

INTERTWINING SEMICLASSICAL BOUND STATES TO A NONLINEAR MAGNETIC SCHRÖDINGER EQUATION

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ABSTRACT. We consider the magnetic NLS equation

$$(0.1) \quad (-\varepsilon i \nabla + A(x))^2 u + V(x)u = |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $2 < p < 2^* := 2N/(N-2)$, $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a magnetic potential and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded electric potential. We consider a group G of orthogonal transformations of \mathbb{R}^N , and we assume that $A(gx) = gA(x)$ and $V(gx) = V(x)$ for any $g \in G$, $x \in \mathbb{R}^N$. Given a group homomorphism $\tau : G \rightarrow \mathbb{S}^1$ into the unit complex numbers, we show the existence of semiclassical solutions $u_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{C}$ to problem (0.1), which satisfy

$$u_\varepsilon(gx) = \tau(g)u_\varepsilon(x)$$

for all $g \in G$, $x \in \mathbb{R}^N$. Moreover, we show that there is a combined effect of the symmetries and the electric potential V on the number of solutions of this type.

1. INTRODUCTION

The stationary nonlinear Schrödinger equation

$$(1.1) \quad (-\hbar i \nabla + A(x))^2 u + (U(x) - E)u = |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

arises in various physical contexts such as nonlinear optics or plasma physics where one simulates the interaction effect among many particles by introducing a nonlinear term, cf. [25, 26]. Here \hbar is the Planck constant, $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a magnetic potential associated to a given magnetic field B , i.e. $\text{curl}A = B$, $U : \mathbb{R}^N \rightarrow \mathbb{R}$ is an electric potential, $E \in \mathbb{R}$, and $p \in (2, 2^*)$, where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

Existence of a solution whose modulus $|u_\hbar|$ vanishes at infinity was first proved by Esteban and Lions [19] using a constrained minimization approach; see also [2, 28, 8, 14, 16] for more recent results.

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The study of semiclassical phenomena for equation (1.1) has been extensively pursued in recent years. From a mathematical point of view, the transition from quantum to classical mechanics can be formally performed by replacing \hbar by ε and letting $\varepsilon \rightarrow 0$. Solutions $u_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{C}$ of (1.1) for small values of $\varepsilon > 0$ are usually referred to as semiclassical bound states.

A considerable amount of work has been done in the case $A = 0$, showing that concentration phenomena of single and multiple spike solutions occur at critical points of the electric potential U when $\varepsilon \rightarrow 0$, see e.g. [20, 29, 17, 1, 23, 4] and the references therein.

In the magnetic case $A \neq 0$ existence and multiplicity of semiclassical solutions to problem (1.1) have been shown in [22, 9, 11, 12, 5, 10]. We emphasize that the presence of the magnetic field does not influence the location of the peaks of $|u_\varepsilon|$ as $\varepsilon \rightarrow 0$, while it does contribute to the phase factor of the solution u_ε .

In this work we consider potentials A and U which are symmetric and we are interested in finding semiclassical solutions of (1.1) having specific symmetries. Symmetries occur in a natural way in this type of problems. They are also a powerful tool for obtaining existence and multiplicity results for nonlinear problems, see e.g. [15, 3, 21, 7, 30].

We start by considering the standard magnetic field $\mathcal{B}(x_1, x_2, x_3) := (0, 0, 2)$ in \mathbb{R}^3 and the standard magnetic potential $\mathcal{A}(x_1, x_2, x_3) := (-x_2, x_1, 0)$ associated to it. It is convenient to identify $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ and to write $\mathcal{A}(z, t) = (iz, 0)$, $z = x_1 + ix_2$. Note that \mathcal{A} satisfies $\mathcal{A}(e^{i\theta}z, t) = e^{i\theta}\mathcal{A}(z, t)$ for every $\theta \in \mathbb{R}$. Given $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we look for solutions to problem

$$(1.2) \quad \begin{cases} (-\varepsilon i \nabla + \mathcal{A})^2 u + \mathcal{V}u = |u|^{p-2} u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \varepsilon \nabla u + i\mathcal{A}u \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

which satisfy

$$(1.3) \quad u(e^{2\pi i k/m} z, t) = e^{2\pi i n k/m} u(z, t) \quad \forall k = 1, \dots, m, \quad x \in \mathbb{R}^N.$$

Solutions of this type arise in a natural way in some problems where the magnetic potential is singular, see [16]. Note that if u satisfies (1.2) and (1.3) then $e^{i\theta}u$ satisfies (1.2) and (1.3) for every $\theta \in \mathbb{R}$. We shall say that u and v are *geometrically distinct* if $e^{i\theta}u \neq v$ for all $\theta \in \mathbb{R}$. We make the following assumptions on the electric potential \mathcal{V} .

(\mathcal{V}_1) $\mathcal{V} : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded continuous function such that

$$\mathcal{V}_0 := \inf_{x \in \mathbb{R}^3} \mathcal{V} < \liminf_{|x| \rightarrow \infty} \mathcal{V}(x).$$

(\mathcal{V}_2) There exists $m_0 \in \mathbb{N}$ such that

$$m_0 \mathcal{V}_0^q < \inf_{t \in \mathbb{R}} \mathcal{V}^q(0, t), \quad q := \frac{p}{p-2} - \frac{N}{2}, \quad \text{and}$$

$$\mathcal{V}(e^{2\pi i k/m_0} z, t) = \mathcal{V}(z, t) \quad \forall k = 1, \dots, m_0, \quad (z, t) \in \mathbb{C} \times \mathbb{R}.$$

For each $m \in \mathbb{N}$ which divides m_0 (for short $m \mid m_0$), we consider the group Γ_m generated by the rotation of angle $2\pi/m$ around the t -axis, i.e. $\Gamma_m = \{e^{2\pi i k/m} : k = 1, \dots, m\}$ acting by multiplication on the z -coordinate of each point $(z, t) \in \mathbb{C} \times \mathbb{R}$. The Γ_m -orbit of (z, t) is the set $\Gamma_m(z, t) := \{(e^{2\pi i k/m} z, t) : k = 1, \dots, m\}$. It consists of a single point if $z = 0$ and of m points if $z \neq 0$. If a subset X of $\mathbb{C} \times \mathbb{R}$ is invariant under every rotation in Γ_m one defines its Γ_m -orbit space to be the set

$$X/\Gamma_m := \{\Gamma_m(z, t) : (z, t) \in X\}$$

of Γ_m -orbits in X with the quotient space topology. Under the above assumptions we will show that the topology of the Γ_m -orbit space of the set

$$\mathcal{M} := \{x \in \mathbb{R}^N : \mathcal{V}(x) = \mathcal{V}_0\}$$

has an effect on the number of solutions to problem (1.2) satisfying (1.3). For the precise statement we need the following notation. Given $\rho > 0$ set

$$\bar{B}_\rho \mathcal{M} := \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{M}) \leq \rho\}.$$

If A is a subset of a topological space Y we write $\text{cat}_Y(A)$ for the Lusternik-Schnirelmann category of A in Y .

Theorem 1. *Assume that \mathcal{V} satisfies (\mathcal{V}_1) and (\mathcal{V}_2). Then, given $\rho, \delta > 0$, $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $m \mid m_0$, there exists $\bar{\varepsilon} > 0$ with the property that for every $\varepsilon \in (0, \bar{\varepsilon})$ problem (1.2) has at least*

$$\text{cat}_{\bar{B}_\rho \mathcal{M}/\Gamma_m}(\mathcal{M}/\Gamma_m)$$

geometrically distinct solutions which satisfy (1.3) and

$$\left| \frac{p-2}{2p} \int |u|^p - \varepsilon^N m \mathcal{V}_0^q \right| < \varepsilon^N \delta.$$

So altogether, for ε small enough, problem (1.2) has at least

$$\sum_{m \mid m_0} m \text{cat}_{\bar{B}_\rho \mathcal{M}/\Gamma_m}(\mathcal{M}/\Gamma_m)$$

geometrically distinct solutions with

$$\frac{p-2}{2p} \int |u|^p < \varepsilon^N (m_0 \mathcal{V}_0^q + \delta).$$

Theorem 1 is a special case of Theorem 2 which we are about to state.

Let G be a closed subgroup of the group $O(N)$ of orthogonal transformations of \mathbb{R}^N . We assume $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a C^1 -function, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded continuous function with $\inf_{\mathbb{R}^N} V > 0$, and they satisfy

$$(1.4) \quad A(gx) = gA(x) \quad \text{and} \quad V(gx) = V(x) \quad \forall g \in G, x \in \mathbb{R}^N.$$

Given a group homomorphism $\tau : G \rightarrow \mathbb{S}^1$ into the unit complex numbers, we look for solutions $u : \mathbb{R}^N \rightarrow \mathbb{C}$ to the problem

$$(1.5) \quad \begin{cases} (-\varepsilon i \nabla + A)^2 u + Vu = |u|^{p-2} u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \varepsilon \nabla u + iAu \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

which satisfy

$$(1.6) \quad u(gx) = \tau(g)u(x) \quad \text{for all } g \in G, x \in \mathbb{R}^N.$$

That is, the absolute value $|u|$ of u satisfies

$$|u(gx)| = |u(x)| \quad \text{for all } g \in G, x \in \mathbb{R}^N,$$

and the phase of $u(gx)$ is that of $u(x)$ multiplied by the phase factor $\tau(g)$. They are called τ -intertwining solutions.

Before stating our main result we introduce some notation. For $x \in \mathbb{R}^N$, let Gx denote the G -orbit of x and G_x denote the G -isotropy subgroup of x , that is,

$$Gx := \{gx : g \in G\}, \quad G_x := \{g \in G : gx = x\}.$$

A subset X of \mathbb{R}^N is G -invariant if $Gx \subset X$ for every $x \in X$. The G -orbit space of X is the set of G -orbits of X ,

$$X/G := \{Gx : x \in X\},$$

with the quotient topology.

Let $q := \frac{p}{p-2} - \frac{N}{2}$ and

$$\ell_G := \min_{x \in \mathbb{R}^N} (\#Gx)V^q(x),$$

where $\#Gx$ denotes the cardinality of Gx , and consider the set

$$M_\tau := \{x \in \mathbb{R}^N : (\#Gx)V^q(x) = \ell_G, G_x \subset \ker \tau\}.$$

Observe that the points in M_τ are not necessarily local minima of V . Assumption (1.4) implies that M_τ is G -invariant. Given $\rho > 0$ let

$$\bar{B}_\rho M_\tau := \{x \in \mathbb{R}^N : \text{dist}(x, M_\tau) \leq \rho\},$$

and let $\text{cat}_{\bar{B}_\rho M_\tau/G}(M_\tau/G)$ denote the Lusternik-Schnirelmann category of M_τ/G in $\bar{B}_\rho M_\tau/G$.

Finally, let $c_{\mathbb{R}^N}$ denote the least energy of a nontrivial solution to the real-valued problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N, \mathbb{R}). \end{cases}$$

We shall prove the following result.

Theorem 2. *Assume there exists $\alpha > 0$ such that the set*

$$(1.7) \quad \{x \in \mathbb{R}^N : (\#Gx)V^q(x) \leq \ell_G + \alpha\}$$

is compact. Then, given $\rho, \delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in (0, \bar{\varepsilon})$, problem (1.5) has at least

$$\text{cat}_{\bar{B}_\rho M_\tau/G}(M_\tau/G)$$

geometrically distinct solutions u which satisfy

$$u(gx) = \tau(g)u(x) \quad \text{for all } g \in G, \quad x \in \mathbb{R}^N,$$

and

$$\left| \frac{p-2}{2p} \int |u|^p - \varepsilon^N \ell_{GC_{\mathbb{R}^N}} \right| < \varepsilon^N \delta.$$

This paper is organized as follows. In Section 2 we discuss the variational problem related to the existence of τ -intertwining solutions to problem (1.5), and we present the strategy for proving Theorem 2. In Sections 3 and 4 we introduce the entrance map and the local baryorbit map which will help us estimate the Lusternik-Schnirelmann category of some sublevel set of the variational functional for ε small enough. Finally, in Section 5 we show that this functional satisfies the Palais-Smale condition below a certain energy value and we prove Theorem 2.

2. THE VARIATIONAL PROBLEM

Let $\nabla_{\varepsilon,A}u := \varepsilon \nabla u + iAu$, and consider the real Hilbert space

$$H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_{\varepsilon,A}u \in L^2(\mathbb{R}^N, \mathbb{C}^N)\}$$

with scalar product

$$\langle u, v \rangle_{\varepsilon,A,V} := \text{Re} \int (\nabla_{\varepsilon,A}u \cdot \overline{\nabla_{\varepsilon,A}v} + V(x)u\bar{v}).$$

We write

$$\|u\|_{\varepsilon,A,V} := \left(\int (|\nabla_{\varepsilon,A}u|^2 + V(x)|u|^2) \right)^{1/2}$$

for the corresponding norm. Since $\inf_{\mathbb{R}^N} V > 0$ and $\sup_{\mathbb{R}^N} V < \infty$, this norm is equivalent to the usual one

$$\|u\|_{\varepsilon,A} := \left(\int (|\nabla_{\varepsilon,A} u|^2 + |u|^2) \right)^{1/2}.$$

If $u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N)$ and

$$(2.1) \quad \varepsilon |\nabla |u|(x)| \leq |\varepsilon \nabla u(x) + iA(x)u(x)| \quad \text{for a.e. } x \in \mathbb{R}^N.$$

This is called the diamagnetic inequality [24]. Together with the Sobolev inequality, it yields

$$|u|_p := \left(\int |u|^p \right)^{1/p} \leq C \|u\|_{\varepsilon,A} \quad \forall u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$$

for some constant $C > 0$, if $p \in [2, 2^*]$.

The action of G on $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$ defined by $(g, u) \mapsto u_g$, where

$$(u_g)(x) := \tau(g)u(g^{-1}x),$$

is an orthogonal action, that is,

$$\langle u_g, v_g \rangle_{\varepsilon,A,V} = \langle u, v \rangle_{\varepsilon,A,V} \quad \forall g \in G, \quad u, v \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}).$$

It also satisfies

$$|u_g|_p = |u|_p \quad \forall g \in G, \quad u \in L^p(\mathbb{R}^N, \mathbb{C}).$$

Therefore, the energy functional,

$$J_{\varepsilon,A,V}(u) = \frac{1}{2} \|u\|_{\varepsilon,A,V}^2 - \frac{1}{p} |u|_p^p, \quad u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}),$$

is G -invariant and, by the principle of symmetric criticality [27, 31], the critical points of the restriction of $J_{\varepsilon,A,V}$ to the fixed point space of the G -action, defined as

$$\begin{aligned} H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau &:= \{u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}) : u_g = u\} \\ &= \{u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}) : u(gx) = \tau(g)u(x) \quad \forall x \in \mathbb{R}^N, \quad g \in G\}, \end{aligned}$$

are the solutions to problem

$$(2.2) \quad \begin{cases} (-\varepsilon i \nabla + A)^2 u + Vu = |u|^{p-2} u, \\ u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}), \\ u(gx) = \tau(g)u(x) \quad \forall x \in \mathbb{R}^N, \quad g \in G. \end{cases}$$

Nontrivial critical points lie on the Nehari manifold

$$\mathcal{N}_{\varepsilon,A,V}^\tau := \{u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau : u \neq 0, \quad \|u\|_{\varepsilon,A,V}^2 = |u|_p^p\},$$

which is a C^2 -manifold, radially diffeomorphic to the unit sphere in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau$. For every $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$ we have that

$$J_{\varepsilon,A,V}(u) = \frac{p-2}{2p} \|u\|_{\varepsilon,A,V}^2 = \frac{p-2}{2p} |u|_p^p,$$

and the critical points of the restriction of $J_{\varepsilon,A,V}$ to $\mathcal{N}_{\varepsilon,A,V}^\tau$ are the nontrivial solutions to (2.2). We denote by

$$\pi_{\varepsilon,A,V} : H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau \setminus \{0\} \rightarrow \mathcal{N}_{\varepsilon,A,V}^\tau$$

the radial projection, given by

$$(2.3) \quad \pi_{\varepsilon,A,V}(u) := \left(\frac{\|u\|_{\varepsilon,A,V}^2}{|u|_p^p} \right)^{1/(p-2)} u.$$

Then,

$$J_{\varepsilon,A,V}(\pi_{\varepsilon,A,V}(u)) = \frac{p-2}{2p} \left(\frac{\|u\|_{\varepsilon,A,V}^2}{|u|_p^p} \right)^{p/(p-2)} \quad \forall u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau \setminus \{0\}.$$

Let

$$c_{\varepsilon,A,V}^\tau := \inf_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}.$$

The group \mathbb{S}^1 of unit complex numbers acts on $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau$ by scalar multiplication $(\gamma, u) \mapsto \gamma u$. This action is orthogonal, that is,

$$\langle \gamma u, \gamma v \rangle_{\varepsilon,A,V} = \langle u, v \rangle_{\varepsilon,A,V} \quad \forall \gamma \in \mathbb{S}^1, \quad u, v \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}).$$

The Nehari manifold $\mathcal{N}_{\varepsilon,A,V}^\tau$ and the functional $J_{\varepsilon,A,V}$ are \mathbb{S}^1 -invariant, that is,

$$\begin{aligned} \gamma u &\in \mathcal{N}_{\varepsilon,A,V}^\tau \quad \forall u \in \mathcal{N}_{\varepsilon,A,V}^\tau, \quad \gamma \in \mathbb{S}^1, \\ J_{\varepsilon,A,V}(\gamma u) &= J_{\varepsilon,A,V}(u) \quad \forall u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau, \quad \gamma \in \mathbb{S}^1. \end{aligned}$$

So, if $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$ is a critical point of $J_{\varepsilon,A,V}$, then every point γu in the \mathbb{S}^1 -orbit of u is a critical point of $J_{\varepsilon,A,V}$. Two critical points of $J_{\varepsilon,A,V}$ are geometrically distinct if their \mathbb{S}^1 -orbits are different. We shall apply \mathbb{S}^1 -equivariant Lusternik-Schnirelmann theory to obtain a lower bound for the number of critical \mathbb{S}^1 -orbits of $J_{\varepsilon,A,V}$. Recall that $J_{\varepsilon,A,V} : \mathcal{N}_{\varepsilon,A,V}^\tau \rightarrow \mathbb{R}$ is said to satisfy the Palais-Smale condition $(PS)_c$ at the level c , if every sequence (u_n) such that

$$u_n \in \mathcal{N}_{\varepsilon,A,V}^\tau, \quad J_{\varepsilon,A,V}(u_n) \rightarrow c, \quad \nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n) \rightarrow 0,$$

contains a convergent subsequence. Here $\nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u)$ denotes the orthogonal projection onto the tangent space to $\mathcal{N}_{\varepsilon,A,V}^\tau$ at u of the

gradient $\nabla_\varepsilon J_{\varepsilon,A,V}(u)$ of $J_{\varepsilon,A,V}$ at u with respect to the scalar product $\langle \cdot, \cdot \rangle_{\varepsilon,A,V}$. Let

$$J_{\varepsilon,A,V}^c := \{u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C}) : J_{\varepsilon,A,V}(u) \leq c\}.$$

The following holds.

Proposition 1. *If $J_{\varepsilon,A,V} : \mathcal{N}_{\varepsilon,A,V}^\tau \rightarrow \mathbb{R}$ satisfies $(PS)_c$ at each level $c \leq d$, then $J_{\varepsilon,A,V}$ has at least*

$$\text{cat} [(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) / \mathbb{S}^1]$$

critical \mathbb{S}^1 -orbits in $\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d$. Here $(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) / \mathbb{S}^1$ denotes the \mathbb{S}^1 -orbit space of $\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d$.

Proof. First, observe that \mathbb{S}^1 acts freely on $\mathcal{N}_{\varepsilon,A,V}^\tau$, that is, $\gamma u \neq u$ for every $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$, $\gamma \in \mathbb{S}^1$, $\gamma \neq 1$. Equivariant Lusternik-Schnirelmann theory guarantees the existence of at least \mathcal{G} -cat $(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d)$ critical \mathbb{S}^1 -orbits of $J_{\varepsilon,A,V}$ in $\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d$, where $\mathcal{G} := \{\mathbb{S}^1\}$, see Theorem 1.1 in [15]. But, since \mathbb{S}^1 acts freely on $\mathcal{N}_{\varepsilon,A,V}^\tau$, one has that

$$\mathcal{G}\text{-cat}(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) = \text{cat} [(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) / \mathbb{S}^1].$$

This proves our claim. \square

In order to obtain our main result we need to find an appropriate level $d = d(\varepsilon) > c_{\varepsilon,A,V}^\tau$ below which $(PS)_c$ holds. Then, we need to estimate $\text{cat} [(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) / \mathbb{S}^1]$. To this purpose, for some appropriate \mathbb{S}^1 -invariant set \mathcal{C} which is the union of connected components of $\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d$, we shall construct maps

$$M_\tau/G \xrightarrow{\iota_\varepsilon} \mathcal{C}/\mathbb{S}^1 \xrightarrow{\beta_\varepsilon} \bar{B}_\rho M_\tau/G,$$

whose composition $\beta_\varepsilon \circ \iota_\varepsilon$ is the inclusion map $M_\tau/G \hookrightarrow \bar{B}_\rho M_\tau/G$. A well known easy argument then gives

$$\text{cat}_{\bar{B}_\rho M_\tau/G} M_\tau/G \leq \text{cat}(\mathcal{C}/\mathbb{S}^1) \leq \text{cat} [(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^d) / \mathbb{S}^1],$$

where $\text{cat}_{\bar{B}_\rho M_\tau/G} M_\tau/G$ stands for the Lusternik-Schnirelmann category of M_τ/G in $\bar{B}_\rho M_\tau/G$. The following two sections are devoted to the construction of ι_ε and β_ε .

3. THE ENTRANCE MAP

Set $q := \frac{p}{p-2} - \frac{N}{2}$. Note that $q > 0$, because $p < 2^*$. Let

$$\ell_G := \min_{x \in \mathbb{R}^N} (\#Gx)V^q(x) \leq V^q(0),$$

and

$$M := \{y \in \mathbb{R}^N : (\#Gy)V^q(y) = \ell_G\}.$$

>From now on we assume that there exists $\alpha > 0$ such that

$$(3.1) \quad \{y \in \mathbb{R}^N : (\#Gy)V^q(y) \leq \ell_G + \alpha\} \text{ is compact.}$$

Then, M is a compact G -invariant set and all G -orbits in M are finite. We split M according to the orbit type of its elements as follows: We choose subgroups G_1, \dots, G_m of G such that the isotropy subgroup G_x of every point $x \in M$ is conjugate to precisely one of the G_i 's, and we set

$$M_i := \{y \in M : G_y = gG_i g^{-1} \text{ for some } g \in G\}.$$

Since isotropy subgroups satisfy $G_{gx} = gG_x g^{-1}$, the sets M_i are G -invariant and, since V is continuous, they are closed and pairwise disjoint, and

$$M = M_1 \cup \dots \cup M_m.$$

Moreover, since

$$|G/G_i| V^q(y) = (\#Gy)V^q(y) = \ell_G \quad \forall y \in M_i,$$

the potential V is constant on each M_i . Here $|G/G_i|$ denotes the index of G_i in G . We denote by V_i the value of V on M_i .

Let $v_{i,\varepsilon}$ be the positive least energy solution to the real-valued problem

$$\begin{cases} -\Delta u + V_i u = |u|^{p-2} u, \\ u \in H_0^1(B(0, 1/\sqrt{\varepsilon}), \mathbb{R}), \end{cases}$$

where $B(\xi, r) := \{x \in \mathbb{R}^N : |x - \xi| < r\}$. Then,

$$\int_{B(0, 1/\sqrt{\varepsilon})} (|\nabla v_{i,\varepsilon}|^2 + V_i |v_{i,\varepsilon}|^2) = |v_{i,\varepsilon}|_p^p.$$

It is well known that $v_{i,\varepsilon}$ is radial, and it is easy to derive from Lemma 3.2 in [6] that

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0} \frac{p-2}{2p} |v_{i,\varepsilon}|_p^p = V_i^q c_{\mathbb{R}^N},$$

where $c_{\mathbb{R}^N}$ is the least energy of a nontrivial solution to problem

$$\begin{cases} -\Delta u + u = |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N, \mathbb{R}). \end{cases}$$

Let $\xi \in M_i$. Following [9] we set

$$\phi_{\varepsilon,\xi}(x) := v_{i,\varepsilon} \left(\frac{x - \xi}{\varepsilon} \right) e^{-iA(\xi) \cdot \left(\frac{x - \xi}{\varepsilon} \right)}.$$

The following holds.

Lemma 1. *Uniformly in $\xi \in M_i$, we have that*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} J_{\varepsilon, A, V} [\pi_{\varepsilon, A, V}(\phi_{\varepsilon, \xi})] = V_i^q c_{\mathbb{R}^N},$$

where $\pi_{\varepsilon, A, V}$ stands for the radial projection onto the Nehari manifold defined in (2.3).

Proof. A change of variable gives

$$\begin{aligned} |\phi_{\varepsilon, \xi}|_p^p &= \int_{B(\xi, \sqrt{\varepsilon})} \left| v_{i, \varepsilon} \left(\frac{x - \xi}{\varepsilon} \right) \right|^p dx \\ &= \varepsilon^N \int_{B(0, 1/\sqrt{\varepsilon})} |v_{i, \varepsilon}(y)|^p dy = \varepsilon^N |v_{i, \varepsilon}|_p^p. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{B(\xi, \sqrt{\varepsilon})} V(x) |\phi_{\varepsilon, \xi}(x)|^2 dx &= \int_{B(\xi, \sqrt{\varepsilon})} V(x) \left| v_{i, \varepsilon} \left(\frac{x - \xi}{\varepsilon} \right) \right|^2 dx \\ &= \varepsilon^N \int_{B(0, 1/\sqrt{\varepsilon})} V(\varepsilon y + \xi) |v_{i, \varepsilon}(y)|^2 dy \\ &= \varepsilon^N \left(\int_{B(0, 1/\sqrt{\varepsilon})} V_i |v_{i, \varepsilon}|^2 + I_1 \right), \end{aligned}$$

where

$$I_1 = \int_{B(0, 1/\sqrt{\varepsilon})} (V(\varepsilon y + \xi) - V(\xi)) |v_{i, \varepsilon}(y)|^2 dy.$$

Hence,

$$|I_1| \leq \left(\max_{|z| \leq \sqrt{\varepsilon}} |V(z + \xi) - V(\xi)| \right) \int_{B(0, 1/\sqrt{\varepsilon})} |v_{i, \varepsilon}(y)|^2 dy.$$

Moreover, since $v_{i, \varepsilon}$ is real valued and A takes values in \mathbb{R}^N ,

$$\begin{aligned} &\int_{B(\xi, \sqrt{\varepsilon})} |\varepsilon \nabla \phi_{\varepsilon, \xi} + iA\phi_{\varepsilon, \xi}|^2 \\ &= \int_{B(\xi, \sqrt{\varepsilon})} \left| \nabla v_{i, \varepsilon} \left(\frac{x - \xi}{\varepsilon} \right) + i(A(x) - A(\xi))v_{i, \varepsilon} \left(\frac{x - \xi}{\varepsilon} \right) \right|^2 dx \\ &= \varepsilon^N \int_{B(0, 1/\sqrt{\varepsilon})} |\nabla v_{i, \varepsilon}(y) + i(A(\varepsilon y + \xi) - A(\xi))v_{i, \varepsilon}(y)|^2 dy \\ &= \varepsilon^N \left(\int_{B(0, 1/\sqrt{\varepsilon})} |\nabla v_{i, \varepsilon}|^2 + I_2 \right), \end{aligned}$$

where

$$I_2 = \int_{B(0, 1/\sqrt{\varepsilon})} |(A(\varepsilon y + \xi) - A(\xi))v_{i, \varepsilon}(y)|^2 dy.$$

Hence,

$$|I_2| \leq \max_{|z| \leq \sqrt{\varepsilon}} |A(z + \xi) - A(\xi)|^2 \int_{B(0,1/\sqrt{\varepsilon})} |v_{i,\varepsilon}(y)|^2 dy.$$

It follows that

$$\begin{aligned} \varepsilon^{-N} J_{\varepsilon,A,V} [\pi_{\varepsilon,A,V}(\phi_{\varepsilon,\xi})] &= \frac{p-2}{2p} \varepsilon^{-N} \left(\frac{\|\phi_{\varepsilon,\xi}\|_{\varepsilon,A,V}^2}{|\phi_{\varepsilon,\xi}|_p^2} \right)^{p/(p-2)} \\ &= \frac{p-2}{2p} \varepsilon^{-N} \left(\frac{\varepsilon^N \left[\int_{B(0,1/\sqrt{\varepsilon})} (|\nabla v_{i,\varepsilon}|^2 + V_i |v_{i,\varepsilon}|^2) + I_1 + I_2 \right]}{\left(\varepsilon^N |v_{i,\varepsilon}|_p^p \right)^{2/p}} \right)^{p/(p-2)} \\ &= \frac{p-2}{2p} \left(\frac{\int_{B(0,1/\sqrt{\varepsilon})} (|\nabla v_{i,\varepsilon}|^2 + V_i |v_{i,\varepsilon}|^2)}{|v_{i,\varepsilon}|_p^2} + o_\varepsilon(1) \right)^{p/(p-2)}, \end{aligned}$$

where $o_\varepsilon(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in $\xi \in M_i$. Using (3.2) we conclude that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} J_{\varepsilon,A,V} [\pi_{\varepsilon,A,V}(\phi_{\varepsilon,\xi})] = V_i^q C_{\mathbb{R}^N},$$

as claimed. \square

The map

$$G/G_\xi \rightarrow G\xi, \quad gG_\xi \mapsto g\xi,$$

is a homeomorphism [18]. So, if $G_i \subset \ker \tau$ and $\xi \in M_i$, then the map

$$G\xi \rightarrow S^1, \quad g\xi \mapsto \tau(g),$$

is well defined and continuous. Define

$$\psi_{\varepsilon,\xi}(x) := \sum_{g\xi \in G\xi} \tau(g) v_{i,\varepsilon} \left(\frac{x - g\xi}{\varepsilon} \right) e^{-iA(g\xi) \cdot \left(\frac{x - g\xi}{\varepsilon} \right)}.$$

Lemma 2. *Assume that $G_i \subset \ker \tau$.*

(a) *For every $\xi \in M_i$ and $\varepsilon > 0$, one has that*

$$\psi_{\varepsilon,\xi}(gx) = \tau(g) \psi_{\varepsilon,\xi}(x) \quad \forall g \in G, x \in \mathbb{R}^N.$$

(b) *For every $\xi \in M_i$ and $\varepsilon > 0$, one has that*

$$\tau(g) \psi_{\varepsilon,g\xi}(x) = \psi_{\varepsilon,\xi}(x) \quad \forall g \in G, x \in \mathbb{R}^N.$$

(c) *One has that*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} J_{\varepsilon,A,V} [\pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})] = \ell_G C_{\mathbb{R}^N}$$

uniformly in $\xi \in M_i$.

Proof. (a) Let $h \in G$. Since h is an orthogonal transformation, A satisfies $A(h^{-1}g\xi) = h^{-1}A(g\xi)$ and $v_{i,\varepsilon}$ is radially symmetric, we have that

$$\begin{aligned}\psi_{\varepsilon,\xi}(hx) &= \sum_{g\xi \in G\xi} \tau(g)v_{i,\varepsilon}\left(\frac{hx - g\xi}{\varepsilon}\right) e^{-iA(g\xi)\cdot\left(\frac{hx - g\xi}{\varepsilon}\right)} \\ &= \sum_{g\xi \in G\xi} \tau(h)\tau(h^{-1}g)v_{i,\varepsilon}\left(\frac{x - h^{-1}g\xi}{\varepsilon}\right) e^{-iA(h^{-1}g\xi)\cdot\left(\frac{x - h^{-1}g\xi}{\varepsilon}\right)} \\ &= \tau(h)\psi_{\varepsilon,\xi}(x),\end{aligned}$$

as claimed.

(b) Let $h \in G$. Then,

$$\begin{aligned}\psi_{\varepsilon,h\xi}(x) &= \sum_{gh\xi \in G(h\xi)} \tau(g)v_{i,\varepsilon}\left(\frac{x - gh\xi}{\varepsilon}\right) e^{-iA(gh\xi)\cdot\left(\frac{x - gh\xi}{\varepsilon}\right)} \\ &= \sum_{gh\xi \in G\xi} \tau(gh)\tau(h^{-1})v_{i,\varepsilon}\left(\frac{x - gh\xi}{\varepsilon}\right) e^{-iA(gh\xi)\cdot\left(\frac{x - gh\xi}{\varepsilon}\right)} \\ &= \tau(h)^{-1}\psi_{\varepsilon,\xi}(x),\end{aligned}$$

as claimed.

(c) Since M_i is compact and its orbits are finite and have the same orbit type, there exists $\varepsilon_i > 0$ such that, for all $\xi \in M_i$,

$$B(g_1\xi, \sqrt{\varepsilon_i}) \cap B(g_2\xi, \sqrt{\varepsilon_i}) = \emptyset \quad \text{if } g_1\xi \neq g_2\xi.$$

Hence, $\phi_{\varepsilon,g_1\xi}$ and $\phi_{\varepsilon,g_2\xi}$ have disjoint supports if $\varepsilon < \varepsilon_i$ and, therefore,

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} J_{\varepsilon,A,V} [\pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})] &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} \sum_{g\xi \in G\xi} J_{\varepsilon,A,V} [\pi_{\varepsilon,A,V}(\phi_{\varepsilon,g\xi})] \\ &= (\#G\xi)V^q(\xi)C_{\mathbb{R}^N} = \ell_G C_{\mathbb{R}^N},\end{aligned}$$

as claimed. □

Let

$$M_\tau := \{y \in M : G_y \subset \ker \tau\} = \bigcup_{G_i \subset \ker \tau} M_i.$$

The following holds.

Proposition 2. *The map $\widehat{v}_\varepsilon : M_\tau \rightarrow \mathcal{N}_{\varepsilon,A,V}^\tau$ given by*

$$\widehat{v}_\varepsilon(\xi) := \pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})$$

is well defined and continuous, and satisfies

$$\tau(g)\widehat{v}_\varepsilon(g\xi) = \widehat{v}_\varepsilon(\xi) \quad \forall \xi \in M_\tau, g \in G.$$

Moreover, given $d > \ell_{GC\mathbb{R}^N}$, there exists $\varepsilon_d > 0$ such that

$$J_{\varepsilon,A,V}(\widehat{v}_\varepsilon(\xi)) \leq \varepsilon^N d \quad \forall \xi \in M_\tau, \varepsilon \in (0, \varepsilon_d).$$

Proof. This follows immediately from Lemma 2. \square

4. A LOCAL BARYORBIT MAP

Next, we consider the real-valued problem

$$(4.1) \quad \begin{cases} -\varepsilon^2 \Delta v + Vv = |v|^{p-2} v, \\ v \in H^1(\mathbb{R}^N, \mathbb{R}), \\ v(gx) = v(x) \quad \forall x \in \mathbb{R}^N, g \in G. \end{cases}$$

We write

$$\langle v, w \rangle_{\varepsilon,V} := \int (\varepsilon \nabla v \cdot \nabla w + V(x)vw), \quad \|v\|_{\varepsilon,V}^2 := \int (|\varepsilon \nabla v|^2 + V(x)|v|^2),$$

and set

$$H^1(\mathbb{R}^N, \mathbb{R})^G := \{v \in H^1(\mathbb{R}^N, \mathbb{R}) : v(gx) = v(x) \quad \forall x \in \mathbb{R}^N, g \in G\}.$$

The nontrivial solutions of (4.1) are the critical points of the energy functional

$$J_{\varepsilon,V}(v) = \frac{1}{2} \|v\|_{\varepsilon,V}^2 - \frac{1}{p} |v|_p^p$$

on the Nehari manifold

$$\mathcal{M}_{\varepsilon,V}^G := \{v \in H^1(\mathbb{R}^N, \mathbb{R})^G : v \neq 0, \|v\|_{\varepsilon,V}^2 = |v|_p^p\}.$$

Let

$$(4.2) \quad c_{\varepsilon,V}^G := \inf_{\mathcal{M}_{\varepsilon,V}^G} J_{\varepsilon,V} = \inf_{\substack{v \in H^1(\mathbb{R}^N, \mathbb{R})^G \\ v \neq 0}} \frac{p-2}{2p} \left(\frac{\|v\|_{\varepsilon,V}^2}{|v|_p^2} \right)^{p/(p-2)}.$$

Lemma 3. $0 < V_0^q c_{\mathbb{R}^N} \leq \varepsilon^{-N} c_{\varepsilon,V}^G$ for every $\varepsilon > 0$, and

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon,V}^G \leq \ell_{GC\mathbb{R}^N}.$$

where $V_0 := \inf_{\mathbb{R}^N} V$.

Proof. Set $v_\varepsilon(x) := v(\varepsilon x)$. Then $\|v_\varepsilon\|_{1,V_0}^2 = \varepsilon^{-N} \|v\|_{\varepsilon,V_0}^2$ and $|v_\varepsilon|_p^p = \varepsilon^{-N} |v|_p^p$. It follows immediately from (4.2) that

$$V_0^q c_{\mathbb{R}^N} \leq c_{1,V_0}^G = \varepsilon^{-N} c_{\varepsilon,V_0}^G \leq \varepsilon^{-N} c_{\varepsilon,V}^G.$$

To prove the second inequality, take $\xi \in \mathbb{R}^N$ such that its G -orbit is finite. Write $G\xi := \{\xi_1, \dots, \xi_m\}$. Fix $0 < \rho < \frac{1}{2} \min_{i \neq j} |\xi_i - \xi_j|$, and

let $V_\rho := \sup_{B(\xi_1, \rho)} V$. Let $v_{\rho, \varepsilon}$ be the positive least energy solution to problem

$$\begin{cases} -\Delta v + V_\rho v = |v|^{p-2} v, \\ v \in H_0^1(B(0, 1/\sqrt{\varepsilon}), \mathbb{R}). \end{cases}$$

Define

$$w_{\rho, \varepsilon}(x) := \sum_{i=1}^m v_{\rho, \varepsilon} \left(\frac{x - \xi_i}{\varepsilon} \right).$$

If $\sqrt{\varepsilon} \leq \rho$, then $\text{supp}(w_{\rho, \varepsilon}) \subset \cup_{i=1}^m B(\xi_i, \rho)$ and, therefore,

$$\varepsilon^{-N} |w_{\rho, \varepsilon}|_p^p = m |v_{\rho, \varepsilon}|_p^p = m \|v_{\rho, \varepsilon}\|_{1, V_\rho}^2 = \varepsilon^{-N} \|w_{\rho, \varepsilon}\|_{\varepsilon, V_\rho}^2.$$

Hence,

$$\varepsilon^{-N} c_{\varepsilon, V}^G \leq \frac{p-2}{2p} \varepsilon^{-N} \left(\frac{\|w_{\rho, \varepsilon}\|_{\varepsilon, V}^2}{|w_{\rho, \varepsilon}|_p^2} \right)^{p/(p-2)} = m \frac{p-2}{2p} |v_{\rho, \varepsilon}|_p^p.$$

Using Lemma 3.2 in [6] we obtain that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G \leq m \lim_{\varepsilon \rightarrow 0} \frac{p-2}{2p} |v_{\rho, \varepsilon}|_p^p = (\#G\xi) V_\rho^q c_{\mathbb{R}^N}$$

and, letting $\rho \rightarrow 0$, we conclude that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G \leq (\#G\xi) V^q(\xi) c_{\mathbb{R}^N}.$$

Therefore, $\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G \leq \ell_G c_{\mathbb{R}^N}$, as claimed. \square

Fix $\bar{\rho} > 0$ such that

$$\begin{aligned} |y - gy| &> 2\bar{\rho} \quad \text{if } gy \neq y \in M, \\ \text{dist}(M_i, M_j) &> 2\bar{\rho} \quad \text{if } i \neq j, \end{aligned}$$

where M, M_i are the sets defined in Section 3. For $\rho \in (0, \bar{\rho})$, let

$$M_i^\rho := \{y \in \mathbb{R}^N : \text{dist}(y, M_i) \leq \rho, \quad G_y = gG_i g^{-1} \text{ for some } g \in G\}.$$

For every $\xi \in M_i^\rho$ and $\varepsilon > 0$, define

$$\theta_{\varepsilon, \xi} := \sum_{g\xi \in G\xi} \omega_i \left(\frac{x - g\xi}{\varepsilon} \right),$$

where ω_i is the positive radial solution to problem

$$(4.3) \quad \begin{cases} -\Delta w + V_i w = |w|^{p-2} w, \\ w \in H^1(\mathbb{R}^N, \mathbb{R}), \end{cases}$$

and let

$$\Theta_{\rho, \varepsilon} := \{\theta_{\varepsilon, \xi} : \xi \in M_1^\rho \cup \dots \cup M_m^\rho\}.$$

We continue to assume that (3.1) holds.

Proposition 3. *Let $\varepsilon_n > 0$ and $v_n \in H^1(\mathbb{R}^N, \mathbb{R})^G$ be such that*

$$\varepsilon_n \rightarrow 0, \quad \varepsilon_n^{-N} J_{\varepsilon_n, V}(v_n) \rightarrow \bar{c}, \quad \varepsilon_n^{-N} \|\nabla_{\varepsilon_n} J_{\varepsilon_n, V}(v_n)\|_{\varepsilon_n}^2 \rightarrow 0,$$

where $\bar{c} := \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G$ and $\nabla_{\varepsilon_n} J_{\varepsilon_n, V}$ is the gradient of $J_{\varepsilon_n, V}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\varepsilon_n, V}$. Then, up to a subsequence, there exist an $i \in \{1, \dots, m\}$ and a sequence (ξ_n) in \mathbb{R}^N such that

- (i) $G_{\xi_n} = G_i$,
- (ii) $\xi_n \rightarrow \xi \in M_i$,
- (iii) $\varepsilon_n^{-N} \| |v_n| - \theta_{\varepsilon_n, \xi_n} \|_p^p \rightarrow 0$,
- (iv) $\bar{c} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G = \ell_{G, \mathbb{R}^N}$.

Proof. Let $\tilde{v}_n \in H^1(\mathbb{R}^N, \mathbb{R})^G$ be given by $\tilde{v}_n(z) := v_n(\varepsilon_n z)$. Then,

$$|\tilde{v}_n|_p^p = \varepsilon_n^{-N} |v_n|_p^p \rightarrow \frac{2p}{p-2} \bar{c} =: c > 0.$$

Fix $R > 0$ and let

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} |\tilde{v}_n|^p.$$

Lions' lemma [31, Lemma 1.21] implies that $\delta > 0$. Choose $z_n \in \mathbb{R}^N$ such that

$$\int_{B(z_n, R)} |\tilde{v}_n|^p \geq \frac{\delta}{2}.$$

Passing to a subsequence and replacing z_n by some point in Gz_n if necessary, we may assume that $Gz_n = H$ for all $n \in \mathbb{N}$. Let

$$K := \{g \in G : (gz_n - z_n) \text{ is bounded}\}.$$

Then K is a closed subgroup of G which contains H . Let ζ_n denote the orthogonal projection of z_n onto

$$(\mathbb{R}^N)^K := \{z \in \mathbb{R}^N : gz = z \quad \forall g \in K\}.$$

We claim that

- (a) $(z_n - \zeta_n)$ is bounded,
- (b) $G_{\zeta_n} = K \quad \forall n \in \mathbb{N}$,
- (c) $(g\zeta_n - g'\zeta_n)$ is unbounded if $g^{-1}g' \notin K$.

To prove (a) let $z'_n := z_n - \zeta_n$ and assume $|z'_n| \rightarrow \infty$. Then, passing to a subsequence, we have that

$$\frac{z'_n}{|z'_n|} \rightarrow z' \notin (\mathbb{R}^N)^K.$$

On the other hand, for every $g \in K$, we have that $gz'_n - z'_n = gz_n - z_n$. Therefore, $(gz'_n - z'_n)$ is bounded. Hence,

$$\left| \frac{gz'_n}{|z'_n|} - \frac{z'_n}{|z'_n|} \right| = \frac{1}{|z'_n|} |gz'_n - z'_n| \rightarrow 0.$$

We conclude that $gz' = z'$ for all $g \in K$, contradicting the fact that $z' \notin (\mathbb{R}^N)^K$. Thus, $(z_n - \zeta_n)$ is bounded.

To prove (b) first note that, by definition, $G_{\zeta_n} \supset K$. If $g\zeta_n = \zeta_n$, then

$$|gz_n - z_n| \leq |gz_n - g\zeta_n| + |\zeta_n - z_n| = 2|\zeta_n - z_n|.$$

So, by (a), $g \in K$. Thus, $G_{\zeta_n} = K$.

To prove (c) observe that, if $g \notin K$, then

$$\begin{aligned} |gz_n - z_n| &\leq |gz_n - g\zeta_n| + |g\zeta_n - \zeta_n| + |\zeta_n - z_n| \\ &= |g\zeta_n - \zeta_n| + 2|\zeta_n - z_n|. \end{aligned}$$

It follows from (a) that $(g\zeta_n - \zeta_n)$ is unbounded. This proves (c).

Let $\bar{v}_n(z) := \tilde{v}_n(z + \zeta_n)$. Since (\bar{v}_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$, up to a subsequence,

$$\begin{aligned} \bar{v}_n &\rightharpoonup \bar{v} \text{ weakly in } H^1(\mathbb{R}^N, \mathbb{R}), \\ \bar{v}_n(x) &\rightarrow \bar{v}(x) \text{ a.e. on } \mathbb{R}^N, \\ \bar{v}_n &\rightarrow \bar{v} \text{ in } L^p_{loc}(\mathbb{R}^N, \mathbb{R}). \end{aligned}$$

Choosing $C \geq |\zeta_n - z_n|$ for all n , we obtain

$$\int_{B(0, C+R)} |\bar{v}_n|^p = \int_{B(\zeta_n, C+R)} |\tilde{v}_n|^p \geq \int_{B(z_n, R)} |\tilde{v}_n|^p \geq \frac{\delta}{2}.$$

Therefore, $\bar{v} \neq 0$. Set $\xi_n := \varepsilon_n \zeta_n$. After passing to a subsequence,

$$\lim_{n \rightarrow \infty} V(\xi_n) =: \hat{V}.$$

It is easy to see that \bar{v} is a solution to problem

$$\begin{cases} -\Delta v + \hat{V}v = |v|^{p-2}v, \\ v \in H^1(\mathbb{R}^N, \mathbb{R}). \end{cases}$$

Note that, since v_n is G -invariant, $v_n(\varepsilon_n x + g\xi_n) = v_n(\varepsilon_n g^{-1}x + \xi_n) = \bar{v}_n(g^{-1}x)$. Note also that property (c) above implies that

$$\bar{v}g^{-1}(\cdot - g\zeta_n + g'\zeta_n) \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N, \mathbb{R}) \text{ if } g^{-1}g' \notin K.$$

Fix $g_1, \dots, g_k \in G$ such that $k = |G/K|$ and $g_i^{-1}g_j \notin K$ if $i \neq j$. Then,

$$(4.4) \quad \bar{v}_n g_j^{-1} - \sum_{i=j+1}^k v g_i^{-1}(\cdot - g_i \zeta_n + g_j \zeta_n) \rightharpoonup \bar{v} g_j^{-1}$$

weakly in $H^1(\mathbb{R}^N, \mathbb{R})$. Performing first the change of variable $y = \varepsilon_n x + g_j \xi_n$, then applying the Brezis-Lieb lemma to (4.4), and finally performing the change of variable $x = \varepsilon_n^{-1}(y - g_j \xi_n)$, for each $j = 1, \dots, k$, we obtain

$$\begin{aligned}
 & \varepsilon_n^{-N} \left| v_n - \sum_{i=j+1}^k \bar{v} g_i^{-1} \left(\frac{\cdot - g_i \xi_n}{\varepsilon_n} \right) \right|_p^p \\
 &= \left| \bar{v}_n g_j^{-1} - \sum_{i=j+1}^k \bar{v} g_i^{-1} (\cdot + g_i \zeta_n - g_j \zeta_n) \right|_p^p \\
 &= \left| \bar{v}_n g_j^{-1} - \bar{v} g_j^{-1} - \sum_{i=j+1}^k \bar{v} g_i^{-1} (\cdot + g_i \zeta_n - g_j \zeta_n) \right|_p^p \\
 &\quad + |\bar{v} g_j^{-1}|_p^p + o(1) \\
 &= \varepsilon_n^{-N} \left| v_n - \sum_{i=j}^k \bar{v} g_i^{-1} \left(\frac{\cdot - g_i \xi_n}{\varepsilon_n} \right) \right|_p^p + |\bar{v}|_p^p + o(1).
 \end{aligned}$$

Iterating this equalities we conclude that

$$c := \lim_{n \rightarrow \infty} \varepsilon_n^{-N} |v_n|_p^p = \lim_{n \rightarrow \infty} \varepsilon_n^{-N} \left| v_n - \sum_{i=1}^k \bar{v} g_i^{-1} \left(\frac{\cdot - g_i \xi_n}{\varepsilon_n} \right) \right|_p^p + |G/K| |\bar{v}|_p^p.$$

Therefore,

$$\begin{aligned}
 \ell_{GC_{\mathbb{R}^N}} &\leq \lim_{n \rightarrow \infty} (\#G\xi_n)V(\xi_n) = |G/K| \hat{V}^q_{C_{\mathbb{R}^N}} \\
 &\leq |G/K| \frac{p-2}{2p} |\bar{v}|_p^p \leq \bar{c} \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G \leq \ell_{GC_{\mathbb{R}^N}}.
 \end{aligned}$$

This proves that $\bar{c} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} c_{\varepsilon, V}^G = \ell_{GC_{\mathbb{R}^N}}$ and that

$$(4.5) \quad \lim_{n \rightarrow \infty} \varepsilon_n^{-N} \left| v_n - \sum_{i=1}^k \bar{v} g_i^{-1} \left(\frac{\cdot - g_i \xi_n}{\varepsilon_n} \right) \right|_p^p = 0.$$

Moreover, $(\#G\xi_n)V(\xi_n) \leq \ell_{GC_{\mathbb{R}^N}} + \alpha$ for n large enough. Thus, assumption (3.1) implies that, up to a subsequence, $\xi_n \rightarrow \xi$. It follows that $V(\xi) = \hat{V}$ and

$$\ell_{GC_{\mathbb{R}^N}} \leq (\#G\xi)V(\xi)_{C_{\mathbb{R}^N}} \leq |G/K| \hat{V}^q_{C_{\mathbb{R}^N}} \leq |G/K| \frac{p-2}{2p} |\bar{v}|_p^p \leq \ell_{GC_{\mathbb{R}^N}}.$$

We conclude that $\xi \in M_i$ for some $i = 1, \dots, m$, and, therefore, $\hat{V} = V_i$ and $K = G_\xi = gG_i g^{-1}$ for some $g \in G$.

Moreover, \bar{v} solves problem (4.3) and

$$\frac{p-2}{2p} \int |\bar{v}|^p = V_i^q C_{\mathbb{R}^N}.$$

Hence, $\bar{v}(z) = \pm\omega_i(z - z_0)$ for some $z_0 \in \mathbb{R}^N$. Observe that \bar{v} is K -invariant because

$$\bar{v}_n(gz) = \tilde{v}_n(gz + \zeta_n) = \tilde{v}_n(z + g^{-1}\zeta_n) = \tilde{v}_n(z + \zeta_n) = \bar{v}_n(z) \quad \forall g \in K, n \in \mathbb{N}.$$

So, if K is nontrivial, then $z_0 = 0$ and, since ω_i is radial, equation (4.5) becomes

$$\lim_{n \rightarrow \infty} \varepsilon_n^{-N} \left| v_n \pm \sum_{i=1}^k \omega_i \left(\frac{\cdot - g_i \xi_n}{\varepsilon_n} \right) \right|_p^p = 0.$$

If K is the trivial group, we replace ξ_n by $\xi'_n := \xi_n + \varepsilon_n z_0$. Since $G\xi_n \cong G$ and $\varepsilon_n \rightarrow 0$, ξ'_n has the same properties as ξ_n for n large enough. Moreover, since ω_i is radially symmetric,

$$\begin{aligned} \bar{v} \left(\frac{g^{-1}z - \xi_n}{\varepsilon_n} \right) &= \pm\omega_i \left(\frac{g^{-1}z - \xi_n}{\varepsilon_n} - z_0 \right) \\ &= \pm\omega_i \left(\frac{g^{-1}z - \xi'_n}{\varepsilon_n} \right) = \pm\omega_i \left(\frac{z - g\xi'_n}{\varepsilon_n} \right). \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \varepsilon_n^{-N} \left| v_n \pm \sum_{i=1}^k \omega_i \left(\frac{\cdot - g_i \xi'_n}{\varepsilon_n} \right) \right|_p^p = 0,$$

as claimed. \square

The proof of the following proposition is completely analogous to that of Proposition 5.5 in [13]. We omit the details.

Proposition 4. *Given $\rho \in (0, \bar{\rho})$ there exist $d_\rho > \ell_{G C_{\mathbb{R}^N}}$ and $\varepsilon_\rho > 0$ with the following property: For every $\varepsilon \in (0, \varepsilon_\rho)$ and every $v \in \mathcal{M}_{\varepsilon, V}^G$ with $J_{\varepsilon, V}(v) \leq \varepsilon^N d_\rho$ there exists precisely one G -orbit $G\xi_{\varepsilon, v}$ with $\xi_{\varepsilon, v} \in M_1^\rho \cup \dots \cup M_m^\rho$ such that*

$$\varepsilon^{-N} \left\| |v| - \theta_{\varepsilon, \xi_{\varepsilon, v}} \right\|_{\varepsilon, V}^2 = \min_{\theta \in \Theta_{\rho, \varepsilon}} \left\| |v| - \theta \right\|_{\varepsilon, V}^2.$$

Proposition 4 allows us to define, for each $\rho \in (0, \bar{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, a local baryorbit map

$$\widehat{\beta}_{\rho, \varepsilon, 0} : \mathcal{M}_{\varepsilon, V}^G \cap J_{\varepsilon, V}^{\varepsilon^N d_\rho} \longrightarrow (M_1^\rho \cup \dots \cup M_m^\rho) / G$$

by taking

$$\widehat{\beta}_{\rho, \varepsilon, 0}(v) := G\xi_{\varepsilon, v},$$

where $G\xi_{\varepsilon,v}$ is the unique G -orbit given by the previous proposition. Here, as usual, $J_{\varepsilon,V}^c := \{v \in H^1(\mathbb{R}^N, \mathbb{R}) : J_{\varepsilon,V}(v) \leq c\}$.

We now come back to our original problem. Let $\pi_{\varepsilon,V} : H^1(\mathbb{R}^N, \mathbb{R})^G \setminus \{0\} \rightarrow \mathcal{M}_{\varepsilon,V}^G$ be the radial projection, which is given by

$$\pi_{\varepsilon,V}(v) := \left(\frac{\|v\|_{\varepsilon,V}^2}{|v|_p^p} \right)^{1/(p-2)} v.$$

Using the diamagnetic inequality we obtain the following.

Corollary 1. *For each $\rho \in (0, \bar{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, the local baryorbit map*

$$\widehat{\beta}_{\rho,\varepsilon} : \mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d_\rho} \longrightarrow (M_1^\rho \cup \dots \cup M_m^\rho) / G,$$

given by

$$\widehat{\beta}_{\rho,\varepsilon}(u) := \widehat{\beta}_{\rho,\varepsilon,0}(\pi_{\varepsilon,V}(|u|)),$$

is well defined and continuous. It satisfies

$$\widehat{\beta}_{\rho,\varepsilon}(\gamma u) = \widehat{\beta}_{\rho,\varepsilon}(u) \quad \forall \gamma \in \mathbb{S}^1,$$

$$\widehat{\beta}_{\rho,\varepsilon}(\widehat{v}_\varepsilon(\xi)) = \xi \quad \forall \xi \in M_\tau \text{ with } J_{\varepsilon,A,V}(\iota_\varepsilon(\xi)) \leq \varepsilon^N d_\rho,$$

where \widehat{v}_ε is the map defined in Proposition 2.

Proof. If $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$, then $\pi_{\varepsilon,V}(|u|) \in \mathcal{M}_{\varepsilon,V}^G$ and

$$J_{\varepsilon,V}(\pi_{\varepsilon,V}(|u|)) = \frac{p-2}{2p} \left(\frac{\| |u| \|_{\varepsilon,V}^2}{|u|_p^2} \right)^{p/(p-2)}.$$

The diamagnetic inequality yields $\| |u| \|_{\varepsilon,V}^2 \leq \|u\|_{\varepsilon,A,V}^2$. Therefore,

$$(4.6) \quad J_{\varepsilon,V}(\pi_{\varepsilon,V}(|u|)) \leq \frac{p-2}{2p} \left(\frac{\|u\|_{\varepsilon,A,V}^2}{|u|_p^2} \right)^{p/(p-2)} = J_{\varepsilon,A,V}(u).$$

So, if $J_{\varepsilon,A,V}(u) \leq \varepsilon^N d_\rho$, then $\pi_{\varepsilon,V}(|u|) \leq \varepsilon^N d_\rho$. Hence, $\widehat{\beta}_{\rho,\varepsilon}$ is well defined. It is straightforward to verify that it has the desired properties. \square

Let

$$M_\tau^\rho := \bigcup_{G_i \subset \ker \tau} M_i^\rho$$

We obtain the following conclusion.

Corollary 2. *For every $\rho \in (0, \bar{\rho})$ and $d > \ell_{GC\mathbb{R}^N}$ there exists $\varepsilon_{\rho,d} > 0$ such that*

$$\text{cat}_{M_\tau^\rho/G} M_\tau/G \leq \text{cat} \left((\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d}) / \mathbb{S}^1 \right)$$

for every $\varepsilon \in (0, \varepsilon_{\rho,d})$.

Proof. Let $d_\rho > \ell_G c_{\mathbb{R}^N}$ and $\varepsilon_\rho > 0$ be as in Proposition 4. We may assume that $d \leq d_\rho$. Let $\varepsilon_{\rho,d} := \min\{\varepsilon_d, \varepsilon_\rho\}$, where ε_d is as in Proposition 2, and fix $\varepsilon \in (0, \varepsilon_{\rho,d})$. Then,

$$J_{\varepsilon,A,V}(\widehat{v}_\varepsilon(\xi)) \leq \varepsilon^N d \quad \text{and} \quad \widehat{\beta}_{\rho,\varepsilon}(\widehat{v}_\varepsilon(\xi)) = \xi \quad \forall \xi \in M_\tau.$$

Since the sets $M_1^\rho, \dots, M_m^\rho$ are G -invariant and pairwise disjoint, the set

$$\mathcal{C} := \{u \in \mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d} : \widehat{\beta}_{\rho,\varepsilon}(u) \in M_\tau^\rho/G\}$$

is a union of connected components of $\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d}$. Therefore,

$$\text{cat}(\mathcal{C}/\mathbb{S}^1) \leq \text{cat}\left(\left(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d}\right)/\mathbb{S}^1\right).$$

By Proposition 2 and Corollary 1, the maps

$$M_\tau/G \xrightarrow{\iota_\varepsilon} \mathcal{C}/\mathbb{S}^1 \xrightarrow{\beta_{\rho,\varepsilon}} M_\tau^\rho/G,$$

given by $\iota_\varepsilon(G\xi) := \widehat{v}_\varepsilon(xi)$ and $\beta_{\rho,\varepsilon}(\mathbb{S}^1 u) := \widehat{\beta}_{\rho,\varepsilon}(u)$, are well defined and satisfy $\beta_{\rho,\varepsilon}(\iota_\varepsilon(\xi)) = \xi$ for all $\xi \in M_\tau$. Therefore,

$$\text{cat}_{M_\tau^\rho/G} M_\tau/G \leq \text{cat}(\mathcal{C}/\mathbb{S}^1).$$

This finishes the proof. \square

We conclude this section with the following observation.

Corollary 3. *If there exists $x \in \mathbb{R}^N$ such that $(\#Gx)V^q(x) = \ell_G$ and $G_x \subset \ker \tau$, then*

$$\lim_{\varepsilon \rightarrow \infty} c_{\varepsilon,A,V}^\tau = \ell_G c_{\mathbb{R}^N}.$$

where $c_{\varepsilon,A,V}^\tau := \inf_{\mathcal{N}_{\varepsilon,A,V}^\tau} \varepsilon^{-N} J_{\varepsilon,A,V}$.

Proof. Inequality (4.6) yields

$$c_{\varepsilon,V}^G := \inf_{\mathcal{M}_{\varepsilon,V}^G} J_{\varepsilon,V} \leq \inf_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V} =: c_{\varepsilon,A,V}^\tau.$$

The result follows from Proposition 3 and Lemma 2(c). \square

5. THE PALAIS-SMALE CONDITION

To prove the Palais-Smale condition we need to look at the real-valued problem

$$(5.1) \quad \begin{cases} -\varepsilon^2 \Delta v + \lambda v = |v|^{p-2} v, \\ v \in H_0^1(\mathbb{R}^N \setminus \overline{B}(0, r), \mathbb{R}), \\ v(gx) = v(x) \quad \forall x \in \mathbb{R}^N \setminus \overline{B}(0, r), \quad g \in G. \end{cases}$$

where $\bar{B}(0, r) := \{x \in \mathbb{R}^N : |x| \leq r\}$, $r > 0$, and λ is a positive constant. The nontrivial solutions of (5.1) are the critical points of the energy functional

$$J_{\varepsilon, \lambda}(v) = \frac{1}{2} \|v\|_{\varepsilon, \lambda}^2 - \frac{1}{p} |v|_p^p = \frac{1}{2} \int_{|x| > r} (\varepsilon^2 |\nabla v|^2 + \lambda |v|^2) - \frac{1}{p} \int_{|x| > r} |v|^p$$

on the Nehari manifold

$$\mathcal{M}_{\varepsilon, \lambda, r}^G := \{v \in H_0^1(\mathbb{R}^N \setminus \bar{B}(0, r), \mathbb{R})^G : v \neq 0, \|v\|_{\varepsilon, \lambda}^2 = |v|_p^p\}.$$

Let

$$c_{\varepsilon, \lambda, r}^G := \inf_{v \in \mathcal{M}_{\varepsilon, \lambda, r}^G} J_{\varepsilon, \lambda}(v).$$

Note that $c_{\varepsilon, \lambda, r}^G \leq c_{\varepsilon, \lambda, s}^G$ if $r \leq s$. Moreover, the following holds.

Lemma 4. $\lim_{r \rightarrow \infty} c_{\varepsilon, \lambda, r}^G = \varepsilon^N (\min_{x \in \mathbb{R}^N \setminus \{0\}} \#Gx) \lambda^q c_{\mathbb{R}^N}$.

Proof. First, we prove this assertion for $\varepsilon = 1$. Set $k := \min_{x \in \mathbb{R}^N \setminus \{0\}} \#Gx$. It is easy to see that $\lambda^q c_{\mathbb{R}^N} \leq c_{1, \lambda, r}^G \leq k \lambda^q c_{\mathbb{R}^N}$, so $c_{1, \lambda, r}^G = \lambda^q c_{\mathbb{R}^N}$ if $k = 1$. It was shown in [7] that, if $k \geq 2$, then $c_{1, \lambda, r}^G < k \lambda^q c_{\mathbb{R}^N}$ and $c_{1, \lambda, r}^G$ is attained for every $r > 0$. It remains to show that $\lim_{r \rightarrow \infty} c_{1, \lambda, r}^G \geq k \lambda^q c_{\mathbb{R}^N}$. Assume $k \geq 2$ and let $u_n \in \mathcal{M}_{1, \lambda, n}^G$ be such that $J_{1, \lambda}(u_n) = c_{1, \lambda, n}^G$, $n \in \mathbb{N}$. Fix $R > 0$ and let

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} |u_n|^p.$$

Since $|u_n|_p^p \geq \frac{2p}{p-2} \lambda^q c_{\mathbb{R}^N} > 0$, Lions' lemma [31, Lemma 1.21] implies that $\delta > 0$. Choose $\xi_n \in \mathbb{R}^N$ such that

$$\int_{B(\xi_n, R)} |u_n|^p \geq \frac{\delta}{2}.$$

Arguing as in the proof of Proposition 3, we may assume

$$(5.2) \quad \begin{aligned} G_{\xi_n} &= K \quad \forall n \in N, \\ |g\xi_n - g'\xi_n| &\rightarrow \infty \quad \text{if } g^{-1}g' \notin K. \end{aligned}$$

We claim that the sequence $(|\xi_n| - n)$ is unbounded. Assume it is not. Then, passing to a subsequence, we have that $|\xi_n| - n \rightarrow d$. Choose a rotation ρ_n about the origin which sends $-\xi_n$ to $(-|\xi_n|, 0, \dots, 0)$, and set $w_n(x) := u_n(\rho_n^{-1}x + \xi_n)$. Then, (w_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$ and, passing to a subsequence, we have that $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$ and $w_n \rightarrow w$ strongly in $L_{loc}^p(\mathbb{R}^N, \mathbb{R})$. Since

$$\int_{B(0, R)} |w_n|^p = \int_{B(\xi_n, R)} |u_n|^p \geq \frac{\delta}{2},$$

we conclude that $w \neq 0$. Let $\mathbb{H}^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > -d\}$. Since every compact subset in the interior of $\mathbb{R}^N \setminus \mathbb{H}^N$ is contained in $B((-|\xi_n|, 0, \dots, 0), n)$ for large enough n , and $w_n \equiv 0$ in $B((-|\xi_n|, 0, \dots, 0), n)$, we have that $w \in H_0^1(\mathbb{H}^N, \mathbb{R})$. Moreover, since every compact subset of \mathbb{H}^N is contained in $\mathbb{R}^N \setminus \bar{B}((-|\xi_n|, 0, \dots, 0), n)$ for large enough n , we have that w solves

$$\begin{aligned} -\Delta w + \lambda w &= |w|^{p-2} w, \\ w &\in H_0^1(\mathbb{H}^N, \mathbb{R}), \end{aligned}$$

contradicting the fact that this problem has only the trivial solution [19]. Therefore, $(|\xi_n| - n)$ must be unbounded.

Passing to a subsequence, we assume that $|\xi_n| - n \rightarrow \infty$. Set $v_n(x) := u_n(x + \xi_n)$. Then, up to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N, \mathbb{R})$, $v_n(x) \rightarrow v(x)$ a.e. in \mathbb{R}^N and $v_n \rightarrow v$ strongly in $L_{loc}^p(\mathbb{R}^N, \mathbb{R})$ and, arguing as above, we conclude that v is a nontrivial solution to problem

$$(5.3) \quad \begin{aligned} -\Delta v + \lambda v &= |v|^{p-2} v, \\ v &\in H^1(\mathbb{R}^N, \mathbb{R}). \end{aligned}$$

Note that, since u_n is G -invariant, $u_n(x + g\xi_n) = u_n(g^{-1}x + \xi_n) = v_n(g^{-1}x)$. By (5.2) we have also that

$$vg^{-1}(\cdot - g\xi_n + g'\xi_n) \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbb{R}^N, \mathbb{R}) \text{ if } g^{-1}g' \notin K.$$

Fix $g_1, \dots, g_k \in G$ such that $g_i^{-1}g_j \notin K$ if $i \neq j$. Then,

$$u_n(\cdot + g_j\xi_n) - \sum_{i=j+1}^k vg_i^{-1}(\cdot - g_i\xi_n + g_j\xi_n) \rightharpoonup vg_j^{-1} \quad \text{weakly in } H^1(\mathbb{R}^N, \mathbb{R})$$

so, using the Brezis-Lieb lemma [31, Lemma 1.32], for each $j = 1, \dots, k$, we have

$$\begin{aligned} \left| u_n - \sum_{i=j+1}^k vg_i^{-1}(\cdot - g_i\xi_n) \right|_p^p &= \left| u_n(\cdot + g_j\xi_n) - \sum_{i=j+1}^k vg_i^{-1}(\cdot - g_i\xi_n + g_j\xi_n) \right|_p^p \\ &= \left| u_n(\cdot + g_j\xi_n) - vg_j^{-1} - \sum_{i=j+1}^k vg_i^{-1}(\cdot - g_i\xi_n + g_j\xi_n) \right|_p^p \\ &\quad + |vg_j^{-1}|_p^p + o(1) \\ &= \left| u_n - \sum_{i=j}^k vg_i^{-1}(\cdot - g_i\xi_n) \right|_p^p + |v|_p^p + o(1). \end{aligned}$$

Iterating this equalities we obtain

$$|u_n|_p^p \geq k|v|_p^p + o(1)$$

and, since v is a nontrivial solution of (5.3), we conclude that

$$c_{1,\lambda,n}^G = J_{1,\lambda}(u_n) = \frac{p-2}{2p} |u_n|_p^p \geq k \frac{p-2}{2p} |v|_p^p + o(1) \geq k\lambda^q c_{\mathbb{R}^N} + o(1).$$

This proves that $\lim_{r \rightarrow \infty} c_{1,\lambda,r}^G = k\lambda^q c_{\mathbb{R}^N}$.

Finally, fix $\varepsilon > 0$. If $u_\varepsilon(x) := u(\varepsilon x)$, then $u_\varepsilon \in \mathcal{M}_{1,\lambda,r}^G$ iff $u \in \mathcal{M}_{\varepsilon,\lambda,\varepsilon r}^G$, and $\|u_\varepsilon\|_{1,\lambda}^2 = \varepsilon^{-N} \|u\|_{\varepsilon,\lambda}^2$. Therefore, $\varepsilon^N c_{1,\lambda,r}^G = c_{\varepsilon,\lambda,\varepsilon r}^G$, and the result for ε follows from the case $\varepsilon = 1$. \square

Now we go back to the variational problem considered in Section 2. Let $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$. We denote by $\nabla_\varepsilon J_{\varepsilon,A,V}(u)$ the gradient of $J_{\varepsilon,A,V}$ at u with respect to the scalar product $\langle \cdot, \cdot \rangle_{\varepsilon,A,V}$, and by $\nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u)$ its orthogonal projection onto the tangent space to $\mathcal{N}_{\varepsilon,A,V}^\tau$ at u . Recall that $J_{\varepsilon,A,V} : \mathcal{N}_{\varepsilon,A,V}^\tau \rightarrow \mathbb{R}$ is said to satisfy the Palais-Smale condition $(PS)_c$ at the level c , if every sequence (u_n) such that

$$u_n \in \mathcal{N}_{\varepsilon,A,V}^\tau, \quad J_{\varepsilon,A,V}(u_n) \rightarrow c, \quad \nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n) \rightarrow 0,$$

contains a convergent subsequence. As usual, the following holds.

Lemma 5. *If (u_n) is a sequence in $\mathcal{N}_{\varepsilon,A,V}^\tau$ such that $J_{\varepsilon,A,V}(u_n) \rightarrow c$ and $\nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n) \rightarrow 0$, then $\nabla_\varepsilon J_{\varepsilon,A,V}(u_n) \rightarrow 0$.*

Proof. Let $F_{\varepsilon,A,V}(u) := J'_{\varepsilon,A,V}(u)u = \|u\|_{\varepsilon,A,V}^2 - |u|_p^p$. Then,

$$\mathcal{N}_{\varepsilon,A,V}^\tau = \{u \in H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau \setminus \{0\} : F_{\varepsilon,A,V}(u) = 0\}.$$

Therefore,

$$\nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n) = \nabla_\varepsilon J_{\varepsilon,A,V}(u_n) - \mu_n \nabla_\varepsilon F_{\varepsilon,A,V}(u_n)$$

for a suitable $\mu_n \in \mathbb{R}$. Since $u_n \in \mathcal{N}_{\varepsilon,A,V}^\tau$ and (u_n) is bounded in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$, taking the scalar product of this equality with u_n we obtain

$$\begin{aligned} o(1) &= \left\langle \nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n), u_n \right\rangle_{\varepsilon,A,V} \\ &= J'_{\varepsilon,A,V}(u_n)u_n - \mu_n F'_{\varepsilon,A,V}(u_n)u_n \\ &= -\mu_n F'_{\varepsilon,A,V}(u_n)u_n. \end{aligned}$$

It is easy to see that $|F'_{\varepsilon,A,V}(u_n)u_n| \geq a > 0$ and that $(\nabla_\varepsilon F_{\varepsilon,A,V}(u_n))$ is bounded in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$. Hence $\mu_n \rightarrow 0$ and, therefore, $\mu_n \nabla_\varepsilon F_{\varepsilon,A,V}(u_n) \rightarrow 0$. We conclude that $\nabla_\varepsilon J_{\varepsilon,A,V}(u_n) \rightarrow 0$. \square

Proposition 5. *For every $\varepsilon > 0$, the functional $J_{\varepsilon,A,V} : \mathcal{N}_{\varepsilon,A,V}^\tau \rightarrow \mathbb{R}$ satisfies $(PS)_c$ at each level*

$$c < \varepsilon^N \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) V_\infty^q c_{\mathbb{R}^N},$$

where $V_\infty := \liminf_{|x| \rightarrow \infty} V(x)$.

Proof. Let (u_n) be a sequence in $\mathcal{N}_{\varepsilon,A,V}^\tau$ which satisfies

$$J_{\varepsilon,A,V}(u_n) \rightarrow c < \varepsilon^N \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) V_\infty^q C_{\mathbb{R}^N}, \quad \nabla_{\mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V}(u_n) \rightarrow 0.$$

Since (u_n) is bounded in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$, it contains a subsequence such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau, \\ u_n(x) &\rightarrow u(x) \quad \text{a.e. in } \mathbb{R}^N, \\ u_n &\rightarrow u \quad \text{strongly in } L_{loc}^p(\mathbb{R}^N, \mathbb{C}). \end{aligned}$$

Since $\nabla_\varepsilon J_{\varepsilon,A,V}(u_n) \rightarrow 0$ we have that

$$\begin{aligned} o(1) &= J'_{\varepsilon,A,V}(u_n)u = \langle u_n, u \rangle_{\varepsilon,A,V} - \operatorname{Re} \int |u_n|^{p-2} u_n \bar{u} \\ &= \|u\|_{\varepsilon,A,V}^2 - |u|_p^p + o(1). \end{aligned}$$

Hence, $\|u\|_{\varepsilon,A,V}^2 = |u|_p^p$.

We claim that, given $\delta > 0$, there exists $R > 0$ such that, passing to a subsequence,

$$(5.4) \quad \int_{|x| > R} |u_n|^p < \delta \quad \forall n > R.$$

First, we show that this suffices to prove the proposition. Indeed, choosing R large enough we obtain

$$\begin{aligned} & \left| \|u_n\|_{\varepsilon,A,V}^2 - \|u\|_{\varepsilon,A,V}^2 \right| = \left| |u_n|_p^p - |u|_p^p \right| \\ & \leq \left| \int_{|x| < R} |u_n|^p - \int_{|x| < R} |u|^p \right| + \int_{|x| > R} |u_n|^p + \int_{|x| > R} |u|^p \\ & < 3\delta \quad \forall n > R. \end{aligned}$$

Hence, $\|u_n\|_{\varepsilon,A} \rightarrow \|u\|_{\varepsilon,A}$ as $n \rightarrow \infty$, which implies strong convergence. To prove claim (5.4) we argue by contradiction. Assume there exist $\delta_0 > 0$ and a subsequence (u_{n_k}) such that

$$(5.5) \quad \int_{|x| > k} |u_{n_k}|^p \geq \delta_0 \quad \forall k.$$

Choose $\lambda < V_\infty$ such that

$$(5.6) \quad c < \varepsilon^N \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) \lambda^q C_{\mathbb{R}^N},$$

and $R_0 > 0$ such that

$$(5.7) \quad V(x) \geq \lambda \quad \text{if } |x| \geq R_0.$$

Let $\eta > 0$. Since the sequence (u_n) is bounded in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$, we may choose $r > R_0$ such that

$$\int_{r < |x| < r+1} (|\nabla_{\varepsilon,A} u_{n_k}|^2 + V(x) |u_{n_k}|^2) < \eta \quad \forall k.$$

Let $\rho \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be radial and such that $\rho(x) = 0$ if $|x| \leq r$, $\rho(x) = 1$ if $|x| \geq r+1$, and $\rho(x) \in [0, 1]$ for all $x \in \mathbb{R}^N$. Set $w_k := \rho u_{n_k}$. A straightforward computation shows that

$$\begin{aligned} |J'_{\varepsilon,A,V}(u_{n_k})w_k - J'_{\varepsilon,A,V}(w_k)w_k| &\leq C \int_{r < |x| < r+1} (|\nabla_{\varepsilon,A} u_{n_k}|^2 + V(x) |u_{n_k}|^2) \\ &< C\eta \quad \forall k. \end{aligned}$$

Here and in the following C is some positive constant, not necessarily the same one, independent of r . Since (w_k) is bounded in $H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})$, we have that $J'_{\varepsilon,A,V}(u_{n_k})w_k \rightarrow 0$. Therefore,

$$\left| \|w_k\|_{\varepsilon,A,V}^2 - |w_k|_p^p \right| \leq C\eta \quad \forall k.$$

Inequality (5.5) implies that $|w_k|_p^p \geq \delta_0$ for k large enough. Hence,

$$(5.8) \quad \left| \frac{\|w_k\|_{\varepsilon,A,V}^2}{|w_k|_p^2} - |w_k|_p^{p-2} \right| \leq C\eta.$$

Let

$$\pi_{\varepsilon,A,\lambda} : H_{\varepsilon,A}^1(\mathbb{R}^N, \mathbb{C})^\tau \setminus \{0\} \rightarrow \mathcal{N}_{\varepsilon,A,\lambda}^\tau$$

be the radial projection. The diamagnetic inequality (2.1) yields $\| |w_k| \|_{\varepsilon,\lambda,r}^2 \leq \|w_k\|_{\varepsilon,A,\lambda}^2$. Therefore, using (5.7) and (5.8), we obtain

$$\begin{aligned} c_{\varepsilon,\lambda,r}^G &\leq J_{\varepsilon,\lambda}(\pi_{\varepsilon,\lambda,r} |w_k|) = \frac{p-2}{2p} \left(\frac{\| |w_k| \|_{\varepsilon,\lambda,r}^2}{|w_k|_p^2} \right)^{p/(p-2)} \\ &\leq \frac{p-2}{2p} \left(\frac{\|w_k\|_{\varepsilon,A,\lambda}^2}{|w_k|_p^2} \right)^{p/(p-2)} \leq \frac{p-2}{2p} \left(\frac{\|w_k\|_{\varepsilon,A,V}^2}{|w_k|_p^2} \right)^{p/(p-2)} \\ &\leq \frac{p-2}{2p} (|w_k|_p^{p-2} + C\eta)^{p/(p-2)} \leq (1 + C\eta)^{p/(p-2)} \frac{p-2}{2p} |u_{n_k}|_p^p. \end{aligned}$$

Letting $k \rightarrow \infty$ and $\eta \rightarrow 0$, and using inequality (5.6), we conclude that

$$\lim_{r \rightarrow \infty} c_{\varepsilon,\lambda,r}^G \leq c < \varepsilon^N \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) \lambda^q c_{\mathbb{R}^N},$$

contradicting Lemma 4. This proves claim (5.4). \square

Proof of Theorem 2. Let $\rho, \delta > 0$. Assumption (1.7) implies that

$$\ell_G < \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) V_\infty^q$$

where $V_\infty := \liminf_{|x| \rightarrow \infty} V(x)$. Without loss of generality, we may assume that $\rho \in (0, \bar{\rho})$, with $\bar{\rho}$ as in Section 4, and that

$$\ell_{GC\mathbb{R}^N} + \delta < \min_{x \in \mathbb{R}^N \setminus \{0\}} (\#Gx) V_\infty^q c_{\mathbb{R}^N}.$$

We choose $d \in (\ell_{GC\mathbb{R}^N}, \ell_{GC\mathbb{R}^N} + \delta)$. By Corollary 3 we may also choose $\varepsilon_0 > 0$ such that

$$\ell_{GC\mathbb{R}^N} - \delta < \varepsilon^{-N} \inf_{u \in \mathcal{N}_{\varepsilon,A,V}^\tau} J_{\varepsilon,A,V} \quad \forall \varepsilon \in (0, \varepsilon_0).$$

By Proposition 5, for every $\varepsilon \in (0, \varepsilon_0)$, the functional $J_{\varepsilon,A,V} : \mathcal{N}_{\varepsilon,A,V}^\tau \rightarrow \mathbb{R}$ satisfies $(PS)_c$ at each level $c \leq \varepsilon^N d$ so, by Proposition 1, $J_{\varepsilon,A,V}$ has at least

$$\text{cat} \left[\left(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d} \right) / \mathbb{S}^1 \right]$$

geometrically distinct solutions $u \in \mathcal{N}_{\varepsilon,A,V}^\tau$ satisfying

$$\varepsilon^N \ell_{GC\mathbb{R}^N} - \varepsilon^N \delta \leq J_{\varepsilon,A,V}(u) = \frac{p-2}{2p} |u|_p^p \leq \varepsilon^N d < \varepsilon^N \ell_{GC\mathbb{R}^N} + \varepsilon^N \delta.$$

Let $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_{\rho,d}\}$ with $\varepsilon_{\rho,d}$ as in Corollary 2. Then,

$$\text{cat}_{M_\tau^p/G} M_\tau/G \leq \text{cat} \left(\left(\mathcal{N}_{\varepsilon,A,V}^\tau \cap J_{\varepsilon,A,V}^{\varepsilon^N d} \right) / \mathbb{S}^1 \right)$$

for every $\varepsilon \in (0, \bar{\varepsilon})$. This concludes the proof. \square

REFERENCES

- [1] A. AMBROSETTI, M. BADIÀLE, S. CINGOLANI, Semiclassical states of nonlinear Schrödinger equations, *Arch. Ration. Mech. Anal.* **140** (1997), 285–300.
- [2] G. ARIOLI, A. SZULKIN, A semilinear Schrödinger equations in the presence of a magnetic field, *Arch. Ration. Mech. Anal.* **170** (2003), 277–295.
- [3] T. BARTSCH, "Topological Methods for Variational Problems with Symmetries", Lecture Notes in Math. 1560, Springer-Verlag, Berlin Heidelberg 1993.
- [4] T. BARTSCH, M. CLAPP, T. WETH, Configuration spaces, transfer, and 2-nodal solutions of a semiclassical nonlinear Schrödinger equation, *Math. Ann.* **338** (2007), 147–185.
- [5] T. BARTSCH, E. N. DANCER, S. PENG, On multi-bump semi-classical bound states of nonlinear Schrödinger equations with electromagnetic fields, *Adv. Differential Equations* **11** (2006), 781–812.
- [6] V. BENCI, G. CERAMI, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology, *Calc. Var. Partial Differential Equations* **2** (1994), 29–48.
- [7] G. CERAMI, M. CLAPP, Sign changing solutions of semilinear elliptic problems in exterior domains, *Calc. Var. Partial Differential Equations* **30** (2007), 353–367.
- [8] J. CHABROWSKI, A. SZULKIN, On the Schrödinger equation involving a critical Sobolev exponent and magnetic field, *Topol. Methods Nonlinear Anal.* **25** (2005), 3–21.

- [9] S. CINGOLANI, Semiclassical stationary states of Nonlinear Schrödinger equations with an external magnetic field, *J. Differential Equations* **188** (2003), 52-79.
- [10] S. CINGOLANI, L. JEANJEAN, S. SECCHI, Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions, *ESAIM Control Optim. Calc. Var.*, to appear.
- [11] S. CINGOLANI, S. SECCHI, Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields, *J. Math. Anal. Appl.* **275** (2002), 108–130.
- [12] S. CINGOLANI, S. SECCHI, Semiclassical states for NLS equations with magnetic potentials having polynomial growths, *J. Math. Phys.* **46** (2005), 1-19.
- [13] M. CLAPP, G. IZQUIERDO, Multiple positive symmetric solutions of a singularly perturbed elliptic equation, *Topol. Meth. Nonl. Anal.* **18** (2001), 17-39.
- [14] M. CLAPP, R. ITURRIAGA, A. SZULKIN, Periodic and Bloch solutions to a magnetic nonlinear Schrödinger equation, *Adv. Nonlinear Stud.* in press.
- [15] M. CLAPP, D. PUPPE, Critical point theory with symmetries, *J. reine angew. Math.* **418** (1991), 1-29.
- [16] M. CLAPP, A. SZULKIN, Multiple solutions to a nonlinear Schrödinger equation with Aharonov-Bohm magnetic field, preprint.
- [17] M. DEL PINO, P. FELMER, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* **4**(1996), 121–137.
- [18] T. TOM DIECK, "Transformation groups", Walter de Gruyter, Berlin-New York 1987.
- [19] M. ESTEBAN, P.L. LIONS, Existence and non-existence results for semilinear elliptic problems in unbounded domains, *Proc. Roy. Soc. Edinburgh* **93A** (1982), 1-14.
- [20] A. FLOER, A. WEINSTEIN, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* **69** (1986), 397–408.
- [21] E. HEBEY, M. VAUGON, Existence and multiplicity of nodal solutions for nonlinear elliptic equations with critical Sobolev growth, *J. Funct. Anal.* **119** (1994) 298-318.
- [22] K. KURATA, Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields, *Nonlinear Anal.* **41** (2000), 763–778.
- [23] Y.Y. LI, On a singularly perturbed elliptic equation, *Adv. Differential Equations* **2** (1997), 955–980.
- [24] E.H. LIEB, M. LOSS, "Analysis", *Graduate Studies in Math.* **14**, Amer. Math. Soc. 1997.
- [25] P. MEYSTRE, "Atom Optics", Springer-Verlag, New York 2001.
- [26] D.L. MILLS, "Nonlinear Optics", Springer-Verlag, New York 1998.
- [27] R. PALAIS, The principle of symmetric criticality, *Comm. Math. Phys.* **69** (1979), 19-30.
- [28] A.A. PANKOV, On nontrivial solutions of the nonlinear Schrödinger equation with a magnetic field, *Funct. Anal. Appl.* **37** (2003), 75–77.
- [29] P. RABINOWITZ, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992), 270–291.

- [30] N. SAINTIER, Blow-up theory for symmetric critical equations involving the p -Laplacian, *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), 227–245.
- [31] M. WILLEM, "Minimax theorems", *PNLDE* **24**, Birkhäuser, Boston-Basel-Berlin 1996.

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