|\omega\|^2 + \varepsilon_\rho + o(\varepsilon_\rho), \\
|\psi\sigma_\rho|_p^p &= |\sigma_\rho|_p^p + \int (\psi^p - 1)\,|\sigma_\rho|^p \geq m\,|\omega|_p^p + (p-1)\varepsilon_\rho + o(\varepsilon_\rho).
\end{aligned}$$

Therefore, since $\|\omega\|^2 = |\omega|_p^p$, there are positive constants C_0, ρ_0 such that

$$\frac{\|\psi\sigma_\rho\|^2}{|\psi\sigma_\rho|_p^2} \leq \frac{m\|\omega\|^2 + \varepsilon_\rho + o(\varepsilon_\rho)}{\left(m\|\omega\|^2 + (p-1)\varepsilon_\rho + o(\varepsilon_\rho)\right)^{2/p}} \leq \left(m\|\omega\|^2\right)^{\frac{p-2}{p}} - C_0\varepsilon_\rho \qquad \text{for } \rho \geq \rho_0.$$

We conclude that

$$J(\pi(\psi\sigma_\rho)) = \frac{p-2}{2p}\left(\frac{\|\psi\sigma_\rho\|^2}{|\psi\sigma_\rho|_p^2}\right)^{\frac{p}{p-2}} < mc_\infty \qquad \text{for } \rho \geq \rho_0,$$

as claimed. ■

Proposition 7. *Assume that $\ell := \min\{\#Gx : x \in \mathbb{R}^N \smallsetminus \{0\}\} \geq 2$. Then $c_\infty < c_0^G < \ell c_\infty$ and c_0^G is attained by J at some $u_0 \in \mathcal{N}^G$ which does not change sign.*

Proof. Let (u_n) be a minimizing sequence for J on $\mathcal{N}^G$. By Ekeland's variational principle we may assume that it is a Palais-Smale sequence. The previous lemma implies that $c_0^G < \ell c_\infty$. Thus, by Lemma 3, (u_n) converges strongly to $u_0 \in \mathcal{N}^G$. ■

3. A variational principle for sign changing solutions

The aim of this section is to prove the following result.

Theorem 8. *Let $\ell := \min\{\#Gy : y \in \mathbb{R}^N \smallsetminus \{0\}\}$. If there exist $v_1, v_2 \in H_0^1(\Omega)^G \smallsetminus \{0\}$ such that $v_1, v_2 \geq 0$, $v_1 \neq \lambda v_2$ for all $\lambda \in \mathbb{R}$, and*

$$\max\{J(sv_1 - tv_2) : s, t \geq 0\} < c_0^G + \ell c_\infty,$$

then problem $(\wp_\Omega)$ has a G-invariant sign changing solution.

The proof is based on the following lemmas.

Lemma 9. *Let $v_1, v_2 \in H_0^1(\Omega)^G \smallsetminus \{0\}$ be such that $v_1 \neq \lambda v_2$ for all $\lambda \in \mathbb{R}$. Then there exists $r_0 > 0$ such that*

$$J(sv_1 - tv_2) \leq 0 \qquad \text{for all } s, t \geq 0,\ s + t \geq r_0.$$

Proof. Set $u_\zeta := (1-\zeta)v_1 - \zeta v_2$, $\zeta \in [0,1]$. By assumption $u_\zeta \neq 0$, hence $r_1 := \min_{\zeta\in[0,1]} \|u_\zeta\| > 0$. Let $r_2 > 0$ be such that

$$J(sv_1 - tv_2) \leq 0 \qquad \text{if } \|sv_1 - tv_2\| \geq r_2.$$

If $s, t \geq 0$ and $s + t \geq r_0 := r_2/r_1$ then, setting $\zeta := t/(s+t)$, we get that $\|sv_1 - tv_2\| = (s+t)\|u_\zeta\| \geq r_2$, and hence that $J(sv_1 - tv_2) \leq 0$. ■

Next, consider the negative gradient flow $\varphi : \mathcal{G} \to H_0^1(\Omega)^G$ of J defined by

$$\begin{cases} \frac{d}{dt}\varphi(t,u) = -\nabla J(\varphi(t,u)) \\ \varphi(0,u) = u \end{cases}$$

where $\mathcal{G} := \{(t,u) : u \in H_0^1(\Omega)^G,\ 0 \leq t < T(u)\}$ and $T(u) \in (0,\infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t,u)$. A subset $\mathcal{D}$ of $H_0^1(\Omega)^G$ is called *strictly positively invariant for the flow φ* if

$$\varphi(t,u) \in \operatorname{int}(\mathcal{D}) \quad \text{for every } u \in \mathcal{D} \text{ and every } t \in (0, T(u)),$$

Here $\mathrm{int}(\mathcal{D})$ denotes the interior of $\mathcal{D}$ in $H_0^1(\Omega)^G$. Let $\mathcal{P}^G := \{u \in H_0^1(\Omega)^G : u \geq 0\}$ be the convex cone of nonnegative functions in $H_0^1(\Omega)^G$, and let

$$B_\alpha(\pm\mathcal{P}^G) := \left\{u \in H_0^1(\Omega)^G : \mathrm{dist}(u, \pm\mathcal{P}^G) \leq \alpha\right\}.$$

Consider the set

$$\mathcal{E}^G := \{u \in H_0^1(\Omega)^G : u^+, u^- \in \mathcal{N}^G\},$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. The following holds.

Lemma 10. *There exists $\alpha > 0$ such that*
a) $\left[B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G)\right] \cap \mathcal{E}^G = \emptyset$, *and*
b) $B_\alpha(\mathcal{P}^G)$ *and* $B_\alpha(-\mathcal{P}^G)$ *are strictly positively invariant for* φ.

The proof is identical to that of Proposition 3.1 in [3]. Define

$$\mathcal{D} := B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G) \cup J^0,$$

where $J^c := \{u \in H_0^1(\Omega)^G : J(u) \leq c\}$ for $c \in \mathbb{R}$. Since $J(u) > 0$ for every nontrivial critical point of J, the set $J^0 \smallsetminus \{0\}$ is strictly positively invariant for φ. Thus, by Lemma 10, $\mathcal{D}$ is strictly positively invariant for φ and

$$\mathcal{D} \cap \mathcal{E}^G = \emptyset. \tag{6}$$

Lemma 11. *If $c < c_0^G + \ell c_\infty$ and J has no G-invariant sign changing critical point in J^c, then there exists a continuous map $\varrho : H_0^1(\Omega)^G \cap J^c \to \mathcal{D}$ such that $\varrho(u) = u$ for every $u \in \mathcal{D} \cap J^c$.*

Proof. For $u \in H_0^1(\Omega)^G$ we consider the entrance time $e(u)$ of u into the set $\mathcal{D}$, that is,

$$e(u) := \inf\{t \in (0, \infty] : \varphi(t, u) \in \mathcal{D}\}.$$

Since $\mathcal{D}$ is strictly positively invariant, a standard argument shows that $e : H_0^1(\Omega)^G \to [0, \infty]$ is continuous. Lemma 3 guarantees that there are no non-convergent G-invariant Palais-Smale sequences in $J^c \smallsetminus \mathcal{D}$ and, since we are assuming that J has no critical points in $J^c \smallsetminus \mathcal{D}$, we have that $e(u) < \infty$ for all $u \in H_0^1(\Omega)^G \cap J^c$. It follows that the map

$$\varrho : H_0^1(\Omega)^G \cap J^c \to \mathcal{D} \qquad \varrho(u) = \varphi(e(u), u)$$

is well defined, and $\varrho(u) = \varphi(0, u) = u$ for every $u \in \mathcal{D} \cap J^c$. ■

Set $Q := \{z = (s, t) \in \mathbb{R}^2 : s, t \geq 0\}$. We need the following topological lemma.

Lemma 12. *Let $f : Q \to Q$ be continuous. Assume there exist $q, r_0 > 0$ such that*
i) $f(s, 0) = (s^q, 0)$ *and* $f(0, t) = (0, t^q)$ *for all* $s, t \geq 0$,
ii) $f(\lambda s, \lambda t) = \lambda^q f(s, t)$ *if* $s + t \geq r_0$ *and* $\lambda \geq 1$,
iii) $f(s, t) \neq 0$ *if* $s + t = r_0$.
Then f is surjective.

Proof. Assume, by contradiction, that f is not surjective. Then there exists $z_0 = (s_0, t_0) \in \mathrm{int}(Q)$ which is not in the image of f. Fix $r_1 > s_0 + t_0$, and consider the triangle $T := \{(s, t) \in Q : s + t \leq r_1\}$. Thus, $z_0 \in \mathrm{int}(T)$ and there is a retraction $\theta : T \smallsetminus \{z_0\} \to \partial T$ given by

$$\{\theta(z)\} = \partial T \cap \{z_0 + \lambda(z - z_0) : \lambda \geq 0\}.$$

One has also a retraction $\eta : Q \to T$ given by $\eta(z) = z$ if $z \in T$ and $\eta(s,t) = r_1(s+t)^{-1}(s,t)$ if $s+t \geq r_1$. Set $|(s,t)|_1 := s+t$, and fix $\lambda_0 \geq 1$ such that $|f(\lambda_0 z)|_1 \geq r_1$ if $|z|_1 = r_0$. The map $h : T \to \partial T$ given by

$$h(z) := \theta \circ \eta \circ f(\lambda_0 r_0 r_1^{-1} z)$$

is the identity on the vertices of T and maps each side of T into itself. Therefore, $h\,|_{\partial T}$ is homotopic to the identity map $id_{\partial T} : \partial T \to \partial T$. This implies that $id_{\partial T}$ is nullhomotopic, which is false. ■

Finally, define $\gamma : H_0^1(\Omega)^G \to \mathbb{R}$ by

$$\gamma(u) = \begin{cases} \frac{|u|_p^p}{\|u\|^2} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

By Sobolev's embedding theorem this function is continuous. Note that $\gamma(u) = 1$ if and only if $u \in \mathcal{N}^G$. Therefore,

$$\gamma(u^+) = 1 = \gamma(u^-) \quad \text{if and only if} \quad u \in \mathcal{E}^G. \tag{7}$$

We are ready to prove Theorem 8.

Proof of Theorem 8. Without loss of generality we may assume that $v_1, v_2 \in \mathcal{N}^G$. We argue by contradiction. Let

$$c := \max\{J(sv_1 - tv_2) : s,t \geq 0\} < c_0^G + \ell c_\infty,$$

and assume that J has no G-invariant sign changing critical point in J^c. Let $\varrho : H_0^1(\Omega)^G \cap J^c \to \mathcal{D}$ be the retraction given by Lemma 11. Consider the maps

$$\iota : Q \to H_0^1(\Omega)^G \cap J^c, \qquad \iota(s,t) := sv_1 - tv_2,$$
$$\kappa : H_0^1(\Omega)^G \to Q, \qquad \kappa(u) := (\gamma(u^+), \gamma(u^-)).$$

Assertions (6) and (7) imply that the point $(1,1)$ is not in the image of the restriction of κ to $\mathcal{D}$. Therefore, the composition

$$f := \kappa \circ \varrho \circ \iota : Q \to Q$$

is not surjective. Note that $\iota(s,0) = sv_1 \in \mathcal{P}^G \subset \mathcal{D}$ and $\iota(0,t) = -tv_2 \in -\mathcal{P}^G \subset \mathcal{D}$ for all $(s,t) \in Q$. Fix $r_0 > 0$ as in Lemma 9. Then $\iota(s,t) \in J^0 \subset \mathcal{D}$ for all $(s,t) \in Q$ with $s+t \geq r_0$. On the other hand, if $u := \iota(s,t) \in \mathcal{D}$, then $f(s,t) = \kappa(u)$. By definition, $\kappa(\lambda u) = \lambda^{p-2}\kappa(u)$ for every $\lambda \geq 0$. Therefore f satisfies conditions *i)* and *ii)* of Lemma 12 with $q := p-2$. Clearly condition *iii)* also holds, in fact, $f(s,t) = 0$ iff $s = 0 = t$. Lemma 12 asserts that f must be surjective. This is a contradiction. ■

4. Sign changing solutions in exterior domains with small complement

Let $Gx = \{y_1, ..., y_\ell\}$ be a minimal G-orbit of $\mathbb{R}^N \smallsetminus \{0\}$, and consider the function

$$\sigma_\rho(x) := \sum_{i=1}^{\ell} \omega(x - \rho y_i) \in H^1(\mathbb{R}^N)^G.$$

Lemma 13. *If Gx satisfies assumption* (1), *then there exists $\rho_0 > 0$ such that*

$$\max\{J(s\sigma_\rho - t\omega) : s, t \geq 0\} < (\ell+1)c_\infty \qquad \text{for } \rho \geq \rho_0.$$

Proof. We may assume that $|y_i| = 1$. For $\rho > 0$, $i, j \in \{1, ..., \ell\}$, $i \neq j$, we set

$$\begin{aligned}
\varepsilon_{\rho,i,j} &:= \int \omega(x-\rho y_i)^{p-1}\omega(x-\rho y_j)dx \\
&= \int \left[\nabla\omega(x-\rho y_i)\nabla\omega(x-\rho y_j) + \omega(x-\rho y_i)\omega(x-\rho y_j)\right]dx, \\
\varepsilon_\rho &:= \sum_{i\neq j}\varepsilon_{\rho,i,j}, \\
\varepsilon_{\rho,0} &:= \int \omega(x-\rho y_i)^{p-1}\omega(x)dx \\
&= \int \left[\nabla\omega(x-\rho y_i)\nabla\omega(x) + \omega(x-\rho y_i)\omega(x)\right]dx, .
\end{aligned}$$

Note that $\varepsilon_{\rho,0}$ does not depend on i. Lemma 5 together with (4) yield

$$\lim_{\rho\to\infty}\varepsilon_{\rho,i,j}\left(|d_{ij}\rho|^{\frac{N-1}{2}}\exp(d_{ij}\rho)\right) = \gamma_0 > 0, \qquad \lim_{\rho\to\infty}\varepsilon_{\rho,0}\left(|\rho|^{\frac{N-1}{2}}\exp(\rho)\right) = \gamma_0 > 0,$$

where $d_{i,j} := |y_i - y_j|$. Since assumption (1) holds, one has that $d_{i,j} < 1$ for some $i \neq j$. Therefore, $\varepsilon_{\rho,0} = o(\varepsilon_\rho)$ and

$$\begin{aligned}
\|s\sigma_\rho - t\omega\|^2 &= s^2\|\sigma_\rho\|^2 + t^2\|\omega\|^2 - 2\ell st\varepsilon_{\rho,0} \\
&= \ell s^2\|\omega\|^2 + t^2\|\omega\|^2 + s^2\varepsilon_\rho + o(\varepsilon_\rho).
\end{aligned}$$

Moreover, applying Lemma 4, we get

$$\begin{aligned}
|s\sigma_\rho - t\omega|_p^p &\geq s^p|\sigma_\rho|_p^p + t^p|\omega|_p^p - p\int(s^{p-1}t\sigma_\rho^{p-1}\omega + st^{p-1}\sigma_\rho\omega^{p-1}) \\
&\geq s^p\ell|\omega|_p^p + t^p|\omega|_p^p + (p-1)s^p\varepsilon_\rho + o(\varepsilon_\rho).
\end{aligned}$$

Consequently, since $\|\omega\|^2 = |\omega|_p^p$, we obtain

$$\frac{\|s\sigma_\rho - t\omega\|^2}{|s\sigma_\rho - t\omega|_p^2} \leq \frac{\ell s^2\|\omega\|^2 + t^2\|\omega\|^2 + s^2\varepsilon_\rho + o(\varepsilon_\rho)}{\left(\ell s^p\|\omega\|^2 + t^p\|\omega\|^2 + (p-1)s^p\varepsilon_\rho + o(\varepsilon_\rho)\right)^{2/p}}.$$

The parameters s, t at which the maximum is attained tend to 1 as $\rho \to \infty$. It follows that there exist $C_0, \rho_0 > 0$ such that

$$\frac{\|s\sigma_\rho - t\omega\|^2}{|s\sigma_\rho - t\omega|_p^2} \leq \left((\ell+1)\|\omega\|^2\right)^{\frac{p-2}{p}} - C_0\varepsilon_\rho \qquad \text{for } \rho \geq \rho_0.$$

Hence,

$$J(s\sigma_\rho - t\omega) = \frac{p-2}{2p}\left(\frac{\|s\sigma_\rho - t\omega\|^2}{|s\sigma_\rho - t\omega|_p^2}\right)^{\frac{p}{p-2}} < (\ell+1)c_\infty \qquad \text{for } \rho \geq \rho_0,$$

as claimed. ■

We assume from now on that $Gx = \{y_1, ..., y_\ell\}$ is a minimal G-orbit of $\mathbb{R}^N \smallsetminus \{0\}$ which satisfies assumption (1), and we fix $\rho > 0$ such that

$$\max\{J(s\sigma_\rho - t\omega) : s, t \geq 0\} < (\ell + 1)c_\infty.$$

We also fix a radially symmetric smooth cut-off function $\psi : \mathbb{R}^N \to [0,1]$ such that $\psi(x) = 0$ if $|x| \leq 1$ and $\psi(x) = 1$ if $|x| \geq 2$, and set $\psi_R(x) := \psi(\frac{x}{R})$, $R > 0$.

Lemma 14. *There exists $R_0 > 0$ such that*

$$\max\{J(s\psi_R\sigma_\rho - t\psi_R\omega) : s, t \geq 0\} \leq (\ell + 1)c_\infty \qquad \text{for } R \leq R_0.$$

Proof. Set $u_\tau := (1 - \tau)\sigma_\rho - \tau\omega$. Then

$$\{s\psi_R\sigma_\rho - t\psi_R\omega : s, t \geq 0\} = \{\psi_R\lambda u_\tau : \tau \in [0,1],\ \lambda \geq 0\}.$$

For each $\lambda \geq 0$ and for all $\tau \in [0,1]$ one has that

$$\begin{aligned}
\|\psi_R\lambda u_\tau - \lambda u_\tau\|^2 &= \lambda^2 \|(1 - \psi_R)u_\tau\|^2 \\
&= \lambda^2 \int \left(|(1 - \psi_R)\nabla u_\tau + \nabla(1 - \psi_R)u_\tau|^2 + |(1 - \psi_R)u_\tau|^2\right) \\
&\leq 4\lambda^2 \int |(1 - \psi_R)|^2 \left(|\nabla u_\tau|^2 + u_\tau^2\right) + |\nabla(1 - \psi_R)|^2 u_\tau^2 \\
&\leq C_1\lambda^2 \|1 - \psi_R\|^2, &(8)\\
|\psi_R\lambda u_\tau - \lambda u_\tau|_p^p &= \lambda^p |(1 - \psi_R)u_\tau|_p^p \leq C_2\lambda^p |1 - \psi_R|_p^p, &(9)
\end{aligned}$$

where C_1, C_2 are positive constants. Hence, there exist $R_1, C_3, C_4 > 0$ such that

$$\|\psi_R u_\tau\|^2 \leq C_3 \ \text{ and } \ |\psi_R u_\tau|_p^p \geq C_4 \qquad \text{for all } R \leq R_1,\ \tau \in [0,1].$$

So, if $\lambda_0 > 0$ satisfies $\frac{C_3}{2}\lambda_0^2 - \frac{C_4}{p}\lambda_0^p = 0$, we have that

$$J(\psi_R\lambda u_\tau) \leq \frac{C_3}{2}\lambda^2 - \frac{C_4}{p}\lambda^p \leq 0 \quad \text{ for all } R \leq R_1,\ \tau \in [0,1],\ \lambda \geq \lambda_0. \tag{10}$$

For all $\lambda \leq \lambda_0$ and $\tau \in [0,1]$ inequalities (8) and (9) yield

$$\begin{aligned}
|\|\psi_R\lambda u_\tau\| - \|\lambda u_\tau\|| &\leq \|\psi_R\lambda u_\tau - \lambda u_\tau\| \leq C_5 \|1 - \psi_R\|, \\
\left||\psi_R\lambda u_\tau|_p - |\lambda u_\tau|_p\right| &\leq |\psi_R\lambda u_\tau - \lambda u_\tau|_p \leq C_6 |1 - \psi_R|_p,
\end{aligned}$$

where C_5, C_6 are positive constants. Hence, there exists $R_0 \in (0, R_1]$ such that

$$\begin{aligned}
|J(\psi_R\lambda u_\tau) - J(\lambda u_\tau)| &\leq \frac{1}{2}\left|\|\psi_R\lambda u_\tau\|^2 - \|\lambda u_\tau\|^2\right| + \frac{1}{p}\left||\psi_R\lambda u_\tau|_p^p - |\lambda u_\tau|_p^p\right| \\
&\leq (\ell + 1)c_\infty - \max\{J(\lambda u_\tau) : \tau \in [0,1],\ \lambda \geq 0\},
\end{aligned}$$

for all $R \leq R_0$, $\tau \in [0,1]$, $\lambda \leq \lambda_0$. This inequality, together with (10), yields

$$J(\psi_R\lambda u_\tau) \leq (\ell + 1)c_\infty \quad \text{ for all } R \leq R_0, \tau \in [0,1], \lambda \geq 0,$$

as claimed. ■

Proof of Theorem 1. Let $R_0 > 0$ be as in the previous lemma and set $v_1 := \psi_{R_0}\sigma_\rho$ and $v_2 := \psi_{R_0}\omega$. If $\mathbb{R}^N \smallsetminus \Omega \subset B_{R_0}(0)$, then $v_1, v_2 \in H_0^1(\Omega)^G$ and v_1, v_2 satisfy the hypotheses of Theorem 8. ■

Note that, if $\ell = \infty$, Theorem 8 yields a G-invariant sign changing solution of $(\wp_\Omega)$ without any restriction on the size of $\mathbb{R}^N \smallsetminus \Omega$. In fact, one can show there are infinitely many G-invariant sign changing solutions in this case.

5. Sign changing solutions in exterior domains

We turn to the proof of Theorem 2. We assume that G is the kernel of a homomorphism $\phi : \Gamma \to \mathbb{Z}/2$, where Γ is a closed subgroup of $O(N)$, and that Ω is Γ-invariant. Then ϕ induces an action of Γ on $H_0^1(\Omega)$ given by

$$(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x). \tag{11}$$

The fixed point space of this action is the space

$$\begin{aligned} H_0^1(\Omega)^\phi \quad &:= \{u \in H_0^1(\Omega) : \gamma u = u \quad \forall \gamma \in \Gamma\} \\ &= \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \quad \forall \gamma \in \Gamma,\ x \in \Omega\}. \end{aligned}$$

The functional J satisfies $J(\gamma u) = J(u)$ for all $\gamma \in \Gamma$ and $u \in H_0^1(\Omega)$. So, by the principle of symmetric criticality [9, Theorem 1.28], the solutions of problem $(\wp_\Omega)$ which satisfy (2) are the critical points of the restriction of J to the space $H_0^1(\Omega)^\phi$. The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}^\phi := \{u \in H_0^1(\Omega)^\phi : u \neq 0,\ \|u\|^2 = |u|_p^p\} = \mathcal{N}^G \cap H_0^1(\Omega)^\phi.$$

As before, let $\ell := \min\{\#Gx : x \in \mathbb{R}^N \smallsetminus \{0\}\}$.

Lemma 15. *Let (u_n) be a sequence in $H_0^1(\Omega)^\phi$ such that $J(u_n) \to c < 2\ell c_\infty$ and $\nabla J(u_n) \to 0$. Then (u_n) has a convergent subsequence.*

Proof. This follows from the representation result for Palais-Smale sequences given in [4], taking the symmetries into account. ■

Let $\psi : \mathbb{R}^N \to [0,1]$ be a radially symmetric smooth cut-off function such that $\psi(x) = 0$ if $|x| \leq R$ and $\psi(x) = 1$ if $|x| \geq 2R$, where R satisfies $\mathbb{R}^N \smallsetminus \Omega \subset B_R(0)$. Let $Gx = \{y_1, ..., y_m\}$ be a G-orbit in $\mathbb{R}^N \smallsetminus \{0\}$. Assume that $\Gamma x \neq Gx$, fix $\gamma \in \phi^{-1}(-1)$ and consider the function

$$\theta_\rho(x) := \sum_{i=1}^m \omega(x - \rho y_i) - \sum_{i=1}^m \omega(x - \rho\gamma y_i) \in H^1(\mathbb{R}^N)^\phi.$$

Then $\psi\theta_\rho \in H_0^1(\Omega)^\phi$ and the following holds.

Lemma 16. *If $m \geq 2$ and $\operatorname{dist}(\gamma x, Gx) > \min\{|x - y| : y \in Gx,\ y \neq x\}$, then there exists $\rho_0 > 0$ such that*

$$J(\pi(\psi\theta_\rho)) = \frac{p-2}{2p}\left(\frac{\|\psi\theta_\rho\|^2}{|\psi\theta_\rho|_p^2}\right)^{\frac{p}{p-2}} < 2mc_\infty \qquad \forall \rho \geq \rho_0.$$

Proof. Without loss of generality we assume that $|y_i| = 1$. For $\rho > 0$, $i, j \in \{1, ..., m\}$, $i \neq j$, we set

$$\begin{aligned}
\varepsilon_{\rho,i,j} &:= \int \omega(x-\rho y_i)^{p-1}\omega(x-\rho y_j)dx \\
&= \int \left[\nabla\omega(x-\rho y_i)\nabla\omega(x-\rho y_j) + \omega(x-\rho y_i)\omega(x-\rho y_j)\right]dx, \\
\varepsilon^{\gamma}_{\rho,i,j} &:= \int \omega(x-\rho\gamma y_i)^{p-1}\omega(x-\rho\gamma y_j)dx \\
&= \int \left[\nabla\omega(x-\rho\gamma y_i)\nabla\omega(x-\rho\gamma y_j) + \omega(x-\rho\gamma y_i)\omega(x-\rho\gamma y_j)\right]dx, \\
\widehat{\varepsilon}^{\gamma}_{\rho,i,j} &:= \int \omega(x-\rho\gamma y_i)^{p-1}\omega(x-\rho y_j)dx \\
&= \int \left[\nabla\omega(x-\rho\gamma y_i)\nabla\omega(x-\rho y_j) + \omega(x-\rho\gamma y_i)\omega(x-\rho y_j)\right]dx, \\
\varepsilon_{\rho} &:= \sum_{i\neq j}\varepsilon_{\rho,i,j}, \qquad \varepsilon^{\gamma}_{\rho} := \sum_{i\neq j}\varepsilon^{\gamma}_{\rho,i,j}, \qquad \widehat{\varepsilon}^{\gamma}_{\rho} := \sum_{i\neq j}\widehat{\varepsilon}^{\gamma}_{\rho,i,j}.
\end{aligned}$$

Note that, since γ is a linear isometry, we have that $\varepsilon_{\rho,i,j} = \varepsilon^{\gamma}_{\rho,i,j}$ and $\varepsilon_{\rho} = \varepsilon^{\gamma}_{\rho}$. Lemma 5 together with (4) yield

$$\begin{aligned}
\lim_{\rho\to\infty}\varepsilon_{\rho,i,j}\left(|d_{ij}\rho|^{\frac{N-1}{2}}\exp(d_{ij}\rho)\right) &= \lim_{\rho\to\infty}\varepsilon^{\gamma}_{\rho,i,j}\left(|d_{ij}\rho|^{\frac{N-1}{2}}\exp(d_{ij}\rho)\right) \\
&= \lim_{\rho\to\infty}\widehat{\varepsilon}^{\gamma}_{\rho,i,j}\left(\left|\widehat{d}_{ij}\rho\right|^{\frac{N-1}{2}}\exp(\widehat{d}_{ij}\rho)\right) = \gamma_0 > 0
\end{aligned}$$

where $d_{i,j} := |y_i - y_j| = |\gamma y_i - \gamma y_j|$, $\widehat{d}_{ij} := |\gamma y_i - y_j|$. Therefore, since $\mathrm{dist}(\gamma x, Gx) > \min\{|x-y| : y \in Gx,\, y \neq x\}$, we have that $\widehat{\varepsilon}^{\gamma}_{\rho} = o(\varepsilon_{\rho})$. Lemma 5 together with (4) also yield

$$\begin{aligned}
&\lim_{\rho\to\infty}\left(\int h(x)\omega(x-\rho y_i)^q dx\right)|q\rho|^{\frac{N-1}{2}}\exp(q\rho) = 0 \quad \text{and} \\
&\lim_{\rho\to\infty}\left(\int h(x)\omega(x-\rho\gamma y_i)^q dx\right)|q\rho|^{\frac{N-1}{2}}\exp(q\rho) = 0, \quad \text{if } h \in C_c(\mathbb{R}^N) \text{ and } q > 1.
\end{aligned}$$

Now, arguing as in the proof of Lemma 6, taking into account that $\psi\Delta\psi$, $\psi^p - 1 \in C_c(\mathbb{R}^N)$ and using Lemma 4 we obtain

$$\begin{aligned}
\|\psi\theta_\rho\|^2 &\leq \|\theta_\rho\|^2 - \int(\psi\Delta\psi)\,\theta_\rho^2 = 2m\|\omega\|^2 + 2\varepsilon_\rho + o(\varepsilon_\rho), \\
|\psi\theta_\rho|_p^p &= |\theta_\rho|_p^p + \int(\psi^p - 1)\,|\theta_\rho|^p \geq 2m\,|\omega|_p^p + 2(p-1)\varepsilon_\rho + o(\varepsilon_\rho).
\end{aligned}$$

Therefore, since $\|\omega\|^2 = |\omega|_p^p$, there are positive constants C_0, ρ_0 such that

$$\frac{\|\psi\theta_\rho\|^2}{|\psi\theta_\rho|_p^2} \leq \frac{2m\|\omega\|^2 + 2\varepsilon_\rho + o(\varepsilon_\rho)}{\left(2m\|\omega\|^2 + 2(p-1)\varepsilon_\rho + o(\varepsilon_\rho)\right)^{2/p}} \leq \left(2m\|\omega\|^2\right)^{\frac{p-2}{p}} - C_0\varepsilon_\rho \qquad \text{for } \rho \geq \rho_0.$$

We conclude that

$$J(\pi(\psi\theta_\rho)) = \frac{p-2}{2p}\left(\frac{\|\psi\theta_\rho\|^2}{|\psi\theta_\rho|_p^2}\right)^{\frac{p}{p-2}} < 2mc_\infty \qquad \text{for } \rho \geq \rho_0,$$

as claimed. ■

Proof of Theorem 2. The previous lemma implies that

$$\inf_{\mathcal{N}^\phi} J < 2\ell c_\infty.$$

Lemma 15 implies that this infimum is attained. ■

References

[1] A. Bahri and Y.Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^N$, Revista Matemática Iberoamericana **6** (1990), 1-15.

[2] A. Bahri and P. L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. Henri Poincaré **14** (1997), 365-413.

[3] T. Bartsch, M. Clapp, and T. Weth, *Configuration spaces, transfer, and 2-nodal solutions of a semiclassical nonlinear Schrödinger equation*, Math. Ann. **338** (2007), 147-185.

[4] V. Benci and G. Cerami, *Positive solutions of semilinear elliptic problems in exterior domains,* Arch. Rat. Mech. Anal. **99** (1987), 283-300.

[5] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations I,* Arch. Rat. Mech. Anal. **82** (1983), 313-345.

[6] G. Cerami, *Some nonlinear elliptic problems in unbounded domains,* Milan J. Math **74** (2005), 1-31.

[7] G. Cerami and D. Passaseo, *Existence and multiplicity results for semilinear elliptic Dirichlet problems in exterior domains,* Nonl. Anal. **24** (1995), 1533-1547.

[8] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$*,* Mathematical analysis and applications, Part A, pp. 369-402, Adv. in Math. Suppl. Stud. 7a, , Academic Press, New York-London, 1981.

[9] M. Willem, *Minimax theorems,* PNLDE **24**, Birkhäuser, Boston-Basel-Berlin 1996.

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