Periodic and Bloch solutions to a magnetic nonlinear Schrödinger equation

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To our friend and colleague Vieri Benci, with great esteem.

Abstract

We study the equation

$$(\wp_A) \qquad (-i\nabla + A)^2 u + Vu = |u|^{p-2}u,$$

where $A \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in C^{0,\alpha}(\mathbb{R}^N)$ are 2π-periodic in each variable, $V > 0$, and $p \in (2, 2^*)$ with $2^* := \infty$ if $N = 2$ and $2^* := \frac{2N}{N-2}$ if $N \geq 3$. We address two questions: First, the gauge-dependence problem for 2π-periodic solutions $u : \mathbb{R}^N \to \mathbb{C}$ and second, the multiplicity of Bloch solutions. Unlike

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the nonperiodic case where problem $(\wp_A)$ is basically independent of A (it is gauge invariant), in the periodic case this is far from being true. Under some assumptions on A we show that, if there exists a one-to-one correspondence between the 2π-periodic solutions of $(\wp_A)$ and those of $(\wp_{A+z})$ preserving their absolute value, then z lies in a subset of measure zero of $\mathbb{R}^N$. We use this fact to show the existence of an uncountable set of Bloch solutions with real quasimomentum.

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1 Introduction and statement of results

The behavior of a charged particle in the presence of an external magnetic field B and an electric field is described by the magnetic Schödinger operator

$$L_{A,V} = (-i\nabla + A)^2 + V,$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is an electric potential and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential associated to B, that is, $\mathrm{curl}A = B$. In the language of differential forms, A is a 1-form $A = A_1 dx_1 + \cdots + A_N dx_N$ and $\mathrm{curl}A := dA = \sum_{j<k} b_{jk} dx_j \wedge dx_k$, where $b_{jk} = (\mathrm{curl}A)_{jk} = \partial_j A_k - \partial_k A_j$.

This paper is concerned with the problem

$$(\wp_A) \quad \begin{cases} (-i\nabla + A)^2 u + Vu = |u|^{p-2}u, \\ u \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}), \quad \nabla u + iAu \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

where $A \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in C^{0,\alpha}(\mathbb{R}^N)$ are 2π-periodic in each variable $x_1, ..., x_N$, $V > 0$, and $p \in (2, 2^*)$ with $2^* := \infty$ if $N = 2$ and $2^* := \frac{2N}{N-2}$ if $N \geq 3$.

Existence of solutions to the equation in $(\wp_A)$ with periodic and nonperiodic data whose absolute value vanishes at infinity has been shown for example in [8, 13, 2, 15, 7, 5, 4, 6], both in the classical and semiclassical regime. There is an extensive literature on this subject in the nonmagnetic case $A = 0$.

Here we shall address two questions: First, the gauge-dependence problem if one considers only 2π-periodic solutions and second, the multiplicity question for solutions whose absolute value is 2π-periodic.

Recall that every closed 2-form B on $\mathbb{R}^N$ is exact, that is, there exists a 1-form A such that $\mathrm{curl}A = B$. Moreover, if $\tilde{A}$ is another 1-form with $\mathrm{curl}\tilde{A} = B$, then $A - \tilde{A}$ is the gradient of a function φ. A straightforward computation shows that

$$u \text{ solves } (\wp_A) \Longleftrightarrow e^{-i\varphi}u \text{ solves } (\wp_{\tilde{A}}). \tag{1.1}$$

This is called the gauge invariance. It says that the choice of A with fixed $\mathrm{curl}A = B$ does not affect the solutions of $(\wp_A)$ in any essential way, as long as we allow arbitrary solutions.

Since our data are periodic, it is natural to consider periodic solutions. Now, if we are interested only in 2π-periodic solutions the situation changes drastically. In this case, problem $(\wp_A)$ can be interpreted as a problem on the N-dimensional flat torus $\mathbb{T}^N := \mathbb{R}^N/2\pi\mathbb{Z}^N$ which has nontrivial topology. This has the effect that a 2π-periodic closed 2-form B might not be the curl of a 2π-periodic 1-form [3]. A necessary condition for this to happen is that the mean value of B over $[0, 2\pi]^N$ is 0 (let us mention that the condition is also sufficient, by an argument which we do not discuss in this paper). Moreover, two 2π-periodic 1-forms A and $\tilde{A}$ do not necessarily differ by the gradient of a 2π-periodic function, so there is no obvious one-to-one correspondence between the 2π-periodic solutions of $(\wp_A)$ and those of $(\wp_{\tilde{A}})$ as given by (1.1). However, $\tilde{A}$ differs from $A + z$ by the gradient of a 2π-periodic function φ for some $z \in \mathbb{R}^N$ (see Proposition 3.2). This leaves us with comparing problems $(\wp_{A+z})$ for different choices of $z \in \mathbb{R}^N$. So the question is, does the choice of $z \in \mathbb{R}^N$ affect the solutions of $(\wp_{A+z})$ in some essential way? Now, in the context of quantum physics, a relevant quantity is the absolute value of the solution: $|u(x)|^2$ can be interpreted as the (unnormalized) probability density of finding a particle at x. Note that (1.1) establishes a one-to-one correspondence which preserves the absolute value of the solutions. So one may ask whether there is a one-to-one correspondence associating to each 2π-periodic solution u_0 of $(\wp_A)$ a 2π-periodic solution u_z of $(\wp_{A+z})$ with the same absolute value, i.e. $|u_z| = |u_0|$. We shall address this question and prove the following.

Theorem 1.1 *Assume that problem $(\wp_A)$ has a nowhere vanishing 2π-periodic solution u_0. Then there exists a quadric $\mathcal{Q}$ of codimension at least one containing the origin with the following property: If $(\wp_{A+z})$ has a 2π-periodic solution u_z such that $|u_z| = |u_0|$, then $z \in \mathcal{Q} + \mathbb{Z}^N$.*

By a quadric we mean, as usual, the set of zeros of a quadratic polynomial. So Theorem 1.1 implies, in particular, that $(\wp_{A+z})$ does not have a 2π-periodic solution u_z with $|u_z| = |u_0|$ for a.e. $z \in \mathbb{R}^N$.

It is also natural to look for solutions to $(\wp_A)$ which are not necessarily periodic but whose absolute value is periodic. Moreover, from the point of view of physics it is desirable that $(\wp_A)$ is invariant with respect to the gauge choice for the vector potential A. This suggests looking at solutions of the form $\psi(x) = e^{iz\cdot x}u(x)$ with $z \in \mathbb{R}^N$ and u a 2π-periodic function. Following the usual terminology, we call them *Bloch solutions.* More precisely, they are Bloch solutions with *real* quasimomentum z (in general, $z \in \mathbb{C}^N$ is admitted, see e.g. [11]). They have 2π-periodic absolute value and render problem $(\wp_A)$ gauge invariant, more precisely,

$$\psi \text{ is a Bloch solution of } (\wp_{A+z}) \Longleftrightarrow e^{iz\cdot x}\psi \text{ is a Bloch solution of } (\wp_A).$$

In quantum mechanical models two solutions ψ and $e^{i\gamma}\psi$, $\gamma \in \mathbb{R}$, represent the same state. So the state space is, in fact, the space of $\mathbb{S}^1$-orbits $[\psi] := \{e^{i\gamma}\psi : \gamma \in \mathbb{R}\}$ of the Sobolev space of complex-valued functions ψ where the solutions are seaked. The group of translations $\mathbb{Z}^N$ acts also on the $\mathbb{S}^1$-orbit space in the obvious way. Bloch solutions are solutions whose $\mathbb{S}^1$-orbit is 2π-periodic (see Proposition 5.1).

A 2π-periodic solution u_z of $(\wp_{A+z})$ gives rise to a Bloch solution $\psi_z := e^{iz\cdot x}u_z$ of $(\wp_A)$ with the same absolute value. Hence, the gauge-dependence question for

2π-periodic solutions is related to the multiplicity question for Bloch solutions. As a consequence of Theorem 1.1 we obtain the following.

Theorem 1.2 *Assume there exists $\varepsilon_0 > 0$ such that for every $|z| < \varepsilon_0$ problem $(\wp_{A+z})$ has a nonwhere vanishing 2π-periodic solution u_z. Then problem $(\wp_A)$ has an uncountable family of Bloch solutions $(\psi_\alpha)_{\alpha\in\mathcal{I}}$ such that*

$$|\psi_\alpha| \neq |\psi_\beta| \quad \text{if } \alpha \neq \beta.$$

Moreover, if u_z is a ground state, then for each $\delta > 0$ there is an uncountable set $\mathcal{J} \subset \mathcal{I}$ such that

$$\left| \int_{[0,2\pi]^N} |\psi_\alpha|^p - \int_{[0,2\pi]^N} |u_0|^p \right| < \delta \quad \forall \alpha \in \mathcal{J}.$$

By a ground state we mean a nontrivial solution with minimal energy, see Section 2.

This paper is organized as follows: In Section 2 we show existence and regularity of 2π-periodic solutions to problem $(\wp_A)$. Section 3 contains several results concerning the periodic gauge-dependence question and the proof of Theorem 1.1. In Section 4 we give a condition for the existence of nowhere vanishing 2π-periodic solutions to $(\wp_A)$. Section 5 is devoted to the proof of Theorem 1.2.

2 Preliminaries

Let $\mathbb{Z}^N$ be the subset of points of $\mathbb{R}^N$ with integer coordinates. A function u defined on $\mathbb{R}^N$ will be called 2π-periodic if it is 2π-periodic in each coordinate, that is, if $u(x + 2\pi m) = u(x)$ for all $m \in \mathbb{Z}^N$.

We assume throughout this paper that $A \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in C^{0,\alpha}(\mathbb{R}^N)$ are 2π-periodic and $V > 0$.

Consider the real Hilbert space

$$H^1_{A,per}(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}) : u \text{ is } 2\pi\text{-periodic}, \nabla u + iAu \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}^N)\},$$

with scalar product

$$\langle u, v\rangle_{A,V} := \operatorname{Re} \int_{[0,2\pi]^N} ((\nabla u + iAu) \cdot (\nabla \bar{v} - iA\bar{v}) + V u\bar{v}).$$

Our assuptions on V imply that the induced norm,

$$\|u\|_{A,V} := \left(\int_{[0,2\pi]^N} (|\nabla u + iAu|^2 + V\,|u|^2) \right)^{1/2},$$

is equivalent to the usual one, obtained by taking $V \equiv 1$ [14].

Let $H^1_{per}(\mathbb{R}^N)$ be the subspace of 2π-periodic functions in the (real-valued) Sobolev space $H^1(\mathbb{R}^N)$. If $u \in H^1_{A,per}(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1_{per}(\mathbb{R}^N)$ and

$$|\nabla |u| (x)| \leq |\nabla u(x) + iA(x)u(x)| \quad \text{for a.e. } x \in \mathbb{R}^N.$$

This is called the diamagnetic inequality [14]. Note that $H^1_{per}(\mathbb{R}^N)$ is isometrically isomorphic to the Sobolev space $H^1(\mathbb{T}^N)$, where $\mathbb{T}^N := \mathbb{R}^N/2\pi\mathbb{Z}^N$ is the flat N-torus obtained by identifying $x \in \mathbb{R}^N$ with $x + 2\pi m$ for every $m \in \mathbb{Z}^N$. Since $\mathbb{T}^N$ is compact, the embedding $H^1(\mathbb{T}^N) \hookrightarrow L^p(\mathbb{T}^N)$ is compact for $p \in [2, 2^*)$ [10]. Using this fact, and the diamagnetic inequality, it is not hard to show that the embedding

$$H^1_{A,per}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p_{per}(\mathbb{R}^N, \mathbb{C}), \qquad p \in [2, 2^*),$$

is also compact, where $L^p_{per}(\mathbb{R}^N, \mathbb{C}) := \{u \in L^p_{loc}(\mathbb{R}^N, \mathbb{C}) : u \text{ is } 2\pi\text{-periodic}\}$. We denote the norm in $L^p_{per}(\mathbb{R}^N, \mathbb{C})$ by

$$|u|_p := \left(\int_{[0,2\pi]^N} |u|^p \right)^{1/p}.$$

Nontrivial 2π-periodic solutions of problem $(\wp_A)$ are in one-to-one correspondence with critical points of the functional

$$I_{A,V}(u) := \|u\|^2_{A,V}$$

on the L^p-sphere

$$S_{A,p} := \left\{ u \in H^1_{A,per}(\mathbb{R}^N, \mathbb{C}) : |u|^p_p = 1 \right\}.$$

More precisely, u is a critical point of $I_{A,V}$ on $S_{A,p}$ if and only if $(I_{A,V}(u))^{1/(p-2)}u$ is a 2π-periodic solution of $(\wp_A)$. A 2π-periodic ground state of problem $(\wp_A)$ is a 2π-periodic solution which corresponds to a minimum of $I_{A,V}$ on $S_{A,p}$. Existence of 2π-periodic ground states is proved in the standard way.

Proposition 2.1 *Problem $(\wp_A)$ has a 2π-periodic ground state.*

Proof. If (u_n) is a minimizing sequence for $I_{A,V}$ in $S_{A,p}$, then (u_n) is bounded in $H^1_{A,per}(\mathbb{R}^N, \mathbb{C})$. So, after passing to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1_{A,per}(\mathbb{R}^N, \mathbb{C})$ and $u_n \to u$ strongly in $L^p_{per}(\mathbb{R}^N, \mathbb{C})$. Thus, $u \in S_{A,p}$. Since the functional $I_{A,V}$ is weakly lower semicontinuous, u is a minimizer of $I_{A,V}$ in $S_{A,p}$. ∎

In fact, for the particular nonlinearity we have chosen, the symmetric mountain pass theorem of Ambrosetti and Rabinowitz [1, 16] provides a sequence of 2π-periodic solutions of $(\wp_A)$ whose L^p-norm is unbounded.

The results in the following sections will require regularity of solutions. This is also proved by standard methods.

Proposition 2.2 *If $u \in H^1_{A,per}(\mathbb{R}^N, \mathbb{C})$ is a weak solution of the equation*

$$(-i\nabla + A)^2 u + Vu = |u|^{p-2} u,$$

then $u \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{C})$.

Proof. Write $u = v + iw$ with v, w real valued. Then v and w are weak solutions of the elliptic equations

$$-\Delta v + Vv = f := |u|^{p-2} v - 2A \cdot \nabla w - |A|^2 v - (\mathrm{div} A) w$$
$$-\Delta w + Vw = g := |u|^{p-2} w + 2A \cdot \nabla v - |A|^2 w + (\mathrm{div} A) v$$

We use the classical boot-strap argument. Since $v, w \in H^1_{per}(\mathbb{R}^N) \subset L^{2^*}_{per}(\mathbb{R}^N)$, we have that $f, g \in L^{q_1}_{per}(\mathbb{R}^N)$ for $q_1 := \min\left\{2^*(p-1)^{-1}, 2\right\}$. L^q regularity theory [9] yields that $v, w \in W^{2,q_1}_{per}(\mathbb{R}^N)$. If $2q_1 < N$ then $v, w \in L^{Nq_1/(N-2q_1)}_{per}(\mathbb{R}^N)$ and $\nabla v, \nabla w \in L^{Nq_1/(N-q_1)}_{per}(\mathbb{R}^N, \mathbb{R}^N)$, as asserted by Sobolev's theorem [9]. Hence, $f, g \in L^{q_2}_{per}(\mathbb{R}^N)$ for

$$q_2 := Nq_1 \min\left\{[(N-2q_1)(p-1)]^{-1}, (N-q_1)^{-1}\right\}.$$

Note that there exists a constant $\kappa > 1$ such that $Nq_1[(N-2q_1)(p-1)]^{-1} \geq \kappa q_1$ and $Nq(N-q)^{-1} \geq \kappa q$ for all $q \in (1, N)$. Hence, $q_2 \geq \kappa q_1$. If $2q_2 < N$ then, arguing as before, we have that $v, w \in L^{Nq_2/(N-2q_2)}_{per}(\mathbb{R}^N)$ and $\nabla v, \nabla w \in L^{Nq_2/(N-q_2)}_{per}(\mathbb{R}^N, \mathbb{R}^N)$. So, after a finite number of steps, we find q_m with $2q_m \geq N$. Then $v, w \in L^q_{per}(\mathbb{R}^N)$ for all $q \in [1, \infty)$, $\nabla v, \nabla w \in L^N_{per}(\mathbb{R}^N)$ and $q_{m+1} = N$. It follows that $\nabla v, \nabla w \in L^q_{per}(\mathbb{R}^N)$ and therefore $v, w \in W^{2,q}_{per}(\mathbb{R}^N)$ for all $q \in [1, \infty)$. By the Sobolev embedding theorem, $f, g \in C^{0,\alpha}_{per}(\mathbb{R}^N)$. Now Schauder theory [9] yields $v, w \in C^{2,\alpha}(\mathbb{R}^N)$. ■

3 Gauge dependence for periodic solutions

Set

$$B := \mathrm{curl} A,$$

with $A \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and 2π-periodic, as in the introduction. This section is devoted to the following question: If $\tilde{A} \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ is also 2π-periodic and satisfies $\mathrm{curl}\tilde{A} = B$, is there a one-to-one correspondence associating to each 2π-periodic solution u of $(\wp_A)$ a 2π-periodic solution $\tilde{u}$ of $(\wp_{\tilde{A}})$ with $|\tilde{u}| = |u|$?

We start with the following easy remark.

Proposition 3.1 *If there exists a 2π-periodic function φ_0 such that $\tilde{A} - A = \nabla\varphi_0$ then*

$$u \text{ is a } 2\pi\text{-periodic solution of } (\wp_A)$$
$$\iff e^{-i\varphi_0} u \text{ is a } 2\pi\text{-periodic solution of } (\wp_{\tilde{A}}).$$

Proof. Note that

$$(\nabla + i\tilde{A})(e^{-i\varphi_0}u) = e^{-i\varphi_0}(\nabla u + i(\tilde{A} - \nabla\varphi_0)u) = e^{-i\varphi_0}(\nabla + iA)u.$$

Hence, $e^{-i\varphi_0}u \in H^1_{\tilde{A},per}(\mathbb{R}^N, \mathbb{C})$ iff $u \in H^1_{A,per}(\mathbb{R}^N, \mathbb{C})$. A straightforward computation shows that $e^{-i\varphi_0}u$ is a solution of $(\wp_{\tilde{A}})$ iff u is a solution of $(\wp_A)$. ■

In general, however, $\tilde{A} - A$ is not the gradient of a 2π-periodic function. The following holds.

Proposition 3.2 *If* $\mathrm{curl} A = 0$, *then* $A = z + \nabla\varphi_0$ *where* φ_0 *is a* 2π*-periodic function and* $z := \frac{1}{(2\pi)^N}\int_{[0,2\pi]^N} A \in \mathbb{R}^N$.

Proof. Replacing A by $A - z$ we may assume that $\int_{[0,2\pi]^N} A = 0$. Expanding $A = (A_1, ..., A_N)$ in a Fourier series, we have

$$A_j(x) = \sum_{m\in\mathbb{Z}^N} a^j_m e^{im\cdot x},$$

where $m = (m_1, \ldots, m_N)$. Since A_j has mean value 0, one has that $a^j_{(0,\ldots,0)} = 0$. Moreover, $\partial_k A_j = \partial_j A_k$ implies that $m_k a^j_m = m_j a^k_m$ for all $k \neq j$. Hence $a^j_m = 0$ whenever $m_j = 0$. Let

$$\varphi_0(x) := \int_\gamma A,$$

where γ is a piecewise smooth curve from 0 to x in $[0, 2\pi]^N$. By Stokes' theorem, φ_0 is well defined. Now we show that it is 2π-periodic. Since A is 2π-periodic, it suffices to show that

$$\varphi_0(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N) = \varphi_0(x_1, \ldots, x_{j-1}, 2\pi, x_{j+1}, \ldots, x_N)$$

for all j. Assume for notational convenience that $j = 1$ and write $x = (x_1, \widehat{x})$, $m = (m_1, \widehat{m})$. Then

$$\varphi_0(2\pi, \widehat{x}) - \varphi_0(0, \widehat{x}) = \int_0^{2\pi} A_1(t, \widehat{x})\, dt = \sum_{m\in\mathbb{Z}^N} \int_0^{2\pi} a^1_m e^{im_1 t} e^{i\widehat{m}\cdot\widehat{x}}\, dt.$$

Since $a^1_m = 0$ whenever $m_1 = 0$, all integrals above are 0. This finishes the proof. ■

Corollary 3.1 *Let* $\tilde{A} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *be* 2π*-periodic and such that* $\mathrm{curl}\tilde{A} = B$ *and let* $z := \frac{1}{(2\pi)^N}\int_{[0,2\pi]^N}(\tilde{A} - A)$. *Then there exists a* 2π*-periodic function* φ_0 *such that*

$$u \text{ is a } 2\pi\text{-periodic solution of } (\wp_{A+z})$$
$$\iff e^{-i\varphi_0}u \text{ is a } 2\pi\text{-periodic solution of } (\wp_{\tilde{A}}).$$

Proof. By the previous proposition, $\tilde{A} - (A + z) = \nabla\varphi_0$ for some 2π-periodic function φ_0. Now the assertion follows from Proposition 3.1. ∎

Corolary 3.1 reduces our original question to the following: Given $z \in \mathbb{R}^N$, is there a one-to-one correspondence associating to each 2π-periodic solution u_0 of $(\wp_A)$ a 2π-periodic solution u_z of $(\wp_{A+z})$ with the same absolute value? An easy case where the answer is positive is the following.

Proposition 3.3 *If $m \in \mathbb{Z}^N$ and $\varphi(x) := m \cdot x$, then*

$$\begin{aligned} &u \textit{ is a } 2\pi\textit{-periodic solution of } (\wp_A) \\ &\Longleftrightarrow e^{-i\varphi} u \textit{ is a } 2\pi\textit{-periodic solution of } (\wp_{A+m}). \end{aligned}$$

Proof. We need only to observe that $(A + m) - A = \nabla\varphi$ and that, although φ is not 2π-periodic, the function $e^{-i\varphi}$ is. The same computation yielding Proposition 3.1 gives this result. ∎

However, the answer to our previous question will be negative for a.e. $z \in \mathbb{R}^N$. The precise statement is given by Theorem 3.1 below. For its proof we shall need the following facts.

Lemma 3.1 *If $u : \mathbb{R}^N \to \mathbb{C}$ is a 2π-periodic continuous function with $|u| > 0$, then there exist a 2π-periodic continuous function $\varphi_0 : \mathbb{R}^N \to \mathbb{R}$ and a vector $m \in \mathbb{Z}^N$ such that $u = |u|\, e^{i\varphi}$ with $\varphi(x) := \varphi_0(x) + m \cdot x$.*

Proof. Set $\theta := \frac{u}{|u|}$. The exponential map $\mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{it}$, is a covering projection and a local diffeomorphism. Hence, θ has a unique lifting $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\varphi(0) = 0$, that is, $\theta = e^{i\varphi}$ [17]. Since θ is 2π-periodic, one has that

$$e^{i\varphi(x+2\pi e_j)} = e^{i\varphi(x)} \quad \text{for all } x \in \mathbb{R}^N$$

and each canonical basis vector $e_j \in \mathbb{R}^N$. It follows that there exist $m_j \in \mathbb{Z}$ such that

$$\varphi(x + 2\pi e_j) - \varphi(x) = 2\pi m_j \quad \text{for all } x \in \mathbb{R}^N.$$

Let $m = (m_1, ..., m_N) \in \mathbb{Z}^N$ and let $\varphi_0(x) := \varphi(x) - m \cdot x$. It is straightforward to check that φ_0 is 2π-periodic. ∎

Lemma 3.2 *Let*

$$Lu := -a(x)\Delta u + \sum_{j=1}^{N} c_j(x) \frac{\partial u}{\partial x_j},$$

where $a \in C^{1,\alpha}_{per}(\mathbb{R}^N)$, $a > 0$, $c_j \in C^{0,\alpha}_{per}(\mathbb{R}^N)$. Then problem

$$\begin{cases} Lu = 0, \\ \int_{[0,2\pi]^N} u = 0, \\ u \in H^1_{per}(\mathbb{R}^N), \end{cases}$$

has only the trivial solution.

Proof. Let $u_0 \in H^1_{per}(\mathbb{R}^N) \setminus \{0\}$ satisfy $Lu_0 = 0$. We wish to show that u_0 is constant. By standard regularity theory [9], $u_0 \in C^{2,\alpha}_{per}(\mathbb{R}^N)$. Let $\alpha_0 \in \mathbb{R}$ be such that $u(x) := 1 + \alpha_0 u_0(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $u(x_0) = 0$ for some $x_0 \in [0, 2\pi]^N$. Assume, by contradiction, that u_0 is not constant. Then there exists $x_1 \in [0, 2\pi]^N$ such that $u(x_1) > 0$. Let $B_r(x_1)$ be the ball centered at x_1 and of radius $r := |x_0 - x_1|$. Then, by a Hopf-type maximum principle [18, Theorem B.5], the normal derivative of u at $x_0 \in \partial B_r(x_1)$ is nonzero. However, this is impossible as u attains its minimum at x_0. It follows that u_0 is constant. So, if $\int_{[0,2\pi]^N} u_0 = 0$, then $u_0 = 0$, as claimed. ■

We are ready to prove the following.

Proposition 3.4 *Assume that* $(\wp_A)$ *has a* 2π*-periodic solution of the form* $u_0 = ve^{i\varphi_0}$ *with* $v > 0$ *and* φ_0 *a* 2π*-periodic function. Then there exist a linearly independent (possibly empty) subset* $\mathcal{B} = \{b_1, ..., b_k\}$ *of* $\mathbb{R}^N$ *and* 2π*-periodic functions* $\varphi_1, ..., \varphi_k \in C^{2,\alpha}(\mathbb{R}^N)$ *with the following property: If* $(\wp_{A+z})$ *has a solution of the form* $u_z = ve^{i\varphi_z}$ *with* φ_z *a* 2π*-periodic function, then* $z \in \operatorname{span}\{b_1, ..., b_k\}$ *and*

$$\left| \sum_{j=1}^{k} (\nabla\varphi_j(x) + b_j) z_j + \nabla\varphi_0(x) + A(x) \right|^2 = |\nabla\varphi_0(x) + A(x)|^2$$

for all $x \in \mathbb{R}^N$*, where* $z = \sum_{j=1}^{k} z_j b_j$*.*

Proof. An easy computation shows that, if $u_0 = ve^{i\varphi_0}$ solves $(\wp_A)$, then

$$-\Delta v + |\nabla\varphi_0 + A|^2 v + Vv = v^{p-1}, \tag{3.2}$$

$$-v\Delta\varphi_0 - 2\nabla v \cdot \nabla\varphi_0 = 2A \cdot \nabla v + (\operatorname{div} A)v. \tag{3.3}$$

Let Z be the set of all $z \in \mathbb{R}^N$ with the property that $(\wp_{A+z})$ has a solution of the form $ve^{i\varphi_z}$ with φ_z a 2π-periodic function. Then φ_z satisfies

$$-\Delta v + |\nabla\varphi_z + A + z|^2 v + Vv = v^{p-1}, \tag{3.4}$$

$$-v\Delta\varphi_z - 2\nabla v \cdot \nabla\varphi_z = 2(A + z) \cdot \nabla v + (\operatorname{div} A)v \tag{3.5}$$

and, subtracting (3.3) from (3.5), we have that

$$-v\Delta(\varphi_z - \varphi_0) - 2\nabla v \cdot \nabla(\varphi_z - \varphi_0) = 2z \cdot \nabla v.$$

Let $\{b_1, ..., b_k\} \subset Z$ be a basis of the vector space spanned by Z and define

$$\varphi_j := \varphi_{b_j} - \varphi_0 - \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} (\varphi_{b_j} - \varphi_0).$$

Then φ_j satisfies

$$-v\Delta\varphi_j - 2\nabla v \cdot \nabla\varphi_j = 2b_j \cdot \nabla v, \quad j = 1, ..., k.$$

Every $z \in Z$ is of the form $z = \sum_{j=1}^{k} z_j b_j$. Since both

$$\varphi_z - \varphi_0 - \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} (\varphi_z - \varphi_0) \quad \text{and} \quad \sum_{j=1}^{k} z_j \varphi_j$$

satisfy

$$\begin{cases} -v\Delta u - 2\nabla v \cdot \nabla u = 2z \cdot \nabla v, \\ \int_{[0,2\pi]^N} u = 0, \\ u \in H^1_{per}(\mathbb{R}^N), \end{cases}$$

Lemma 3.2 implies that

$$\varphi_z - \varphi_0 - \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} (\varphi_z - \varphi_0) = \sum_{j=1}^{k} z_j \varphi_j.$$

Hence,

$$\nabla \varphi_z = \nabla \varphi_0 + \sum_{j=1}^{k} z_j \nabla \varphi_j,$$

and from (3.2) and (3.4) we obtain that

$$|\nabla \varphi_0 + A|^2 = |\nabla \varphi_z + A + z|^2 = \left| \sum_{j=1}^{k} (\nabla \varphi_j + b_j) z_j + \nabla \varphi_0 + A \right|^2,$$

as claimed. ■

Now we prove the main result of this section.

Theorem 3.1 *Assume that* $(\wp_A)$ *has a* 2π*-periodic nowhere vanishing solution* u_0*. Then there exist a basis* $\{b_1, ..., b_N\}$ *of* $\mathbb{R}^N$ *and* 2π*-periodic functions* $\varphi_0, \varphi_1, ..., \varphi_N \in C^{2,\alpha}(\mathbb{R}^N)$ *with the following property: If* $(\wp_{A+z})$ *has a* 2π*-periodic solution* u_z *such that* $|u_z| = |u_0|$*, then there exist* $n_0, m \in \mathbb{Z}^N$ *such that*

$$\mathcal{Q}(x, z + m) = 0 \quad \text{for all } x \in \mathbb{R}^N,$$

where

$$\mathcal{Q}(x, \zeta) := \left| \sum_{j=1}^{N} (\nabla \varphi_j(x) + b_j) \zeta_j + \nabla \varphi_0(x) + A(x) + n_0 \right|^2 - |\nabla \varphi_0(x) + A(x) + n_0|^2 = 0,$$

with $\zeta = \zeta_1 b_1 + \cdots + \zeta_N b_N$.

Proof. By Lemma 3.1 there exist a 2π-periodic function φ_0 and a vector $n_0 \in \mathbb{Z}^N$ such that $u_0 = |u_0|\, e^{i\varphi}$ with $\varphi(x) := \varphi_0(x) + n_0 \cdot x$. Then, $u_1 := e^{-in_0 \cdot x} u_0 = |u_0|\, e^{i\varphi_0}$ is a 2π-periodic solution of problem $(\wp_{A+n_0})$.
Take $b_1, ..., b_k \in \mathbb{R}^N$ and $\varphi_1, ..., \varphi_k \in C^{2,\alpha}_{per}(\mathbb{R}^N)$ as in Proposition 3.4 (with A replaced by $A + n_0$ and u_1 instead of u_0), complete the set of vectors to a basis $\{b_1, ..., b_N\}$ of $\mathbb{R}^N$ and define $\varphi_j = 0$ for $j = k+1, ..., N$.
By Lemma 3.1 again, $u_z = |u_0|\, e^{i\theta}$ with $\theta(x) := \theta_0(x) + n_1 \cdot x$, for some 2π-periodic function $\theta_0 \in C^{2,\alpha}(\mathbb{R}^N)$ and $n_1 \in \mathbb{Z}^N$. It follows that $u_2 := e^{-in_1 \cdot x} u_z = |u_0|\, e^{i\theta_0}$ is a solution of $(\wp_{A+z+n_1})$.
Set $m := n_1 - n_0$. Then $A + z + n_1 = A + n_0 + z + m$, $m = \sum_{j=1}^N m_j b_j$ and Proposition 3.4 asserts that $z + m = \sum_{j=1}^N (z_j + m_j) b_j$ with $z_j + m_j = 0$ if $j = k+1, ..., N$ and that

$$\left| \sum_{j=1}^N (\nabla\varphi_j + b_j)(z_j + m_j) + \nabla\varphi_0 + A + n_0 \right|^2 - |\nabla\varphi_0 + A + n_0|^2 = 0,$$

as claimed. ■

Remark 3.1 Note that, if φ, θ above are 2π-periodic, then $n_0 = m = 0$. Note also that n_0 is determined by u_0 and not by z.

Theorem 1.1 follows easily from Theorem 3.1.

Proof of Theorem 1.1. According to Theorem 3.1 we need only to show that $\mathcal{Q}(x_0, \zeta)$ is a quadratic polynomial in ζ for some $x_0 \in \mathbb{R}^N$.
If the set $\mathcal{B}$ of Proposition 3.4 is empty, then $\varphi_j = 0$ for all $j = 1, ..., N$ and

$$\mathcal{Q}(x, \zeta) = |\zeta + \nabla\varphi_0(x) + A(x) + n_0|^2 - |\nabla\varphi_0(x) + A(x) + n_0|^2,$$

which is indeed a quadratic polynomial in ζ for every x. If $\mathcal{B} \neq \emptyset$ we take b_1 and φ_1 as in Proposition 3.4. Since φ_1 is 2π-periodic, it has a critical point x_0. The coefficient of ζ_1^2 in $\mathcal{Q}(x_0, \zeta)$ is $|b_1|^2 \neq 0$, so $\mathcal{Q}(x_0, \zeta)$ is a quadratic polynomial in ζ. ■

Theorem 1.1 implies that, if $(\wp_A)$ has a 2π-periodic nowhere vanishing solution, then for a.e. $z \in \mathbb{R}^N$ there is no one-to-one correspondence associating to each 2π-periodic solution of $(\wp_{A+z})$ a 2π-periodic solution of $(\wp_A)$ with the same absolute value. In the nonmagnetic case $A = 0$ one has a stronger result.

Theorem 3.2 *Let $v \in H^1(\mathbb{R}^N)$, $v > 0$, be a 2π-periodic solution of*

$$(\wp_0) \qquad -\Delta u + Vu = |u|^{p-2} u.$$

Then, problem

$$(\wp_z) \qquad -\Delta u - 2iz \cdot \nabla u + |z|^2 u + Vu = |u|^{p-2} u$$

has a 2π-periodic solution $u_z \in H^1(\mathbb{R}^N, \mathbb{C})$ such that $|u_z| = v$ if and only if $z \in \mathbb{Z}^N$.

Proof. Proposition 3.3 asserts that, if $z \in \mathbb{Z}^N$, then $(\wp_z)$ has a 2π-periodic solution u_z with $|u_z| = v$. Let us prove the converse. By Theorem 3.1 there exists $m \in \mathbb{Z}^N$ such that

$$\mathcal{Q}(x, z+m) = \left| \sum_{j=1}^{N} (\nabla\varphi_j(x) + b_j)(z_j + m_j) \right|^2 = 0 \quad \forall x \in \mathbb{R}^N.$$

The function

$$f(x) := \sum_{j=1}^{N} \varphi_j(x)(z_j + m_j)$$

is 2π-periodic. Hence, it has a critical point x_0. It follows that

$$\mathcal{Q}(x_0, z+m) = |z+m|^2 = 0$$

and, therefore, that $z = -m \in \mathbb{Z}^N$. ■

The statement above is not true if we replace $A = 0$ by $A = y$ with $y \neq 0$. A straightforward computation shows that

Example 3.1 *For every $y \in \mathbb{R}^N$, u is a 2π-periodic solution of $(\wp_y)$ if and only if its conjugate $\overline{u}$ is a 2π-periodic solution of $(\wp_{-y})$.*

Remark 3.2 At this point we would like to say a few words about the eigenvalue problem

$$(-i\nabla + A + z)^2 u + Vu = \lambda u, \qquad u \in H^1_{per}(\mathbb{R}^N, \mathbb{C}).$$

The spectrum of the operator $L_{A+z,V} := (-i\nabla + A + z)^2 + V$ in $L^2(\mathbb{T}^N)$ consists of a sequence $\lambda_k(z)$ of real eigenvalues. The function $\lambda_k : \mathbb{R}^N \to \mathbb{R}$ is continuous, 2π-periodic, and its image is a non-trivial closed interval $[a_k, b_k]$ [11, 12]. For most values $\mu \in [a_k, b_k]$, the set $\lambda_k^{-1}(\mu)$ is rather large. So, given $z_0 \in \lambda_k^{-1}(\mu)$ and an eigenfunction $u_0 \in H^1_{per}(\mathbb{R}^N, \mathbb{C})$ with $L_{A+z_0,V}(u_0) = \mu u_0$, it makes sense to ask how large is the set of $z \in \lambda_k^{-1}(\mu)$ for which there exists a function $u_z \in H^1_{per}(\mathbb{R}^N, \mathbb{C})$ satisfying $L_{A+z,V}(u_z) = \mu u_z$ and $|u_z| = |u_0|$. One can easily adapt the proof of Theorem 3.1 to obtain an analogous answer to this question for $k = 1$. However, without further knowledge on the codimension of the quadrics, this result will not give much information because, generically, $\lambda_k^{-1}(\mu)$ is a set of codimension one in $\mathbb{R}^N$.

We believe that it should be true generically (with respect to the magnetic field) that an absolute value preserving one-to-one correspondence between 2π-periodic solutions of $(\wp_A)$ and 2π-periodic solutions of $(\wp_{A+z})$ exists only if $z \in \mathbb{Z}^N$. However, by Example 3.1, this cannot be true for all magnetic potentials A.

4 Existence of nowhere vanishing solutions

Theorems 1.1, 1.2 and 3.1 require that problem $(\wp_A)$ has nowhere vanishing 2π-periodic solutions. We prove this for small enough magnetic potentials.

Recall the notation introduced in Section 2. Let

$$\mu_A := \inf_{u \in S_{A,p}} I_{A,V}(u). \tag{4.6}$$

Proposition 4.1 *Assume every 2π-periodic ground state of problem $(\wp_A)$ is nowhere vanishing. Then there exists $\varepsilon_0 > 0$ such that, if $|A - \tilde{A}|_\infty + |\mathrm{div}(A - \tilde{A})|_\infty < \varepsilon_0$, then every 2π-periodic ground state of problem $(\wp_{\tilde{A}})$ is nowhere vanishing.*

Proof. Since A is uniformly bounded, it is not hard to see that $H^1_{A,per}(\mathbb{R}^N, \mathbb{C}) = H^1_{per}(\mathbb{R}^N, \mathbb{C})$. Hence,

$$S_{A,p} = \left\{ u \in H^1_{per}(\mathbb{R}^N, \mathbb{C}) : |u|_p^p = 1 \right\} =: S_p$$

and

$$\mu_A = \inf_{u \in S_p} I_{A,V}(u) = \inf_{u \in S_p} \|u\|^2_{A,V}.$$

Arguing by contradiction, assume that for each n we can find $A_n \in C^{1,\alpha}_{per}(\mathbb{R}^N, \mathbb{R}^N)$, $u_n \in S_p$, and $x_n \in [0, 2\pi]^N$ such that

$$|A - A_n|_\infty + |\mathrm{div}(A - A_n)|_\infty \to 0, \quad |u_n|_p = 1, \quad I_{A_n,V}(u_n) = \mu_{A_n}, \quad u_n(x_n) = 0.$$

Let $v \in S_p$ be a minimizer for $I_{A,V}$. Then,

$$\int_{[0,2\pi]^N} |\nabla v + iA_n v|^2 = \int_{[0,2\pi]^N} |\nabla v + iAv|^2 + o(1),$$

so, using the L^2-boundedness of u_n and ∇u_n, we obtain

$$\begin{aligned} \mu_A \le \|u_n\|^2_{A,V} &= \|u_n\|^2_{A_n,V} + o(1) \le \|v\|^2_{A_n,V} + o(1) \\ &= \|v\|^2_{A,V} + o(1) = \mu_A + o(1). \end{aligned}$$

Hence

$$\|u_n\|^2_{A_n,V} = \mu_{A_n} \to \mu_A.$$

Since the embedding $H^1_{per}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^p_{per}(\mathbb{R}^N, \mathbb{C})$ is compact, after passing to a subsequence we have that $u_n \rightharpoonup u$ weakly in $H^1_{per}(\mathbb{R}^N, \mathbb{C})$ and $u_n \to u$ strongly in $L^p_{per}(\mathbb{R}^N, \mathbb{C})$. It follows that $u \in S_p$ and

$$\lim_{n \to \infty} \|u_n\|^2_{A,V} = \|u\|^2_{A,V} = \mu_A.$$

Therefore, $u_n \to u$ strongly in $H^1_{per}(\mathbb{R}^N, \mathbb{C})$.
Next we show that $u_n \to u$ strongly in $C^0_{per}(\mathbb{R}^N, \mathbb{C})$. Since

$$-\Delta u + Vu = 2iA \cdot \nabla u - |A|^2 u + i(\mathrm{div}A)u + \mu_A |u|^{p-2} u =: f,$$
$$-\Delta u_n + Vu_n = 2iA_n \cdot \nabla u_n - |A_n|^2 u_n + i(\mathrm{div}A_n)u_n + \mu_{A_n} |u_n|^{p-2} u_n =: f_n,$$

we have that $v_n := u_n - u$ solves

$$-\Delta v_n + Vv_n = f_n - f.$$

By Proposition 2.2, $v_n \in C^{2,\alpha}_{per}(\mathbb{R}^N, \mathbb{C})$ and $f_n, f \in C^{0,\alpha}_{per}(\mathbb{R}^N, \mathbb{C})$. Hence, for each $q \in (1, \infty)$ there is a constant C_q such that

$$\|v_n\|_{W^{2,q}(\mathbb{R}^N,\mathbb{C})} \leq C_q(|v_n|_q + |f - f_n|_q) \quad \forall n \in \mathbb{N}$$

[9, Theorem 9.11]. Since $v_n \to 0$ in $H^1_{per}(\mathbb{R}^N, \mathbb{C})$ and in $L^{2^*}_{per}(\mathbb{R}^N, \mathbb{C})$, using a boot-strap argument as we did for the proof of Proposition 2.2, we conclude that $f_n - f \to 0$ in $L^N_{per}(\mathbb{R}^N, \mathbb{C})$ and $v_n \to 0$ in $W^{2,N}_{per}(\mathbb{R}^N, \mathbb{C})$ (it is here that the assumption $|\mathrm{div}(A_n - A)|_\infty \to 0$ is employed). So, by Sobolev's embedding theorem, $v_n \to 0$ in $C^0_{per}(\mathbb{R}^N, \mathbb{C})$.
Finally, passing to a subsequence, we have that $x_n \to x$ in $[0, 2\pi]^N$. Since $u_n \to u$ in $C^0_{per}(\mathbb{R}^N, \mathbb{C})$ and $u_n(x_n) = 0$, we obtain that $u(x) = 0$. This contradicts our assumption that 2π-periodic ground states of problem $(\wp_A)$ are nowhere vanishing. ■

As a consequence, we obtain examples of potentials A for which every 2π-periodic ground state of $(\wp_A)$ is nowhere vanishing.

Corollary 4.1 *There exists $\varepsilon_0 > 0$ such that, if $|\tilde{A}|_\infty + |\mathrm{div}\tilde{A}|_\infty < \varepsilon_0$, then every 2π-periodic ground state of problem $(\wp_{\tilde{A}})$ is nowhere vanishing. In particular, the conclusion holds if $\tilde{A} = z \in \mathbb{R}^N$ and $|z| < \varepsilon_0$.*

Proof. Observe that, if $A = 0$ and u is a 2π-periodic minimizer of $I_{0,V}$ on S_p, then so is $|u|$, i.e. $|u|$ satisfies

$$-\Delta |u| + V |u| = \mu_0 |u|^{p-1}.$$

Hence, by Harnack's inequality [9, Theorem 8.20], $|u|$ is strictly positive in $[0, 2\pi]^N$. The assertion now follows from Proposition 4.1 with $A = 0$. ■

We do not know whether it is true in general that $(\wp_A)$ has a nowhere vanishing solution. For some magnetic potentials, like the classical Aharonov-Bohm magnetic potentials, this is always true. The proof will be given in a forthcoming paper. By analogy with the nonmagnetic case, it seems reasonable to expect nowhere vanishing solutions if the quadratic part of the functional is positive.

5 Uncountably many Bloch solutions

Now we consider the multiplicity question for Bloch solutions. By a *Bloch function* we mean a function ψ of the form $\psi(x) = e^{iy\cdot x}u(x)$ with $y \in \mathbb{R}^N$ and u a 2π-periodic function. As we have mentioned in the introduction, it is natural to consider ψ and $e^{i\theta}\psi$, $\theta \in \mathbb{R}$, to be the same solution of $(\wp_A)$. Bloch solutions arise in a natural way when one takes as state space for problem $(\wp_A)$ the space of $\mathbb{S}^1$-orbits of the appropriate Sobolev space. Let us explain.

The group $\mathbb{S}^1 := \{e^{i\gamma} : \gamma \in \mathbb{R}\}$ acts by multiplication on the vector space of complex-valued functions $\psi : \mathbb{R}^N \to \mathbb{C}$. We denote by $[\psi] := \{e^{i\gamma}\psi : \gamma \in \mathbb{R}\}$ the $\mathbb{S}^1$-orbit of ψ. For $m \in \mathbb{Z}^N$ let τ_m be the translation $\tau_m(x) := x + 2\pi m$. Clearly,

$$[\psi_1 \circ \tau_m] = [\psi_2 \circ \tau_m] \quad \text{if} \quad [\psi_1] = [\psi_2],$$

so the group of translations $\mathbb{Z}^N$ acts on the $\mathbb{S}^1$-orbit space of complex-valued functions. We shall say that $[\psi]$ *is 2π-periodic* if

$$[\psi \circ \tau_m] = [\psi] \quad \forall m \in \mathbb{Z}^N.$$

The following holds.

Proposition 5.1 *(a) If ψ is a Bloch function, then $[\psi]$ is 2π-periodic.*
(b) If ψ is a nowhere vanishing C^1-function and $[\psi]$ is 2π-periodic, then ψ is a Bloch function.

Proof. *(a)* If $\psi(x) = e^{iz\cdot x}u(x)$ with $z \in \mathbb{R}^N$ and 2π-periodic u, then

$$(\psi \circ \tau_m)(x) = \psi(x + 2\pi m) = e^{i2\pi z\cdot m}\psi(x) \quad \forall x \in \mathbb{R}^N,\ \forall m \in \mathbb{Z}^N.$$

Hence, $[\psi]$ is 2π-periodic.
(b) If $[\psi]$ is 2π-periodic then for every $m \in \mathbb{Z}^N$ there exists $\gamma_m \in \mathbb{R}$ such that $\psi \circ \tau_m = e^{i\gamma_m}\psi$. It is then easy to see that, for some $y \in \mathbb{R}^N$,

$$\psi(x + 2\pi m) = e^{iy\cdot m}\psi(x) \quad \forall x \in \mathbb{R}^N.$$

On the other hand, if ψ is a nowhere vanishing C^1-function, then $\psi = e^{i\varphi}|\psi|$ for some C^1-function $\varphi : \mathbb{R}^N \to \mathbb{R}$. It follows that $\varphi(x) + y \cdot m = \varphi(x + 2\pi m) \bmod 2\pi$. Hence, $\nabla\varphi$ is 2π-periodic. By Proposition 3.2 there exist $z \in \mathbb{R}^N$ and a 2π-periodic function φ_0 such that $\nabla\varphi = z + \nabla\varphi_0$. Therefore, $\psi(x) = e^{i\varphi(x)}|\psi(x)| = e^{iz\cdot x}u(x)$ with $u(x) := e^{i\varphi_0(x)}|\psi(x)|$ 2π-periodic. ■

Note that, for every $z \in \mathbb{R}^N$, ψ is a Bloch function iff $e^{iz\cdot x}\psi$ is a Bloch function. So, since $(A + z) - A = z = \nabla\varphi$ with $\varphi(x) := z \cdot x$, the usual (nonperiodic) gauge invariance yields

$$\psi \text{ is a Bloch solution of } (\wp_{A+z}) \iff e^{iz\cdot x}\psi \text{ is a Bloch solution of } (\wp_A).$$

In other words, problem $(\wp_A)$ for Bloch solutions is gauge invariant. Note also that, if ψ is a Bloch solution, then $|\psi|$ and $|\nabla\psi + iA\psi|$ are 2π-periodic.

Bloch solutions and 2π-periodic solutions are related as follows: If u_z is a 2π-periodic solution of $(\wp_{A+z})$, then $\psi_z := e^{iz\cdot x}u_z$ is a Bloch solution of $(\wp_A)$. Conversely, if ψ is a solution of $(\wp_A)$ of the form $\psi = e^{iz\cdot x}u$ with $z \in \mathbb{R}^N$ and u a 2π-periodic function, then $u = e^{-iz\cdot x}\psi$ is a 2π-periodic solution of $(\wp_{A+z})$.

Theorem 1.2 will follow from Theorem 1.1. Recall the definition of μ_A given in (4.6).

Proof of Theorem 1.2. Let $\varepsilon \in (0, \varepsilon_0)$. We define an equivalence relation in $U := \{z : |z| < \varepsilon\}$ as follows:

$$z \sim y \Longleftrightarrow |u_z| = |u_y|.$$

By Theorem 1.1, the equivalence class $[z] := \{y \in U : |u_y| = |u_z|\}$ of every $z \in U$ is contained in a countable union of quadrics of codimension 1. Hence, $[z]$ is a set of measure zero in $\mathbb{R}^N$, which implies there are uncountably many equivalence classes. We fix a z in $[z]$ and define

$$\psi_{[z]}(x) := e^{iz\cdot x}u_z(x).$$

Then $\left|\psi_{[z]}\right| = |u_z|$ and, therefore, $\left|\psi_{[z]}\right| \neq \left|\psi_{[y]}\right|$ if $[z] \neq [y]$. Suppose now u_z is a ground state. Since $|u_z|_p^p = \mu_{A+z}^{1/(p-2)}$ and $\mu_{A+z} \to \mu_A$ as $z \to 0$ (by the argument of Proposition 4.1) we obtain that

$$\left|\int_{[0,2\pi]^N} \left|\psi_{[z]}\right|^p - \int_{[0,2\pi]^N} |u_0|^p\right| = \left|\int_{[0,2\pi]^N} |u_z|^p - \int_{[0,2\pi]^N} |u_0|^p\right| < \delta$$

if ε is chosen small enough.

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