Infinitely Many Solutions of Elliptic Problems With Perturbed Symmetries in Symmetric Domains

Mónica Clapp *

Instituto de Matemáticas, UNAM,
Circuito Exterior, C.U., 04510 México D.F.
e-mail: mclapp@math.unam.mx

Eric Hernández-Martínez †

Instituto de Matemáticas, UNAM,
Circuito Exterior, C.U., 04510 México D.F.
e-mail: erichdz@math.unam.mx

Abstract

We study superlinear elliptic boundary value problems with perturbed symmetries in domains which are invariant under the action of a group G. We give conditions on the growth of the nonlinearity which guarantee the existence of infinitely many G-invariant solutions. These conditions improve those obtained by Bahri and Lions (1988) and Bolle, Ghoussoub and Tehrani (2000) if the domain contains a G-orbit of large enough dimension.

2000 Mathematics Subject Classification. 35J20, 35J65.

Key words. Elliptic boundary value problems, perturbed symmetries, multiple symmetric solutions.

1 Introduction

Consider the problem

$$(\wp) \quad \begin{cases} -\Delta u = |u|^{p-2}\, u + f & \text{in } \Omega \\ \quad u = u_0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $p \in (2, 2^*)$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, $f \in C(\overline{\Omega})$, and $u_0 \in C^2(\partial\Omega)$.

*The author acknowledges the support of PAPIIT IN110902-3 and CONACYT 43724.

†The author acknowledges the support of PAPIIT IN110902-3 and CONACYT 43724.

If $f = u_0 = 0$ this problem is symmetric (i.e. u is a solution if and only if $-u$ is), and well known equivariant variational methods yield infinitely many solutions. If either f or u_0 are nontrivial, the problem is not symmetric, and those methods do not apply any more.

The first results for the nonsymmetric case were obtained by Bahri and Berestycki [2], Struwe [19], and Rabinowitz [17] in the early 1980s. They established the existence of infinitely many solutions if p is sufficiently small and $u_0 = 0$. Similar results for $u_0 \neq 0$ were obtained by Candela and Salvatore [8]. The best condition on p, so far, is due to Bahri and Lions [3], who showed that problem $(\wp)$ has infinitely many solutions if $u_0 = 0$ and $p \in (2, \frac{2N-2}{N-2})$. For $u_0 \neq 0$ Bolle, Ghoussoub and Tehrani [4] established the existence of infinitely many solutions if $p \in (2, \frac{2N}{N-1})$.

Here we consider domains Ω which are invariant under the action of a group G. If some G-orbit in Ω has large enough dimension we improve the conditions on p given in [3] and [4]. Before stating our result we need some definitions.

Let G be a closed subgroup of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$. Assume that Ω is G-invariant, and that f and u_0 are G-invariant functions. Recall that a subset X of $\mathbb{R}^N$ is *G-invariant* if $gx \in X$ for all $x \in X$, $g \in G$, and a function $h : X \to \mathbb{R}$ is *G-invariant* if $h(gx) = h(x)$ for all $x \in X$, $g \in G$. Consider the problem

$$(\wp_G) \qquad \begin{cases} -\Delta u = |u|^{p-2}\, u + f & \text{in } \Omega \\ \quad u = u_0 & \text{on } \partial\Omega \\ u(gx) = u(x) & \forall x \in \Omega,\ g \in G. \end{cases}$$

Let $m := \max\{\dim Gx : x \in \Omega\}$, where $Gx := \{gx : g \in G\}$ is the G-orbit of x, and let $p'_{N,m}$ be the maximal root of the equation

$$(2N - m - 2)p^2 - (5N - 2m - 2)p + 2N = 0.$$

Set

$$p_{N,m} := \max\left\{p'_{N,m}, \frac{2N-2}{N-2}\right\}, \qquad \widetilde{p}_{N,m} := \max\left\{\frac{6N-4m}{3N-2m-2}, \frac{2N}{N-1}\right\}.$$

For $u \in H_0^1(\Omega)$ we consider the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2}.$$

We prove the following result.

Theorem 1.1 *If either $u_0 = 0$ and $p \in (2, p_{N,m})$, or if $u_0 \neq 0$ and $p \in (2, \widetilde{p}_{N,m})$, then problem $(\wp_G)$ has an unbounded sequence (u_k) of G-invariant solutions which satisfy*

$$\|u_k\|^2 \leq Ck^\gamma, \tag{1.1}$$

where C is a positive constant and $\gamma = 2p/(N-m)(p-2)$.

One has that $p_{N,m} > \frac{2N-2}{N-2}$ if $m > \frac{N^2}{2(N-1)}$, and $\widetilde{p}_{N,m} > \frac{2N}{N-1}$ if $m > \frac{N}{2}$, so our conditions on p are better than those in [3] and [4] in these cases. Note that $p_{N,m} < 2^*$ and $\widetilde{p}_{N,m} < 2^*$. One expects problem $(\wp)$ to have infinitely many solutions for every $p \in (2, 2^*)$. Bahri [1] showed that this is indeed true for almost every $f \in L^2(\Omega)$ and $u_0 = 0$. In the radial case $G = O(N)$, $N \geq 4$, an optimal result was obtained by Candela, Palmieri and Salvatore [7]. They showed that problem $(\wp_{O(N)})$ has infinitely many radial solutions for every $p \in (2, 2^*)$ if Ω is a ball.

The main ingredient in the proof of Theorem 1.1 is an asymptotic formula for the G-invariant Dirichlet eigenvalues of $-\Delta$ in Ω obtained by Brüning and Heinze [5, 6] and Donnelly [12] (see Theorem 3.1). If G is the trivial group, the estimate (1.1) on the norm of the solutions was obtained in [9].

2 The variational problem

Let G be a closed subgroup of $O(N)$. We assume Ω to be G-invariant, and $f \in C(\overline{\Omega})$ and $u_0 \in C^2(\partial\Omega)$ to be G-invariant functions. We denote also by $u_0 \in C^2(\overline{\Omega})$ the unique extension of the given u_0 which satisfies $-\Delta u_0 = 0$ in Ω. Note that this extension is also G-invariant. We write $|\cdot|_p$ for the norm in $L^p(\Omega)$. The group G acts on $H_0^1(\Omega)$ by

$$(g, u) \mapsto gu, \qquad (gu)(x) := u(g^{-1}x).$$

The functional

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u + u_0|_p^p - \int_\Omega fu,$$

is invariant with respect to this action, i.e. $J(gu) = J(u)$ for every $g \in G$, $u \in H_0^1(\Omega)$. So, by the principle of symmetric criticality [15], the critical points of its restriction $J : H_0^1(\Omega)^G \to \mathbb{R}$ to the fixed point set

$$H_0^1(\Omega)^G = \{u \in H_0^1(\Omega) : u(gx) = u(x) \;\; \forall x \in \Omega, \;\; g \in G\}$$

are precisely the solutions of problem $(\wp_G)$.

Next, consider the path of functionals $I_t : H_0^1(\Omega)^G \to \mathbb{R}$, $t \in [0, 1]$, given by

$$I_t(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u + tu_0|_p^p - t\int_\Omega fu. \tag{2.1}$$

Note that $I_1 = J$, and that $I := I_0$ is an even functional, i.e. $I(u) = I(-u)$. It is well known that I has an unbounded sequence of critical values. Bolle, Ghoussoub, and Tehrani established conditions on the path I_t which guarantee that I_1 has also an unbounded sequence of critical values. We recall their result.

Let X be an infinite-dimensional Hilbert space and let $I : X \times [0, 1] \to \mathbb{R}$ be a C^2-functional. We think of I as being a path of functionals

$$I_t : X \to \mathbb{R}, \qquad I_t(u) := I(u, t), \qquad 0 \leq t \leq 1,$$

and denote by $I_t'(u) = \frac{\partial}{\partial u} I(u, t)$ the derivative of I_t. Assume the following holds.

(P1) Every sequence $(u_n, t_n) \in X \times [0,1]$ such that $(I_{t_n}(u_n))$ is bounded and $\left\|I'_{t_n}(u_n)\right\| \to 0$ has a convergent subsequence.

(P2) For every $b \in \mathbb{R}$ there is a constant C such that

$$\left|\frac{\partial}{\partial t} I(u,t)\right| \leq C(\|I'_t(u)\| + 1)(\|u\| + 1) \qquad \text{if } |I_t(u)| \leq b.$$

(P3) There exist two continuous functions $\theta_1, \theta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\theta_1 \leq \theta_2$, which are Lipschitz continuous on the second variable and such that

$$\theta_1(t, I_t(u)) \leq \frac{\partial}{\partial t} I(u,t) \leq \theta_2(t, I_t(u)) \quad \text{if } I'_t(u) = 0.$$

(P4) I_0 is even and, for every finite dimensional subspace W of X,

$$\sup_{0 \leq t \leq 1} I_t(w) \to -\infty \qquad \text{as } w \in W, \ \|w\| \to \infty.$$

Fix a sequence of linear subspaces $X_1 \subset \cdots \subset X_k \subset \cdots$ of X with $\dim X_k = k$, and define

$$c_k := \inf_{\varphi \in \Gamma} \sup_{u \in X_k} I_0(\varphi(u))$$

where

$$\Gamma = \{\varphi \in C(X,X) : \varphi \text{ is odd and } \exists R > 0 \text{ such that } \varphi(u) = u \text{ for } \|u\| > R\}.$$

Let $\zeta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be defined by

$$\begin{cases} \zeta_i(0,s) = s \\ \frac{\partial}{\partial t}\zeta_i(t,s) = \theta_i(t, \zeta_i(t,s)) \end{cases} \tag{2.2}$$

The following result was proved in [4] (see Proof of Theorem 2.2 in [4]).

Theorem 2.1 (Bolle, Ghoussoub, Tehrani) *Assume that I satisfies properties* (P1)-(P4)*. If $\zeta_2(1, c_k + \varepsilon) < \zeta_1(1, c_{k+1})$ for some $\varepsilon > 0$, then for every $\varphi \in \Gamma$ such that $\sup_{\varphi(X_k)} I_0 < c_k + \varepsilon$ there is a critical value $\widetilde{c}_k$ of I_1 which satisfies*

$$\zeta_2(1, c_k) < \zeta_1(1, c_{k+1}) \leq \widetilde{c}_k \leq \zeta_2(1, \sup_{u \in X_{k+1}} I_0(\varphi(u))). \tag{2.3}$$

Moreover, if the sequence

$$\left(\frac{c_{k+1} - c_k}{\max\limits_{0 \leq t \leq 1} |\theta_1(t, c_{k+1})| + \max\limits_{0 \leq t \leq 1} |\theta_2(t, c_k)| + 1} \right) \tag{2.4}$$

is unbounded, then $(\widetilde{c}_k)$ is unbounded.

We wish to apply this result to the path of functionals defined in (2.1). The main step consists in estimating the growth of the values c_k.

3 An unbounded sequence of solutions

Consider the G-invariant eigenvalue problem

$$-\Delta u = \lambda u, \qquad u \in H_0^1(\Omega)^G.$$

Let

$$0 < \lambda_1^G \leq \lambda_2^G \leq \ldots \leq \lambda_k^G \leq \ldots$$

be the G-invariant eigenvalues, counted with their multiplicities, and let $e_k^G \in H_0^1(\Omega)^G$ be the corresponding eigenfunctions with $\left|e_k^G\right|_2 = 1$. Set

$$X_k := \mathrm{span}\{e_1^G, \ldots, e_k^G\}.$$

Consider the even functional $I : H_0^1(\Omega)^G \to \mathbb{R}$,

$$I(u) := \frac{1}{2}\left\|u\right\|^2 - \frac{1}{p}\left|u\right|_p^p,$$

and define

$$c_k^G := \inf_{\varphi \in \Gamma^G} \sup_{u \in X_k} I(\varphi(u)), \tag{3.1}$$

where

$$\Gamma^G := \{\varphi \in C(H_0^1(\Omega)^G, H_0^1(\Omega)^G) : \varphi \text{ is odd, and } \varphi(u) = u \text{ if } \|u\| \text{ is large enough}\}.$$

If G is the trivial group we denote these values by c_k^1. Bahri and Lions [3] showed that there are positive constants b_1, b_2 and $k_0 \in \mathbb{N}$ such that

$$b_1 k^{2p/N(p-2)} \leq c_k^1 \leq b_2 k^{2p/N(p-2)} \qquad \forall k \geq k_0. \tag{3.2}$$

We now wish to estimate the values c_k^G.

Brüning and Heinze [5, 6], and Donnelly [12] studied the asymptotic behavior of the G-invariant eigenvalues. They proved the following result.

Theorem 3.1 (Brüning-Heintze, Donnelly) *For $k \to \infty$ one has the asymptotic formula*

$$\lambda_k^G \sim \beta_0 k^{2/(N-m)},$$

where β_0 is a positive constant depending only on Ω and G, and $m = m(G, \Omega)$ is the dimension of the principal G-orbits of Ω, i.e.

$$m := \max\{\dim(Gx) : x \in \Omega\}, \qquad \text{where } Gx := \{gx : g \in G\}. \tag{3.3}$$

We shall use this result to obtain estimates for c_k^G. We need the following lemma.

Lemma 3.1 *For every $\rho > 0$ one has that*

$$c_k^G \geq \inf\{I(u) : u \in X_{k-1}^\perp,\ \|u\| = \rho\},$$

where $X_k^\perp$ is the orthogonal complement of X_k in $H_0^1(\Omega)^G$.

Proof. Assume, by contradiction, that there exist $\rho, \varepsilon > 0$ such that $c_k^G + \varepsilon < \inf\{I(u) : u \in S_\rho X_{k-1}^\perp\}$, where $S_\rho X_{k-1}^\perp := \{u \in X_{k-1}^\perp : \|u\| = \rho\}$. By (3.1) we may choose an odd map $\varphi : H_0^1(\Omega)^G \to H_0^1(\Omega)^G$ with $I(\varphi(u)) \leq c_k^G + \varepsilon$ for all $u \in X_k$, and $\varphi(u) = u$ if $\|u\| \geq R$. Thus, $\varphi(X_k) \cap S_\rho X_{k-1}^\perp = \emptyset$. Set $E_R := \{u \in H_0^1(\Omega)^G : \|u\| \geq R\}$. We consider the map

$$\eta : H_0^1(\Omega)^G \setminus S_\rho X_{k-1}^\perp \longrightarrow X_{k-1} \cup E_R =: Y$$

defined as follows: Let $\pi(u)$ be the orthogonal projection of u onto $X_{k-1}^\perp$. If $u \notin X_{k-1}$, let $\pi_\rho(u)$ be the positive multiple of $\pi(u)$ of length ρ, and let $t_u := \min\{t \geq 0 : \pi_\rho(u) + t(u - \pi_\rho(u)) \in Y\}$. Set

$$\eta(u) := \begin{cases} \pi_\rho(u) + t_u(u - \pi_\rho(u)) & \text{if } u \in H_0^1(\Omega)^G \setminus (S_\rho X_{k-1}^\perp \cup Y) \\ u & \text{if } u \in Y. \end{cases}$$

Note that η is an odd map. The composition $\eta \circ \varphi : X_k \to Y$ is well defined, odd and continuous, and satisfies $(\eta \circ \varphi)(u) = u$ if $\|u\| \geq R$. Therefore, $\eta \circ \varphi$ induces a map of the quotient spaces obtained by identifying E_R to a point and, composing both ends with radial homeomorphisms, we obtain a map

$$X_k \cup \{\infty\} \cong X_k / X_k \cap E_R \xrightarrow{\eta \circ \varphi} Y/E_R \cong X_{k-1} \cup \{\infty\}$$

which is odd in X_k and maps ∞ to ∞. This contradicts the Borsuk-Ulam theorem for representation spheres with fixed points [14, 10]. ■

Corollary 3.1 *There exist $\beta_1 > 0$ and $k_1 \in \mathbb{N}$ such that*

$$\beta_1 k^\alpha \leq c_k^G \qquad \forall k \geq k_1,$$

where $\alpha := \max\left\{\frac{(N+2)-(N-2)(p-1)}{(N-m)(p-2)}, \frac{2p}{N(p-2)}\right\}$ *with m as in* (3.3).

Proof. Let $v \in X_{k-1}^\perp$. Using the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} I(v) &= \frac{1}{2}\|v\|^2 - \frac{1}{p}|v|_p^p \\ &\geq \frac{1}{2}\|v\|^2 - \frac{a_1}{p}\|v\|^{p(1-s)}|v|_2^{ps} \\ &\geq \frac{1}{2}\|v\|^2 - \frac{a_1}{p}\left(\lambda_k^G\right)^{-ps/2}\|v\|^p, \end{aligned} \tag{3.4}$$

where a_1 is a positive constant and $s \in (0,1)$ satisfies $\frac{1}{p} = \frac{1-s}{2^*} + \frac{s}{2}$. Set

$$\rho := \left(a_1^{-1}\left(\lambda_k^G\right)^{ps/2}\right)^{1/(p-2)}.$$

If $v \in X_{k-1}^\perp$ and $\|v\| = \rho$, we obtain from (3.4) that

$$I(v) \geq \left(\frac{1}{2} - \frac{a_1}{p}\left(\lambda_k^G\right)^{-ps/2}\rho^{p-2}\right)\rho^2 = \frac{p-2}{2p}\rho^2 = a_2\left(\lambda_k^G\right)^{ps/(p-2)},$$

and Lemma 3.1 yields

$$c_k^G \geq a_2 \left(\lambda_k^G\right)^{ps/(p-2)}.$$

By Theorem 3.1 there exist $k_1 \in \mathbb{N}$ and $a_3 > 0$ such that $\lambda_k^G \geq a_3 k^{2/(N-m)}$ for all $k \geq k_1$. Therefore,

$$c_k^G \geq \beta_1 k^{2ps/(N-m)(p-2)} \qquad \forall k \geq k_1.$$

Observe that $2ps = (N+2) - (N-2)(p-1)$. On the other hand, the Bahri-Lions estimate (3.2) yields

$$c_k^G \geq c_k^1 \geq b_1 k^{2p/N(p-2)},$$

and our claim is proved. ■

Corollary 3.2 *There exists $\beta_2 > 0$ and $k_2 \in \mathbb{N}$ such that*

$$c_k^G \leq \beta_2 k^\gamma \qquad \forall k \geq k_2,$$

where $\gamma := \dfrac{2p}{(N-m)(p-2)}$ *with m as in* (3.3).

Proof. Let $u \in X_k$. Then, for some constant $a_1 > 0$, we have that

$$\|u\|^2 \leq \lambda_k^G |u|_2^2 \leq a_1 \lambda_k^G |u|_p^2.$$

It follows that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p \\ &\leq \frac{1}{2}\|u\|^2 - \frac{a_2}{p}\left(\lambda_k^G\right)^{-p/2}\|u\|^p \\ &\leq \frac{p-2}{2p} a_2^{-2/p-2}\left(\lambda_k^G\right)^{p/p-2}. \end{aligned}$$

Hence,

$$c_k^G \leq \sup_{u \in X_k} I(u) \leq a_3 \left(\lambda_k^G\right)^{p/p-2}.$$

On the other hand, by Theorem 3.1, there exists $a_4 > 0$and $k_2 \in \mathbb{N}$ such that $\lambda_k^G \leq a_4 k^{2/(N-m)}$ for all $k \geq k_2$. Therefore,

$$c_k^G \leq \beta_2 k^\gamma,$$

for some positive constant β_2 and $\gamma = 2p/(N-m)(p-2)$, as claimed. ■

We apply Corollary 3.1 to prove the following results.

Theorem 3.2 *For every $p \in (2, \widetilde{p}_{N,m})$ the functional $J : H_0^1(\Omega)^G \to \mathbb{R}$ given by*

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u + u_0|_p^p - \int_\Omega fu$$

has an unbounded sequence of critical values $\widetilde{c}_k^G$.

Proof. Bolle, Ghoussoub and Tehrani [4] showed that the path of functionals

$$I_t(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u + tu_0|_p^p - t\int_\Omega fu$$

satisfies properties (P1)-(P4) with $\theta_2(t,s) = a(s^2+1)^{1/4} = -\theta_1(t,s)$, $a > 0$. Now we show that, if $p \in (2, \widetilde{p}_{N,m})$, the sequence

$$\left(\frac{c_{k+1}^G - c_k^G}{a((c_{k+1}^G)^2+1)^{1/4} + a((c_k^G)^2+1)^{1/4}+1}\right) \tag{3.5}$$

is unbounded. Assume, by contradiction, that it is bounded. Then $c_k^G \leq a_1 k^2$ for large enough k (see [16] p.68) and, by Corollary 3.1, we have that $\alpha \leq 2$, that is,

$$p \geq \frac{6N-4m}{3N-2m-2} \quad \text{and} \quad p \geq \frac{2N}{N-1}.$$

Hence $p \geq \widetilde{p}_{N,m}$, contradicting our assumption. Thus, the sequence (3.5) is unbounded, and Theorem 2.1 gives an unbounded sequence $\widetilde{c}_k^G$ of critical values for $J = I_1$. ■

Theorem 3.3 *For every $p \in (2, p_{N,m})$ the functional $J : H_0^1(\Omega)^G \to \mathbb{R}$ given by*

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p - \int_\Omega fu$$

has an unbounded sequence of critical values $\widetilde{c}_k^G$.

Proof. It is easy to see that the path of functionals

$$I_t(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p - t\int_\Omega fu$$

satisfies properties (P1)-(P4) with $\theta_2(t,s) = a(s^2+1)^{1/2p} = -\theta_1(t,s)$. Now, if the sequence

$$\left(\frac{c_{k+1}^G - c_k^G}{a((c_{k+1}^G)^2+1)^{1/2p} + a((c_k^G)^2+1)^{1/2p}+1}\right)$$

were bounded, we would have that $c_k^G \leq a_1 k^{p/(p-1)}$ for large enough k. Then, by Corollary 3.1, we would have $\alpha \leq \frac{p}{p-1}$, that is,

$$(2N-m-2)p^2 - (5N-2m-2)p + 2N \geq 0 \quad \text{and} \quad p \geq \frac{2N-2}{N-2},$$

contradicting our assumption. Now we apply Theorem 2.1 to obtain an unbounded sequence $\widetilde{c}_k^G$ of critical values for $J = I_1$. ■

4 Estimates on the energy of the solutions

Let $\widetilde{c}_k^G$ be the critical values of $J : H_0^1(\Omega)^G \to \mathbb{R}$ given by Theorems 3.2 and 3.3. We shall prove the following.

Proposition 4.1 *There is a positive constant β_3 and $k_3 \in \mathbb{N}$ such that*

$$\widetilde{c}_k^G \leq \beta_3 k^\gamma \qquad \forall k \geq k_3,$$

where $\gamma = 2p/(N-m)(p-2)$.

We start with some lemmas. Set

$$I_C^\#(u) := \frac{C}{2} \|u\|^2 - \frac{1}{p} |u|_p^p.$$

Lemma 4.1 *If $C \geq 1$, then there exists a continuous function $\varrho : H_0^1(\Omega)^G \to [0,\infty)$ with the following properties:*
(i) $I([(1-s)+s\varrho(u)]\,u) \leq I(u)$ for every $u \in H_0^1(\Omega)^G$, $s \in [0,1]$.
(ii) If $I_C^\#(u) \leq 0$ then $\varrho(u) = 1$.
(iii) If $2I(u) \leq \max_{t\geq 0} I(tu)$ then $I_C^\#(\varrho(u)u) \leq 0$.
(iv) If $2I(u) \geq \max_{t\geq 0} I(tu)$ then $I_C^\#(\varrho(u)u) \leq \kappa_0 I(u)$ with $\kappa_0 \geq 1$.

Proof. For each $v \in H_0^1(\Omega)^G$ with $\|v\| = 1$ consider the numbers $0 < t_v^- < \widehat{t}_v < t_v^+ < T_v < \infty$ which satisfy the following:

$$I(\widehat{t}_v v) = \max_{t\geq 0} I(tv), \qquad I_C^\#(T_v v) = 0,$$
$$2I(tv) \geq \max_{t\geq 0} I(tv) \Longleftrightarrow t \in [t_v^-, t_v^+].$$

Let $\rho_v : [0,\infty) \to [0,\infty)$ be such that $\rho_v(t) = 0$ if $t \in [0, t_v^-]$, ρ_v maps $[t_v^-, \widehat{t}_v]$ linearly onto $[0, \widehat{t}_v]$, and maps $[\widehat{t}_v, t_v^+]$ linearly onto $[\widehat{t}_v, T_v]$, $\rho_v(t) = T_v$ if $t \in [t_v^+, T_v]$, and $\rho_v(t) = t$ if $t \geq T_v$. Now define $\varrho : H_0^1(\Omega)^G \to [0,\infty)$ by

$$\varrho(u) := \begin{cases} \frac{1}{\|u\|} \rho_{u/\|u\|}(\|u\|) & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

It is straightforward to verify that ϱ has the desired properties. ■

Lemma 4.2 *There exist a map*

$$H_0^1(\Omega)^G \times [0,2] \to H_0^1(\Omega)^G, \qquad (u,s) \mapsto u_s,$$

and a $\kappa_1 > 0$ such that $u_0 = u$, $\Omega \smallsetminus \mathrm{supp}(u_2) \neq \emptyset$, and $I(u_s) \leq \max\{\kappa_1 I(u), 0\}$ for every $u \in H_0^1(\Omega)^G$, $s \in [0,2]$.

Proof. Let $R := \max\{|x| : x \in \overline{\Omega}\}$. Fix $r \in (0, R)$ such that $tx \in \Omega$ if $x \in \partial\Omega$, $|x| > r$, and $t \in (r\,|x|^{-1}, 1)$. Choose a nondecreasing C^∞ function $\chi : [0,\infty) \to \mathbb{R}$ such that $\chi(t) = 0$ if $t \in [0, r]$ and $\chi(R) = 1$. For each $s \in [1, 2]$ consider the function $\tau_s : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\tau_s(x) := (1 + (s-1)\chi(|x|))\, x.$$

Then, $\tau_s(x) = x$ if $|x| \leq r$ or $s = 1$, $\tau_s(\partial\Omega) \subset \mathbb{R}^N \smallsetminus \Omega$, and there exists a $d \in (r, R)$ such that $|\tau_2(x)| \geq R$ if $|x| \geq d$. Moreover, $\tau_s(gx) = g\tau_s x$ for all $g \in G$. Let $C_1 := \max\{\|D\tau_s(x)\|^2 : x \in \overline{\Omega},\ s \in [1,2]\}$, and $C_2 := \max\{|\det D\tau_s(x)| : x \in \overline{\Omega}, s \geq 1\}$. For $C := C_1C_2$ we consider the function $\varrho : H_0^1(\Omega)^G \to [0,\infty)$ given by the previous lemma. If $u \in H_0^1(\Omega)^G$ we consider it as an element of $H^1(\mathbb{R}^N)^G$ by extending it to cero outside Ω, and define

$$u_s(x) := \begin{cases} [(1-s) + s\varrho(u)]\, u(x) & \text{if } s \in [0,1] \\ \varrho(u)u(\tau_s x) & \text{if } s \in [1,2]. \end{cases}$$

Then $u_s \in H_0^1(\Omega)^G$, $u_0 = u$, and $\Omega \smallsetminus \mathrm{supp}(u_2) \supset \{x \in \Omega : |x| \geq d\} \neq \emptyset$. Let $s \in [1, 2]$. Then

$$|\nabla u_s(x)|^2 \leq \varrho(u)^2 |\nabla u(\tau_s(x))|^2 \|D\tau_s(x)\|^2 \leq C_1\varrho(u)^2 |\nabla u(\tau_s(x))|^2 .$$

Note that $\min\{|\det D\tau_s(x)| : x \in \overline{\Omega},\ s \geq 1\} \geq 1$. Therefore,

$$\begin{aligned} \int |\nabla u_s|^2 &\leq \int |\nabla u_s(x)|^2 |\det D\tau_s(x)|\, dx \\ &\leq C_1 \int |\varrho(u)\nabla u(\tau_s(x))|^2 |\det D\tau_s(x)|\, dx = C_1 \int |\nabla(\varrho(u)u)|^2 . \end{aligned}$$

On the other hand,

$$\begin{aligned} \int |\varrho(u)u|^p &= \int |\varrho(u)u(\tau_s(x))|^p |\det D\tau_s(x)|\, dx \\ &= \int |u_s(x)|^p |\det D\tau_s(x)|\, dx \leq C_2 \int |u_s|^p . \end{aligned}$$

It follows that

$$\begin{aligned} I(u_s) &= \frac{1}{2}\|u_s\|^2 - \frac{1}{p}|u_s|_p^p \leq C_2^{-1}\left[\frac{C}{2}\|\varrho(u)u\|^2 - \frac{1}{p}|\varrho(u)u|_p^p\right] \\ &= C_2^{-1} I_C^{\#}(\varrho(u)u) \leq \max\{\kappa_1 I(u), 0\} \qquad \text{if } s \in [1,2], \end{aligned}$$

with $\kappa_1 := \max\{C_2^{-1}\kappa_0, 1\}$ and κ_0 as in Lemma 4.1. That lemma also implies that

$$I\left([(1-s) + s\varrho(u)]\, u\right) \leq I(u) \leq \max\{\kappa_1 I(u), 0\} \quad \text{if } s \in [0,1].$$

This concludes the proof. ■

Lemma 4.3 *There are constants $\kappa_1, \kappa_2 > 0$, depending only on G, Ω and p, with the following property: For each pair of finite dimensional subspaces $V \subset W$ of $H_0^1(\Omega)^G$ with $\dim W = \dim V + 1$, and every odd map $\varphi : V \to H_0^1(\Omega)^G$ and $R > 0$ satisfying $\varphi(v) = v$ if $\|v\| \geq R$, there exist $\widetilde{R} > R$ and an odd map $\widetilde{\varphi} : W \to H_0^1(\Omega)^G$ such that*
(i) $\widetilde{\varphi}(v) = \varphi(v)$ *for every* $v \in V$,
(ii) $\widetilde{\varphi}(w) = w$ *for every* $w \in W$ *with* $\|w\| \geq \widetilde{R}$,
(iii) $\max_{w \in W} I(\widetilde{\varphi}(w)) \leq \kappa_1 \max_{v \in V} I(\varphi(v)) + \kappa_2$.

Proof. Fix $\omega \in H_0^1(\Omega)^G$ such that $\operatorname{supp}(\omega) \subset \Omega \smallsetminus \operatorname{supp}(u_2)$, and $\max_{t \geq 0} I(t\omega) \leq I(\omega) =: \kappa_2$. Fix an $e \in W$, orthogonal to V, with $\|e\| = 1$. We extend φ to the halfspace $W^+ = \{v + re : v \in V,\ r \geq 0\}$ as follows

$$\psi(v + se) = \begin{cases} \varphi(v)_s & \text{if } v \in V,\ s \in [0,2] \\ \varphi(v)_2 + (s-2)\omega & \text{if } v \in V,\ s \in [2, \infty) \end{cases}$$

where u_s is as in Lemma 4.2. Since $\varphi(v)_2$ and ω have disjoint supports, we have that

$$I(\psi(v + se)) \leq \begin{cases} \max\{\kappa_1 I(\varphi(v)), 0\} & \text{if } v \in V,\ s \in [0,2] \\ \max\{\kappa_1 I(\varphi(v)), 0\} + I((s-2)\omega) & \text{if } v \in V,\ s \in [2, \infty). \end{cases}$$

Therefore,

$$I(\psi(v + se)) \leq \max\{\kappa_1 I(\varphi(v)), 0\} + \kappa_2 \qquad \forall v \in V,\ s \in [0, \infty). \tag{4.1}$$

Since $\dim V < \infty$, there exists $R_1 \geq R$ such that $\kappa_1 I(v) \leq -\kappa_2$ if $v \in V$ and $\|v\| \geq R_1$, and $I((s-2)\omega) \leq -\max_{v \in V} \kappa_1 I(\varphi(v))$ if $s \geq R_1$. Then,

$$I(\psi(v + se)) \leq 0 \quad \text{if either } v \in V \text{ and } \|v\| \geq R_1, \text{ or } s \geq R_1,$$

so we may fix $R_2 \geq R_1$ such that $I(\psi(w)) \leq 0$ for every $w \in W^+$ with $\|w\| \geq R_2$. Since $\{u \in H_0^1(\Omega)^G : u \neq 0,\ I(u) \leq 0\}$ is homotopy equivalent to the unit sphere in $H_0^1(\Omega)^G$, it is contractible. Hence, there is a homotopy

$$\Psi : \{w \in W^+ : \|w\| = R_2\} \times [0,1] \to \{u \in H_0^1(\Omega)^G : u \neq 0, I(u) \leq 0\}$$

such that $\Psi(w, 0) = \psi(w)$, $\Psi(w, 1) = w$ and $\Psi(v, t) = v$ if $v \in V$ and $t \in [0,1]$. We define $\widetilde{\varphi} : W^+ \to H_0^1(\Omega)^G$ by

$$\widetilde{\varphi}(w) := \begin{cases} \psi(w) & \text{if } w \in W^+,\ \|w\| \leq R_2 \\ \frac{\|w\|}{R_2} \Psi(R_2 \frac{w}{\|w\|}, \|w\| - R_2) & \text{if } w \in W^+,\ R_2 \leq \|w\| \leq R_2 + 1 \\ w & \text{if } w \in W^+,\ R_2 + 1 \leq \|w\| \end{cases}$$

and extend it to an odd map $\widetilde{\varphi} : W \to H_0^1(\Omega)^G$ by setting $\widetilde{\varphi}(w) =: -\widetilde{\varphi}(-w)$ if $-w \in W^+$. Since φ is odd, $\widetilde{\varphi}$ is well defined and continuous. It satisfies *(i)* and *(ii)* with $\widetilde{R} := R_2 + 1$. Moreover, since I is even and $I(\widetilde{\varphi}(w)) \leq 0$ if $\|w\| \geq R_2$, inequality (4.1) yields

$$\max_{w \in W} I(\widetilde{\varphi}(w)) \leq \kappa_1 \max_{v \in V} I(\varphi(v)) + \kappa_2,$$

so *(iii)* holds. ■

Proof of Proposition 4.1. Let $\zeta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be given by

$$\begin{cases} \zeta_1(0,s) = s \\ \frac{\partial}{\partial t}\zeta_1(t,s) = -\theta(\zeta_1(t,s)) \end{cases} \qquad \begin{cases} \zeta_2(0,s) = s \\ \frac{\partial}{\partial t}\zeta_2(t,s) = \theta(\zeta_2(t,s)) \end{cases}$$

where $\theta(s) = a\left(s^2+1\right)^{1/4}$ if $u_0 \neq 0$, and $a\left(s^2+1\right)^{1/2p}$ if $u_0 = 0$. Let c_k^G be the values defined in (3.1). If $\zeta_2(1, c_k^G + \varepsilon) < \zeta_1(1, c_{k+1}^G)$ for some $\varepsilon \in (0,1)$, we fix $\varphi \in \Gamma^G$ such that

$$\sup_{u \in X_k} I(\varphi(u)) < c_k^G + \varepsilon$$

and apply Lemma 4.3 to obtain an odd map $\tilde{\varphi} : X_{k+1} \to H_0^1(\Omega)^G$ such that $\tilde{\varphi}(u) = \varphi(u)$ for $u \in X_k$, $\tilde{\varphi}(u) = u$ if $\|u\| > R$ and

$$\sup_{u \in X_{k+1}} I\left(\tilde{\varphi}(u)\right) \leq \kappa_1\left(c_k^G + \varepsilon\right) + \kappa_2 < \kappa_1 c_k^G + \kappa_1 + \kappa_2.$$

By Tietze's extension theorem, $\tilde{\varphi}$ can be extended to an odd map $\tilde{\varphi} : H_0^1(\Omega)^G \to H_0^1(\Omega)^G$ which satisfies $\tilde{\varphi}(u) = u$ if $\|u\| > R$. Theorem 2.1 asserts that

$$\widetilde{c}_k^G \leq \zeta_2(1, \sup_{u \in X_{k+1}} I\left(\tilde{\varphi}(u)\right)) \leq \zeta_2(1, \kappa_1 c_k^G + \kappa_1 + \kappa_2). \tag{4.2}$$

On the other hand, by definition of ζ_2, we have that $|s - \zeta_2(t,s)| \leq a_1 |\theta(s)|$ with $a_1 > 0$. Hence,

$$\widetilde{c}_k^G \leq a_1\theta\left(\kappa_1 c_k^G + \kappa_1 + \kappa_2\right) + \kappa_1 c_k^G + \kappa_1 + \kappa_2 \leq \kappa_3(c_k^G + 1) \tag{4.3}$$

Corollary 3.2 now yields that

$$\widetilde{c}_k^G \leq \beta_3 k^\gamma,$$

as claimed. ■

Theorems 3.2 and 3.3, together with Proposition 4.1 give Theorem 1.1.

References

[1] A. Bahri, *Topological results on certain class of functionals and applications,* J. Funct. Anal. **41** (1981), 397-427.

[2] A. Bahri, H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), 1-32.

[3] A. Bahri, P.L. Lions, *Morse index of some min-max critical points. I. Application to multiplicity results*, Comm. Pure Appl. Math. **41** (1988),1027-1037.

[4] P. Bolle, N. Ghoussoub, and H. Tehrani, *The multiplicity of solutions to non-homogeneous boundary value problems*, Manuscripta Math. **101** (2000), 325-350.

[5] J. Brüning and E. Heintze, *Représentations des groupes d'isométries dans les sous-espaces propres du laplacien,* C. R. Acad. Sc. Paris **286** (1978), 921-923.

[6] J. Brüning and E. Heintze, *Representations of compact Lie groups and elliptic operators*, Inventiones math. **50** (1979), 169-203.

[7] A.M. Candela, G. Palmieri, A. Salvatore, *Radial solutions of semilinear elliptic equations with broken symmetry,* Topol. Meth. Nonl. Anal. **27** (2006), 117-132.

[8] A. M. Candela, A. Salvatore, *Multiplicity results of an elliptic eqaution with non-homogeneus boundary conditions,* Topol. Meth. Nonl. Anal. **11** (1998) 1-18.

[9] A. Castro, M. Clapp, *Upper estimates for the energy of solutions of nonhomogeneous boundary value problems*, Proc. Amer. Math. Soc. **134** (2006), 167-175.

[10] M. Clapp, *Borsuk-Ulam theorems for perturbed symmetric problems*, Nonl. Anal. **47** (2001), 3749-3758.

[11] M. Cwickel, *Weak type estimates and the number of bound states of Schrödinger operators,* Ann. Math. **106** (1977), 93-102.

[12] H. Donnelly, *G -spaces, the asymptotic splitting of $L^2(M)$ into irreducibles*, Math. Ann. **237** (1978), 23-40.

[13] E. H. Lieb, *Bounds on the eigenvlues of the Laplace and Schrödinger operators,* Bull. Amer. Math. Soc. **82** (1976), 751-753.

[14] W. Marzantowicz, *A Borsuk-Ulam theorem for orthogonal T^k and Z_p^k actions with applications*, J. Math. Anal. and Appl. **137** (1989), 99-121.

[15] R. Palais, *The principle of symmetric criticality,* Comm. Math. Phys. **69** (1979), 19-30.

[16] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Reg. Conf. Ser. Math. **65**, Amer. Math. Soc., Providence, RI 1986.

[17] P.H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals,* Trans. Amer. Math. Soc. **272** (1982), 753-770.

[18] G. Rosenbljum, *The distribution of the discrete spectrum for singular differential operators*, Soviet Math. Dokl. **13** (1972), 245-249.

[19] M. Struwe, *Infinitely many critical points for functionals wich are not even and applications to superlinear boundary value problems,* Manuscripta Math. **32** (1980), 335-364.