

**SINGULAR LIMITS FOR THE BI-LAPLACIAN OPERATOR
WITH EXPONENTIAL NONLINEARITY IN \mathbb{R}^4**

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ABSTRACT. Let Ω be a bounded smooth domain in \mathbb{R}^4 such that for some integer $d \geq 1$ its d -th singular cohomology group with coefficients in some field is not zero, then problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x)e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution blowing-up, as $\rho \rightarrow 0$, at m points of Ω , for any given number m .

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let Ω be a bounded smooth domain in \mathbb{R}^4 . We are interested in studying existence and qualitative properties of positive solutions to the following boundary value problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x)e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $k \in C^2(\bar{\Omega})$ is a non-negative, not identically zero function, and $\rho > 0$ is a small, positive parameter which tends to 0.

In a four-dimensional manifold, this type of equations and similar ones arise from the problem of prescribing the so-called Q -curvature, which was introduced in [7]. More precisely, given (M, g) a four-dimensional Riemannian manifold, the problem consists in finding a conformal metric \tilde{g} for which the corresponding Q -curvature $Q_{\tilde{g}}$ is a-priori prescribed. The Q -curvature for the metric g is defined as

$$Q_g = -\frac{1}{2} (\Delta_g R_g - R_g^2 + 3|\text{Ric}_g|^2),$$

where R_g is the scalar curvature and Ric_g is the Ricci tensor of (M, g) . Writing $\tilde{g} = e^{2w}g$, the problem reduces to finding a scalar function w which satisfies

$$P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \quad (1.2)$$

where P_g is the Paneitz operator [32, 10] defined as

$$P_g w = \Delta_g^2 w + \text{div} \left(\frac{2}{3} R_g g - 2\text{Ric}_g \right) dw.$$

Problem (1.2) is thus an elliptic fourth-order partial differential equation with exponential non-linearity. Several results are already known for this problem [9, 10] and related ones [1, 18, 30]. When the metric g is not Riemannian, the problem has been recently treated by Djadli and Malchiodi in [19] via variational methods.

In the special case where the manifold is the Euclidean space and g is the Euclidean metric, we recover the equation in (1.1), since (1.2) takes the simplified form

$$\Delta^2 w - 2Qe^{4w} = 0.$$

Problem (1.1) has a variational structure. Indeed, solutions of (1.1) correspond to critical points in $H^2(\Omega) \cap H_0^1(\Omega)$ of the following energy functional

$$J_\rho(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 - \rho^4 \int_\Omega k(x)e^u.$$

For any ρ sufficiently small, the functional above has a local minimum which represents a solution to (1.1) close to 0. Furthermore, the Moser-Trudinger inequality assures the existence of a second solution, which can be obtained as a mountain pass critical point for J_ρ . Thus, as $\rho \rightarrow 0$, this second solution turns out not to be bounded. The aim of the present paper is to study multiplicity of solutions to (1.1), for ρ positive and small, under some topological assumption on Ω , and to describe the asymptotic behaviour of such solutions as the parameter ρ tends to zero. Indeed, we prove that, if some cohomology group of Ω is not zero, then given any integer m we can construct solutions to (1.1) which concentrate and blow-up, as $\rho \rightarrow 0$, around some given m points of the domain. These are the singular limits.

Let us mention that concentration phenomena of this type, in domains with topology, appear also in other problems. As a first example, the two-dimensional version of problem (1.1) is the boundary value problem associated to Liouville's equation [25]

$$\begin{cases} \Delta u + \rho^2 k(x)e^u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $k(x)$ is a non-negative function and now Ω is a smooth bounded domain in \mathbb{R}^2 . In [14] it is proved that problem (1.3) admits solutions concentrating, as $\rho \rightarrow 0$, around some given set of m points of Ω , for any given integer m , provided that Ω is not simply connected. See also [5, 6, 21, 20, 11, 8, 24, 29, 31, 35, 38, 36, 37] for related results. A similar result holds true for another semilinear elliptic problem, still in dimension 2, namely

$$\begin{cases} \Delta u + u^p = 0, & u > 0, \quad \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where p now is a parameter converging to $+\infty$. Again in this situation, if Ω is not simply connected, then for p large there exists a solution to (1.4) concentrating around some set of m points of Ω , for any positive integer m [22].

In higher dimensions, the analogy is with the classical Bahri-Coron problem. In [2], Bahri and Coron show that, if $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the presence of topology in the domain guarantees existence of solutions to

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}} = 0, & u > 0, \quad \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

Partial results in this direction are also known in the slightly super critical version of Bahri-Coron's problem, namely

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}+\varepsilon} = 0, & \text{in } \Omega, \\ u > 0, \quad u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

with $\varepsilon > 0$ small. In [12] it is proved that, under the assumption that Ω is a bounded smooth domain in \mathbb{R}^N with a sufficiently small hole, a solution to (1.6) exhibiting concentration in two points is present. See also [3, 23, 34, 13, 33].

The main point of this paper is to show that the presence of topology in the domain implies strongly existence of blowing-up solutions for problem (1.1).

We denote by $H^d(\Omega)$ the d -th cohomology group of Ω with coefficients in some field \mathbb{K} . We shall prove the following

Theorem 1. *Assume that there exists $d \geq 1$ such that $H^d(\Omega) \neq 0$ and that $\inf_{\Omega} k > 0$. Then, given any integer $m \geq 1$, there exists a family of solutions u_{ρ} to Problem (1.1), for ρ small enough, with the property that*

$$\lim_{\rho \rightarrow 0} \rho^4 \int_{\Omega} k(x) e^{u_{\rho}(x)} dx = 64 \pi^2 m.$$

Furthermore, there are m points $\xi_1^{\rho}, \dots, \xi_m^{\rho}$ in Ω , separated at uniform positive distance from each other and from the boundary as $\rho \rightarrow 0$, for which u_{ρ} remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B_{\delta}(\xi_j^{\rho})$ and

$$\sup_{B_{\delta}(\xi_j^{\rho})} u_{\rho} \rightarrow +\infty,$$

for any $\delta > 0$.

The general behaviour of arbitrary families of solutions to (1.1) has been studied by C.S. Lin and J.-C. Wei in [26], where they show that, when blow-up occurs for (1.1) as $\rho \rightarrow 0$, then it is located at a finite number of peaks, each peak being isolated and carrying the energy $64\pi^2$ (at a peak, $u \rightarrow +\infty$ and outside a peak, u is bounded). See [27] and [28] for related results.

We shall see that the sets of points where the solution found in Theorem 1 blows-up can be characterized in terms of Green's function for the biharmonic operator in Ω with the appropriate boundary conditions. Let $G(x, \xi)$ be the Green function defined by

$$\begin{cases} \Delta_x^2 G(x, \xi) = 64\pi^2 \delta_{\xi}(x), & x \in \Omega, \\ G(x, \xi) = \Delta_x G(x, \xi) = 0, & x \in \partial\Omega \end{cases} \quad (1.7)$$

and let $H(x, \xi)$ be its *regular part*, namely, the smooth function defined as

$$H(x, \xi) := G(x, \xi) + 8 \log |x - \xi|.$$

The location of the points of concentration is related to the set of critical points of the function

$$\varphi_m(\xi) = - \sum_{j=1}^m \{2 \log k(\xi_j) + H(\xi_j, \xi_j)\} - \sum_{i \neq j} G(\xi_i, \xi_j), \quad (1.8)$$

defined for points $\xi = (\xi_1, \dots, \xi_m)$ such that $\xi_i \in \Omega$ and $\xi_i \neq \xi_j$ if $i \neq j$.

In [4] the authors prove that for each *nondegenerate* critical point of φ_m there exists a solution to (1.1), for any small ρ , which concentrates exactly around such

critical point as $\rho \rightarrow 0$. We shall show the existence of a solution under a weaker assumption, namely, that φ_m has a *minimax value in an appropriate subset*.

More precisely, we consider the following situation. Let Ω^m denote the cartesian product of m copies of Ω . Note that in any compact subset of Ω^m , we may define, without ambiguity,

$$\varphi_m(\xi_1, \dots, \xi_m) = -\infty \quad \text{if } \xi_i = \xi_j \text{ for some } i \neq j.$$

We shall assume that there exists an open subset U of Ω with smooth boundary, compactly contained in Ω , and such that $\inf_U k > 0$, with the following properties:

P1) U^m contains two closed subsets $B_0 \subset B$ such that

$$\sup_{\xi \in B_0} \varphi_m(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \varphi_m(\gamma(\xi)) =: c_0,$$

where $\Gamma := \{\gamma \in \mathcal{C}(B, \overline{U^m}) : \gamma(\xi) = \xi \text{ for every } \xi \in B_0\}$.

P2) For every $\xi = (\xi_1, \dots, \xi_m) \in \partial U^m$ with $\varphi_m(\xi) = c_0$, there exists an $i \in \{1, \dots, m\}$ such that

$$\begin{aligned} \nabla_{\xi_i} \varphi_m(\xi) &\neq 0 && \text{if } \xi_i \in U, \\ \nabla_{\xi_i} \varphi_m(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial U) && \text{if } \xi_i \in \partial U, \end{aligned}$$

where $T_{\xi_i}(\partial U)$ denotes the tangent space to ∂U at the point ξ_i .

We will show that, under these assumptions, φ_m has a critical point $\xi \in U^m$ with critical value c_0 . Moreover, the same is true for any small enough \mathcal{C}^1 -perturbation of φ_m . Property **P1)** is a common way of describing a change of topology of the sublevel sets of φ_m at the level c_0 , and c_0 is called a minimax value of φ_m . It is a critical value if U^m is invariant under the negative gradient flow of φ_m . If this is not the case, we use property **P2)** to modify the gradient vector field of φ_m near ∂U^m at the level c_0 and thus obtain a new vector field with the same stationary points, and such that $\overline{U^m}$ is invariant and φ_m is a Lyapunov function for the associated negative flow near the level c_0 (see Lemmas 6.3 and 6.4 below). This allows us to prove Theorem 1 and the following.

Theorem 2. *Let $m \geq 1$ and assume that there exists an open subset U of Ω with smooth boundary, compactly contained in Ω , with $\inf_U k > 0$, which satisfies P1) and P2). Then, for ρ small enough, there exists a solution u_ρ to Problem (1.1) with*

$$\lim_{\rho \rightarrow 0} \rho^4 \int_{\Omega} k(x) e^{u_\rho} = 64 \pi^2 m.$$

Moreover, there is an m -tuple $(x_1^\rho, \dots, x_m^\rho) \in U^m$, such that as $\rho \rightarrow 0$

$$\nabla \varphi_m(x_1^\rho, \dots, x_m^\rho) \rightarrow 0, \quad \varphi_m(x_1^\rho, \dots, x_m^\rho) \rightarrow c_0,$$

for which u_ρ remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B_\delta(x_j^\rho)$, and

$$\sup_{B_\delta(x_i^\rho)} u_\rho \rightarrow +\infty,$$

for any $\delta > 0$.

We will show that, for every $m \geq 1$, the set $U := \{\xi \in \Omega : \text{dist}(\xi, \partial\Omega) > \delta\}$ has property **P2)** at a given c_0 , for δ small enough (see Lemma 6.2). Thus, if $\inf_\Omega k > 0$, and if there exist closed subsets $B_0 \subset B$ of Ω^m with

$$\sup_{\xi \in B_0} \varphi_m(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \varphi_m(\gamma(\xi)),$$

then both conditions **P1)** and **P2)** hold. Condition **P1)** holds, for example, if φ_m has a (possibly degenerate) local minimum or local maximum. So a direct consequence of Theorem 2 is that in any bounded domain Ω with $\inf_{\Omega} k > 0$, Problem (1.1) has at least one solution concentrating exactly at one point, which corresponds to the minimum of the regular Green function H . Moreover if, for example, Ω is a contractible domain obtained by joining together m disjoint bounded domains through thin enough tubes, then the function φ_m has a (possibly degenerate) local minimum, which gives rise to a solution exhibiting m points of concentration.

Finally, recall that Problem (1.1) corresponds to a standard case of *uniform singular convergence*, in the sense that the associated nonlinear coefficient in Problem (1.1) $-\rho^4 k(x)$ goes to 0 uniformly in $\bar{\Omega}$ as $\rho \rightarrow 0$, property that is also present in Problem (1.3). Nontrivial topology strongly determines existence of solutions. However, we expect that this strong influence should decay under an inhomogeneous and *non-uniform* singular behavior, where critical points of an *external* function determine existence and multiplicity of solutions. See [16] for a recent two dimensional case of this phenomenon.

The paper is organized as follows. Section 2 is devoted to describing a first approximation for the solution and to estimating the error. Furthermore, Problem (1.1) is written as a fixed point problem, involving a linear operator. In Section 3 we study the invertibility of the linear problem. In Section 4 we solve a projected nonlinear problem. In Section 5 we show that solving the entire nonlinear problem reduces to finding critical points of a certain functional. Section 6 is devoted to the proofs of Theorems 1 and 2.

2. PRELIMINARIES AND ANSATZ FOR THE SOLUTION

This section is devoted to construct a reasonably good approximation U for a solution of (1.1). The shape of this approximation will depend on some points ξ_i , which we leave as parameters yet to be adjusted, where the spikes are meant to take place. As we will see, a convenient set to select $\xi = (\xi_1, \dots, \xi_m)$ is

$$\mathcal{O} := \left\{ \xi \in \mathbb{R}^m : \text{dist}(\xi_j, \partial\Omega) \geq 2\delta_0, \forall j = 1, \dots, m, \text{ and } \min_{i \neq j} |\xi_i - \xi_j| \geq 2\delta_0 \right\} \quad (2.1)$$

where $\delta_0 > 0$ is a small fixed number. We thus fix $\xi \in \mathcal{O}$.

For numbers $\mu_j > 0$, $j = 1, \dots, m$, yet to be chosen, $x \in \mathbb{R}^4$ and $\varepsilon > 0$ we define

$$u_j(x) = 4 \log \frac{\mu_j(1 + \varepsilon^2)}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2} - \log k(\xi_j), \quad (2.2)$$

so that u_j solves

$$\Delta^2 u - \rho^4 k(\xi_j) e^u = 0 \text{ in } \mathbb{R}^4, \quad (2.3)$$

with

$$\rho^4 = \frac{384 \varepsilon^4}{(1 + \varepsilon^2)^4}, \quad (2.4)$$

that is, $\rho \sim \varepsilon$ as $\varepsilon \rightarrow 0$.

Since u_j and Δu_j are not zero on the boundary $\partial\Omega$, we will add to it a bi-harmonic correction so that the boundary conditions are satisfied. Let $H_j(x)$ be the smooth

solution of

$$\begin{cases} \Delta^2 H_j = 0 & \text{in } \Omega, \\ H_j = -u_j & \text{on } \partial\Omega, \\ \Delta H_j = -\Delta u_j & \text{on } \partial\Omega. \end{cases}$$

We define our first approximation $U(\xi)$ as

$$U(\xi) \equiv \sum_{j=1}^m U_j, \quad U_j \equiv u_j + H_j. \quad (2.5)$$

As we will rigorously prove below, $(u_j + H_j)(x) \sim G(x, \xi_j)$ where $G(x, \xi)$ is the Green function defined in (1.7).

While u_j is a good approximation to a solution of (1.1) near ξ_j , it is not so much the case for U , unless the remainder $U - u_j = (H_j + \sum_{k \neq j} u_k)$ vanishes at main order near ξ_j . This is achieved through the following precise choice of the parameters μ_k

$$\log \mu_j^4 = \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j). \quad (2.6)$$

We thus fix μ_j *a priori* as a function of ξ . We write

$$\mu_j = \mu_j(\xi)$$

for all $j = 1, \dots, m$. Since $\xi \in \mathcal{O}$,

$$\frac{1}{C} \leq \mu_j \leq C, \quad \text{for all } j = 1, \dots, m, \quad (2.7)$$

for some constant $C > 0$.

The following lemma expands U_j in Ω .

Lemma 2.1. *Assume $\xi \in \mathcal{O}$. Then we have*

$$H_j(x) = H(x, \xi_j) - 4 \log \mu_j(1 + \varepsilon^2) + \log k(\xi_j) + O(\mu_j^2 \varepsilon^2), \quad (2.8)$$

uniformly in Ω , and

$$u_j(x) = 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - 8 \log|x - \xi_j| + O(\mu_j^2 \varepsilon^2), \quad (2.9)$$

uniformly in the region $|x - \xi_j| \geq \delta_0$, so that in this region,

$$U_j(x) = G(x, \xi_j) + O(\mu_j^2 \varepsilon^2). \quad (2.10)$$

Proof. Let us prove (2.8). Define $z(x) = H_j(x) + 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - H(x, \xi_j)$. Then z is a bi-harmonic function which satisfies

$$\begin{cases} \Delta^2 z = 0 & \text{in } \Omega, \\ z = -u_j + 4 \log \mu_j(1 + \varepsilon^2) - \log k(\xi_j) - 8 \log|\cdot - \xi_j| & \text{on } \partial\Omega, \\ \Delta z = -\Delta u_j - \frac{16}{|\cdot - \xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Let us define $w \equiv -\Delta z$. Thus w is harmonic in Ω and

$$\sup_{\Omega} |w| \leq \sup_{\partial\Omega} |w| \leq C \mu_j^2 \varepsilon^2.$$

We also have $\sup_{\partial\Omega} |z| \leq C \mu_j^2 \varepsilon^2$. Standard elliptic regularity implies

$$\sup_{\Omega} |z| \leq C(\sup_{\Omega} |w| + \sup_{\partial\Omega} |z|) \leq C \mu_j^2 \varepsilon^2,$$

as desired. The second estimate is direct from the definition of u_j . \square

Now, let us write

$$\Omega_\varepsilon = \varepsilon^{-1}\Omega, \quad \xi'_j = \varepsilon^{-1}\xi_j. \quad (2.11)$$

Then u solves (1.1) if and only if $v(y) \equiv u(\varepsilon y) + 4 \log \rho \varepsilon$ satisfies

$$\begin{cases} \Delta^2 v - k(\varepsilon y)e^v = 0, & \text{in } \Omega_\varepsilon, \\ v = 4 \log \rho \varepsilon, \quad \Delta v = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (2.12)$$

Let us define $V(y) = U(\varepsilon y) + 4 \log \rho \varepsilon$, with U our approximate solution (2.5). We want to measure the size of the error of approximation

$$R \equiv \Delta^2 V - k(\varepsilon y)e^V. \quad (2.13)$$

It is convenient to do so in terms of the following norm

$$\|v\|_* = \sup_{y \in \Omega_\varepsilon} \left| \left[\sum_{j=1}^m \frac{1}{(1 + |y - \xi'_j|^2)^{7/2}} + \varepsilon^4 \right]^{-1} v(y) \right| \quad (2.14)$$

Here and in what follows, C denotes a generic constant independent of ε and of $\xi \in \mathcal{O}$.

Lemma 2.2. *The error R in (2.13) satisfies*

$$\|R\|_* \leq C \varepsilon \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We assume first $|y - \xi'_k| < \frac{\delta_0}{\varepsilon}$, for some index k . We have

$$\Delta^2 V(y) = \rho^4 \sum_{j=1}^m k(\xi_j) e^{u_j(\varepsilon y)} = \frac{384 \mu_k^4}{(\mu_k^2 + |y - \xi'_k|^2)^4} + O(\varepsilon^8).$$

Let us estimate $k(\varepsilon y)e^{V(y)}$. By (2.8) and the definition of μ'_j 's,

$$\begin{aligned} H_k(x) &= H(\xi_k, \xi_k) - 4 \log \mu_k + \log k(\xi_j) + O(\mu_k^2 \varepsilon^2) + O(|x - \xi_k|) \\ &= - \sum_{j \neq k} G(\xi_j, \xi_k) + O(\mu_k^2 \varepsilon^2) + O(|x - \xi_k|), \end{aligned}$$

and if $j \neq k$, by (2.10)

$$U_j(x) = u_j(x) + H_j(x) = G(\xi_j, \xi_k) + O(|x - \xi_k|) + O(\mu_j^2 \varepsilon^2).$$

Then

$$H_k(x) + \sum_{j \neq k} U_j(x) = O(\varepsilon^2) + O(|x - \xi_k|). \quad (2.15)$$

Therefore,

$$\begin{aligned} k(\varepsilon y)e^{V(y)} &= k(\varepsilon y)\varepsilon^4 \rho^4 \exp \left\{ u_k(\varepsilon y) + H_k(\varepsilon y) + \sum_{j \neq k} U_j(\varepsilon y) \right\} \\ &= \frac{384 \mu_k^4 k(\varepsilon y)}{(\mu_k^2 + |y - \xi'_k|^2)^4 k(\xi_k)} \left\{ 1 + O(\varepsilon |y - \xi'_k|) + O(\varepsilon^2) \right\} \\ &= \frac{384 \mu_k^4}{(\mu_k^2 + |y - \xi'_k|^2)^4} \left\{ 1 + O(\varepsilon |y - \xi'_k|) \right\} \end{aligned}$$

We can conclude that in this region

$$|R(y)| \leq C \frac{\varepsilon |y - \xi'_k|}{(1 + |y - \xi'_k|^2)^4} + O(\varepsilon^4).$$

If $|y - \xi'_j| \geq \frac{\delta_0}{\varepsilon}$ for all j , using (2.8), (2.9) and (2.10) we obtain

$$\Delta^2 V = O(\varepsilon^4 \rho^4) \quad \text{and} \quad k(\varepsilon y) e^{V(y)} = O(\varepsilon^4 \rho^4).$$

Hence, in this region,

$$R(y) = O(\varepsilon^8)$$

so that finally

$$\|R\|_* = O(\varepsilon).$$

□

Next we consider the energy functional associated with (1.1)

$$J_\rho[u] = \frac{1}{2} \int_\Omega (\Delta u)^2 - \rho^4 \int_\Omega k(x) e^u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega). \quad (2.16)$$

We will give an asymptotic estimate of $J_\rho[U]$, where $U(\xi)$ is the approximation (2.5). Instead of ρ , we use the parameter ε (defined in (2.4)) to obtain the following expansion:

Lemma 2.3. *With the election of μ_j 's given by (2.6),*

$$J_\rho[U] = -128 \pi^2 m + 256 \pi^2 m |\log \varepsilon| + 32 \pi^2 \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \quad (2.17)$$

where $\Theta_\varepsilon(\xi)$ is uniformly bounded together with its derivatives if $\xi \in \mathcal{O}$, and φ_m is the function defined in (1.8).

Proof. We have

$$\begin{aligned} J_\rho[U] &= \frac{1}{2} \sum_{j=1}^m \int_\Omega (\Delta U_j)^2 + \frac{1}{2} \sum_{j \neq i} \int_\Omega \Delta U_j \Delta U_i - \rho^4 \int_\Omega k(x) e^U \\ &\equiv I_1 + I_2 + I_3; \end{aligned}$$

Note that $\Delta^2 U_j = \Delta^2 u_j = \rho^4 k(\xi_j) e^{u_j}$ in Ω and $U_j = \Delta U_j = 0$ in $\partial\Omega$. Then

$$I_1 = \frac{1}{2} \rho^4 \sum_{j=1}^m k(\xi_j) \int_\Omega e^{u_j} U_j \quad \text{and} \quad I_2 = \frac{1}{2} \rho^4 \sum_{j \neq i} k(\xi_j) \int_\Omega e^{u_j} U_i.$$

Let us define the change of variables $x = \xi_j + \mu_j \varepsilon y$, where $x \in \Omega$ and $y \in \Omega_j \equiv (\mu_j \varepsilon)^{-1}(\Omega - \xi_j)$. Using Lemma 2.1 and the definition of ρ in terms of ε in (2.4) we obtain

$$\begin{aligned} I_1 &= 192 \sum_{j=1}^m \int_{\Omega_j} \frac{1}{(1 + |y|^2)^4} \left\{ 4 \log \frac{1}{1 + |y|^2} - 8 \log \mu_j \varepsilon + H(\xi_j, \xi_j) + O(\mu_j \varepsilon |y|) \right\} \\ &= 32 \pi^2 \sum_{j=1}^m \left\{ H(\xi_j, \xi_j) - 8 \log \mu_j \varepsilon \right\} - 64 \pi^2 m + O \left(\varepsilon \mu_j \int_{\Omega_j} \frac{|y|}{(1 + |y|^2)^4} \right) \\ &= 32 \pi^2 \sum_{j=1}^m \left\{ H(\xi_j, \xi_j) - 8 \log \mu_j \varepsilon \right\} - 64 \pi^2 m + \varepsilon \Theta(\xi), \end{aligned}$$

where $\Theta_\varepsilon(\xi)$ is bounded together with its derivatives if $\xi \in \mathcal{O}$. Besides we have used the explicit values $\int_{\mathbb{R}^4} \frac{1}{(1 + |y|^2)^4} = \frac{\pi^2}{6}$, and $\int_{\mathbb{R}^4} \frac{\log(1 + |y|^2)}{(1 + |y|^2)^4} = \frac{\pi^2}{12}$.

We consider now I_2 . As above,

$$\begin{aligned}
\frac{1}{2}\rho^4 \int_{\Omega} e^{u_j} U_i &= \int_{\Omega_j} \frac{192}{(1+|y|^2)^4} \left\{ u_i(\xi_j + \mu_j \varepsilon y) + H_i(\xi_j + \mu_j \varepsilon y) \right\} \\
&= \int_{\Omega_j} \frac{192}{(1+|y|^2)^4} \left\{ u_i(\xi_j + \mu_j \varepsilon y) - 4 \log \mu_i (1 + \varepsilon^2) + \log k(\xi_i) + 8 \log |\xi_j - \xi_i| \right\} \\
&\quad + \int_{\Omega_j} \frac{192}{(1+|y|^2)^4} \left\{ H_i(\xi_j + \mu_j \varepsilon y) - H_i(\xi_j) \right\} \\
&\quad + \int_{\Omega_j} \frac{192}{(1+|y|^2)^4} \left\{ H_i(\xi_j) - H(\xi_j, \xi_i) + 4 \log \mu_i (1 + \varepsilon^2) - \log k(\xi_i) \right\} \\
&\quad + G(\xi_j, \xi_i) \int_{\Omega_j} \frac{192}{(1+|y|^2)^4} \\
&= 32\pi^2 G(\xi_i, \xi_j) + O\left(\varepsilon \mu_j \int_{\Omega_j} \frac{|y|}{(1+|y|^2)^4}\right) + O(\mu_j^2 \varepsilon^2) \\
&= 32\pi^2 G(\xi_i, \xi_j) + \varepsilon \Theta_{\varepsilon}(\xi).
\end{aligned}$$

Thus

$$I_2 = 32\pi^2 \sum_{j \neq i} G(\xi_i, \xi_j) + \varepsilon \Theta_{\varepsilon}(\xi). \quad (2.18)$$

Finally we consider I_3 . Let us denote $A_j \equiv B(\xi_j, \delta_0)$ and $x = \xi_j + \mu_j \varepsilon y$. Then using again Lemma 2.1

$$\begin{aligned}
I_3 &= -\rho^4 \sum_{j=1}^m \int_{A_j} k(x) e^U + O(\varepsilon^4) \\
&= -\rho^4 \sum_{j=1}^m \int_{B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{k(\xi_j + \mu_j \varepsilon y)}{k(\xi_j) (1+|y|^2)^4} \frac{(1+\varepsilon^2)^4}{\varepsilon^4} (1 + O(\varepsilon \mu_j |y|)) + O(\varepsilon^4) \\
&= -384 m \int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^4} + O\left(\varepsilon \mu_j \int_{\mathbb{R}^4} \frac{|y|}{(1+|y|^2)^4}\right) \\
&= -64\pi^2 m + \varepsilon \Theta_{\varepsilon}(\xi),
\end{aligned}$$

uniformly in $\xi \in \mathcal{O}$. Thus, we can conclude the following expansion of $J_{\rho}[U]$:

$$J_{\rho}[U] = -128 m \pi^2 + 256 m \pi^2 |\log \varepsilon| + 32 \pi^2 \varphi_m(\xi) + \varepsilon \Theta_{\varepsilon}(\xi), \quad (2.19)$$

where $\Theta_{\varepsilon}(\xi)$ is a bounded function together with its derivatives in the region $\xi \in \mathcal{O}$, φ_m defined as in (1.8) and $\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}$. \square

In the subsequent analysis we will stay in the expanded variable $y \in \Omega_{\varepsilon}$ so that we will look for solutions of problem (2.12) in the form $v = V + \psi$, where ψ will represent a lower order correction. In terms of ψ , problem (2.12) now reads

$$\begin{cases} \mathcal{L}_{\varepsilon}(\psi) \equiv \Delta^2 \psi - W\psi = -R + N(\psi) & \text{in } \Omega_{\varepsilon}, \\ \psi = \Delta \psi = 0 & \text{on } \partial\Omega_{\varepsilon}, \end{cases} \quad (2.20)$$

where

$$N(\psi) = W[e^{\psi} - \psi - 1] \quad \text{and} \quad W = k(\varepsilon y) e^V. \quad (2.21)$$

Note that

$$W(y) = \sum_{j=1}^m \frac{384 \mu_j^4}{(\mu_j^2 + |y - \xi_j'|)^4} (1 + O(\varepsilon |y - \xi_j'|)) \quad \text{for } y \in \Omega_\varepsilon. \quad (2.22)$$

This fact, together with the definition of $N(\psi)$ given in (2.21), give the validity of the following

Lemma 2.4. *For $\xi \in \mathcal{O}$, $\|W\|_* = O(1)$ and $\|N(\psi)\|_* = O(\|\psi\|_\infty^2)$ as $\|\psi\|_\infty \rightarrow 0$.*

3. THE LINEARIZED PROBLEM

In this section we develop a solvability theory for the fourth-order linear operator \mathcal{L}_ε defined in (2.20) under suitable orthogonality conditions. We consider

$$\mathcal{L}_\varepsilon(\psi) \equiv \Delta^2 \psi - W(y)\psi, \quad (3.1)$$

where $W(y)$ was introduced in (2.20). By expression (2.22) and setting $z = y - \xi_j'$, one can easily see that formally the operator \mathcal{L}_ε approaches, as $\varepsilon \rightarrow 0$, the operator in \mathbb{R}^4

$$\mathcal{L}_j(\psi) \equiv \Delta^2 \psi - \frac{384 \mu_j^4}{(\mu_j^2 + |z|^2)^4} \psi, \quad (3.2)$$

namely, equation $\Delta^2 v - e^v = 0$ linearized around the radial solution $v_j(z) = \log \frac{384 \mu_j^4}{(\mu_j^2 + |z|^2)^4}$. Thus the key point to develop a satisfactory solvability theory for the operator \mathcal{L}_ε is the non-degeneracy of v_j up to the natural invariances of the equation under translations and dilations. In fact, if we set

$$Y_{0j}(z) = 4 \frac{|z|^2 - \mu_j^2}{|z|^2 + \mu_j^2}, \quad (3.3)$$

$$Y_{ij}(z) = \frac{8z_i}{\mu_j^2 + |z|^2}, \quad i = 1, \dots, 4, \quad (3.4)$$

the only bounded solutions of $\mathcal{L}_j(\psi) = 0$ in \mathbb{R}^4 are linear combinations of Y_{ij} , $i = 0, \dots, 4$; see Lemma 3.1 in [4] for a proof.

We define for $i = 0, \dots, 4$ and $j = 1, \dots, m$,

$$Z_{ij}(y) \equiv Y_{ij}(y - \xi_j'), \quad i = 0, \dots, 4.$$

Additionally, let us consider R_0 a large but fixed number and χ a radial and smooth cut-off function with $\chi \equiv 1$ in $B(0, R_0)$ and $\chi \equiv 0$ in $\mathbb{R}^4 \setminus B(0, R_0 + 1)$. Let

$$\chi_j(y) = \chi(|y - \xi_j'|), \quad j = 1, \dots, m.$$

Given $h \in L^\infty(\Omega_\varepsilon)$, we consider the problem of finding a function ψ such that for certain scalars c_{ij} one has

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h + \sum_{i=1}^4 \sum_{j=1}^m c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4, j = 1, \dots, m. \end{cases} \quad (3.5)$$

We will establish a priori estimates for this problem. To this end we shall introduce an adapted norm in Ω_ε , which has been introduced previously in [15]. Given $\psi : \Omega_\varepsilon \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{N}^m$ we define

$$\|\psi\|_{**} \equiv \sum_{j=1}^m \|\psi\|_{C^{4,\alpha}(r_j < 2)} + \sum_{j=1}^m \sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \psi\|_{L^\infty(r_j \geq 2)}, \quad (3.6)$$

with $r_j = |y - \xi'_j|$.

Proposition 3.1. *There exist positive constants $\varepsilon_0 > 0$ and $C > 0$ such that for any $h \in L^\infty(\Omega_\varepsilon)$, with $\|h\|_* < \infty$, and any $\xi \in \mathcal{O}$, there is a unique solution $\psi = T(h)$ to Problem (3.5) for all $\varepsilon \leq \varepsilon_0$, which defines a linear operator of h . Besides, we have the estimate*

$$\|T(h)\|_{**} \leq C |\log \varepsilon| \|h\|_*. \quad (3.7)$$

The proof will be split into a serie of lemmas which we state and prove next. The first step is to obtain a priori estimates for the problem

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 0, \dots, 4, j = 1, \dots, m. \end{cases} \quad (3.8)$$

which involves more orthogonality conditions than those in (3.5). We have the following estimate.

Lemma 3.1. *There exist positive constants $\varepsilon_0 > 0$ and $C > 0$ such that for any ψ solution of Problem (3.8) with $h \in L^\infty(\Omega_\varepsilon)$, $\|h\|_* < \infty$, and $\xi \in \mathcal{O}$, then*

$$\|\psi\|_{**} \leq C \|h\|_* \quad (3.9)$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. We carry out the proof by a contradiction argument. If the above fact were false, then, there would exist a sequence $\varepsilon_n \rightarrow 0$, points $\xi^n = (\xi_1^n, \dots, \xi_m^n) \in \mathcal{O}$, functions h_n with $\|h_n\|_* \rightarrow 0$ and associated solutions ψ_n with $\|\psi_n\|_{**} = 1$ such that

$$\begin{cases} \mathcal{L}_{\varepsilon_n}(\psi_n) = h_n, & \text{in } \Omega_{\varepsilon_n}, \\ \psi_n = \Delta\psi_n = 0, & \text{on } \partial\Omega_{\varepsilon_n}, \\ \int_{\Omega_{\varepsilon_n}} \chi_j Z_{ij} \psi_n = 0, & \text{for all } i = 0, \dots, 4, j = 1, \dots, m. \end{cases} \quad (3.10)$$

Let us set $\tilde{\psi}_n(x) = \psi_n(x/\varepsilon_n)$, $x \in \Omega$. It is directly checked that for any $\delta' > 0$ sufficiently small $\tilde{\psi}_n$ solves the problem

$$\begin{cases} \Delta^2 \tilde{\psi}_n = O(\varepsilon_n^4) + \varepsilon_n^{-4} h_n = o(1), & \text{uniformly in } \Omega \setminus \cup_{k=1}^m B(\xi_j^n, \delta'), \\ \tilde{\psi}_n = \Delta\tilde{\psi}_n = 0, & \text{on } \partial\Omega, \end{cases}$$

together with $\|\tilde{\psi}_n\|_\infty \leq 1$ and $\|\Delta\tilde{\psi}_n\|_\infty \leq C\delta'$, in the considered region. Passing to a subsequence, we then get that $\xi^n \rightarrow \xi^* \in \mathcal{O}$ and $\tilde{\psi}_n \rightarrow 0$ in the $C^{3,\alpha}$ sense over compact subsets of $\Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$. In particular

$$\sum_{|\alpha| \leq 3} \frac{1}{\varepsilon_n^{|\alpha|}} |D^\alpha \psi_n(y)| \rightarrow 0, \quad \text{uniformly in } |y - (\xi_j^n)'| \geq \frac{\delta'}{2\varepsilon_n},$$

for any $\delta' > 0$ and $j \in \{1, \dots, m\}$. We obtain thus that

$$\sum_{j=1}^m \sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \psi_n\|_{L^\infty(r_j \geq \delta'/\varepsilon_n)} \rightarrow 0, \quad (3.11)$$

for any $\delta' > 0$. In conclusion, the *exterior portion* of $\|\psi_n\|_{**}$ goes to zero, see (3.6).

Let us consider now a smooth radial cut-off function $\hat{\eta}$ with $\hat{\eta}(s) = 1$ if $s < \frac{1}{2}$, $\hat{\eta}(s) = 0$ if $s \geq 1$, and define

$$\hat{\psi}_{n,j}(y) = \hat{\eta}_j(y) \psi_n(y) \equiv \hat{\eta}\left(\frac{\varepsilon_n}{\delta_0} |y - (\xi_j^n)'|\right) \psi_n(y),$$

such that

$$\text{supp } \hat{\psi}_{n,j} \subseteq B\left((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\right).$$

We observe that

$$\mathcal{L}_{\varepsilon_n}(\hat{\psi}_{n,j}) = \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n),$$

where

$$\begin{aligned} F(f, g) &= g \Delta^2 f + 2 \Delta f \Delta g + 4 \nabla(\Delta f) \cdot \nabla g + 4 \nabla f \cdot \nabla(\Delta g) \\ &\quad + 4 \sum_{i,j=1}^4 \frac{\partial^2 f}{\partial y_i \partial y_j} \frac{\partial^2 g}{\partial y_i \partial y_j}. \end{aligned} \quad (3.12)$$

Thus we get

$$\begin{cases} \Delta^2 \hat{\psi}_{n,j} = W_n(y) \hat{\psi}_{n,j} + \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n) & \text{in } B\left((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\right), \\ \hat{\psi}_{n,j} = \Delta \hat{\psi}_{n,j} = 0 & \text{on } \partial B\left((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\right). \end{cases} \quad (3.13)$$

The following intermediate result provides an outer estimate. For notational simplicity *we omit* the subscript n in the quantities involved.

Lemma 3.2. *There exist constants $C, R_0 > 0$ such that for large n*

$$\sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geq R_0)} \leq C \{\|\hat{\psi}_j\|_{L^\infty(r_j < 2R_0)} + o(1)\}. \quad (3.14)$$

Proof. We estimate the right-hand side of (3.13). If $2 < r_j < \delta_0/\varepsilon$ we get

$$\Delta^2 \hat{\psi}_j = O\left(\frac{1}{r_j^8}\right) \hat{\psi}_j + \frac{1}{r_j^7} o(1) + O(\varepsilon^4) + O\left(\frac{\varepsilon^3}{r_j}\right) + O\left(\frac{\varepsilon^2}{r_j^2}\right) + O\left(\frac{\varepsilon}{r_j^3}\right).$$

From (3.13) and standard elliptic estimates we have

$$\sum_{|\alpha| \leq 3} |D^\alpha \hat{\psi}_j| \leq C \left\{ \frac{1}{r_j^8} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} + \frac{1}{r_j^7} o(1) + O\left(\frac{\varepsilon}{r_j^3}\right) \right\}, \quad \text{in } 2 \leq r_j \leq \frac{\delta_0}{\varepsilon}.$$

Now, if $r_j \geq 2$

$$\|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\| \leq C \left\{ \frac{1}{r_j^5} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} + o(1) \right\}, \quad |\alpha| \leq 3.$$

Finally

$$\frac{1}{r_j^5} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} \leq \|\hat{\psi}_j\|_{L^\infty(1 < r_j < R_0)} + \frac{1}{R_0^5} \|\hat{\psi}_j\|_{L^\infty(r_j > R_0)},$$

thus fixing R_0 large enough we have

$$\sum_{|\alpha| \leq 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geq R_0)} \leq C \{\|\hat{\psi}_j\|_{L^\infty(1 < r_j < R_0)} + o(1)\}, \quad 2 < r_j < \frac{\delta_0}{\varepsilon},$$

and then (3.14). \square

We continue with the proof of Lemma 3.1.

Since $\|\psi_n\|_{**} = 1$ and using (3.11) and Lemma 3.2 we have that there exists an index $j \in \{1, \dots, m\}$ such that

$$\liminf_{n \rightarrow \infty} \|\psi_n\|_{L^\infty(r_j < R_0)} \geq \alpha > 0. \quad (3.15)$$

Let us set $\tilde{\psi}_n(z) = \psi_n((\xi_j^n)' + z)$. We notice that $\tilde{\psi}_n$ satisfies

$$\Delta^2 \tilde{\psi}_n - W((\xi_j^n)' + z) \tilde{\psi}_n = h_n((\xi_j^n)' + z), \quad \text{in } \Omega_n \equiv \Omega_\varepsilon - (\xi_j^n)'$$

Since $\psi_n, \Delta \psi_n$ are bounded uniformly, standard elliptic estimates allow us to assume that $\tilde{\psi}_n$ converges uniformly over compact subsets of \mathbb{R}^4 to a bounded, non-zero solution $\tilde{\psi}$ of

$$\Delta^2 \psi - \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4} \psi = 0.$$

This implies that $\tilde{\psi}$ is a linear combination of the functions $Y_{ij}, i = 0, \dots, 4$. But orthogonality conditions over $\tilde{\psi}_n$ pass to the limit thanks to $\|\tilde{\psi}_n\|_\infty \leq 1$ and dominated convergence. Thus $\tilde{\psi} \equiv 0$, a contradiction with (3.15). This concludes the proof. \square

Now we will deal with problem (3.8) lifting the orthogonality constraints $\int_{\Omega_\varepsilon} \chi_j Z_{0j} \psi = 0, \quad j = 1, \dots, m$, namely

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta \psi = 0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4 \quad j = 1, \dots, m. \end{cases} \quad (3.16)$$

We have the following a priori estimates for this problem.

Lemma 3.3. *There exist positive constants ε_0 and C such that, if ψ is a solution of (3.16), with $h \in L^\infty(\Omega_\varepsilon)$, $\|h\|_* < \infty$ and with $\xi \in \mathcal{O}$, then*

$$\|\psi\|_{**} \leq C |\log \varepsilon| \|h\|_* \quad (3.17)$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let $R > R_0 + 1$ be a large and fixed number. Let us consider \hat{Z}_{0j} be the following function

$$\hat{Z}_{0j}(y) = Z_{0j}(y) - 1 + a_{0j} G(\varepsilon y, \xi_j), \quad (3.18)$$

where $a_{0j} = (H(\xi_j, \xi_j) - 8 \log(\varepsilon R))^{-1}$. It is clear that if ε is small enough

$$\begin{aligned} \hat{Z}_{0j}(y) &= Z_{0j}(y) + a_{0j} (G(\varepsilon y, \xi_j) - H(\xi_j, \xi_j) + 8 \log(\varepsilon R)) \\ &= Z_{0j}(y) + \frac{1}{|\log \varepsilon|} \left(O(\varepsilon r_j) + 8 \log \frac{R}{r_j} \right). \end{aligned} \quad (3.19)$$

and $Z_{0j}(y) = O(1)$. Next we consider radial smooth cut-off functions η_1 and η_2 with the following properties:

$$\begin{aligned} 0 \leq \eta_1 \leq 1, \quad \eta_1 &\equiv 1 \text{ in } B(0, R), \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^4 \setminus B(0, R+1), \quad \text{and} \\ 0 \leq \eta_2 \leq 1, \quad \eta_2 &\equiv 1 \text{ in } B\left(0, \frac{\delta_0}{3\varepsilon}\right), \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^4 \setminus B\left(0, \frac{\delta_0}{2\varepsilon}\right). \end{aligned}$$

Then we set

$$\eta_{1j}(y) = \eta_1(r_j), \quad \eta_{2j}(y) = \eta_2(r_j), \quad (3.20)$$

and define the test function

$$\tilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1 - \eta_{1j})\eta_{2j}\hat{Z}_{0j}.$$

Note the \tilde{Z}_{0j} 's behavior throught Ω_ε

$$\tilde{Z}_{0j} = \begin{cases} Z_{0j}, & r_j \leq R \\ \eta_{1j}(Z_{0j} - \hat{Z}_{0j}) + \hat{Z}_{0j}, & R < r_j \leq R+1 \\ \hat{Z}_{0j}, & R+1 < r_j \leq \frac{\delta_0}{3\varepsilon} \\ \eta_{2j}\hat{Z}_{0j}, & \frac{\delta_0}{3\varepsilon} < r_j \leq \frac{\delta_0}{2\varepsilon} \\ 0 & \text{otherwise.} \end{cases}$$

In the subsequent, we will label these four regions as

$$\begin{aligned} \Omega_0 &\equiv \{r_j \leq R\}, \quad \Omega_1 \equiv \{R < r_j \leq R+1\}, \quad \Omega_2 \equiv \left\{R+1 < r_j \leq \frac{\delta_0}{3\varepsilon}\right\}, \\ \text{and} \quad \Omega_3 &\equiv \left\{\frac{\delta_0}{3\varepsilon} < r_j \leq \frac{\delta_0}{2\varepsilon}\right\}. \end{aligned}$$

Let ψ be a solution to problem (3.16). We will modify ψ so that the extra orthogonality conditions with respect to Z_{0j} 's hold. We set

$$\tilde{\psi} = \psi + \sum_{j=1}^m d_j \tilde{Z}_{0j}. \quad (3.21)$$

We adjust the constants d_j so that

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \tilde{\psi} = 0, \quad \text{for all } i = 0, \dots, 4; j = 1, \dots, m. \quad (3.22)$$

Then,

$$\mathcal{L}_\varepsilon(\tilde{\psi}) = h + \sum_{j=1}^m d_j \mathcal{L}_\varepsilon(\tilde{Z}_{0j}). \quad (3.23)$$

If (3.22) holds, the previous lemma allows us to conclude

$$\|\tilde{\psi}\|_{**} \leq C \left\{ \|h\|_* + \sum_{j=1}^m |d_j| \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \right\}. \quad (3.24)$$

Estimate (3.17) is a direct consequence of the following claim:

Claim 1. *The constants d_j are well defined,*

$$|d_j| \leq C |\log \varepsilon| \|h\|_* \quad \text{and} \quad \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \leq \frac{C}{|\log \varepsilon|}, \quad \text{for all } j = 1, \dots, m. \quad (3.25)$$

After these facts have been established, using the fact that

$$\|\tilde{Z}_{0j}\|_{**} \leq C,$$

we obtain (3.17), as desired.

Let us prove now Claim 1. First we find d_j . From definition (3.21), orthogonality conditions (3.22) and the fact that $\text{supp } \chi_j \eta_{1k} = \emptyset$ and $\text{supp } \chi_j \eta_{2k} = \emptyset$ if $j \neq k$, we can write

$$d_j \int_{\Omega_\varepsilon} \chi_j Z_{0j}^2 = - \int_{\Omega_\varepsilon} \chi_j Z_{0j} \psi, \quad \forall j = 1, \dots, m. \quad (3.26)$$

Thus d_j is well defined. Note that the orthogonality conditions in (3.22) for $i = 1, \dots, 4$ are also satisfied for $\tilde{\psi}$ thanks to the fact that $R > R_0 + 1$.

We prove now the second inequality in (3.25). From (??), (3.18) and estimate (2.22) we obtain,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right), & \text{in } \Omega_0 \\ \eta_{1j} \mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) + \mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}), & \text{in } \Omega_1 \\ \mathcal{L}_\varepsilon(\hat{Z}_{0j}), & \text{in } \Omega_2 \\ \eta_{2j} \mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{2j}, \hat{Z}_{0j}), & \text{in } \Omega_3, \end{cases} \quad (3.27)$$

and where F was defined in (3.12). We compute now $\mathcal{L}_\varepsilon(\tilde{Z}_{0j})$ in Ω_i , $i = 1, 2, 3$. In Ω_1 , thanks to (3.19) (we consider R here because we will need this dependence below to prove estimate (3.37))

$$|Z_{0j} - \hat{Z}_{0j}|, |R \nabla(Z_{0j} - \hat{Z}_{0j})| \text{ and } |R^2 \Delta(Z_{0j} - \hat{Z}_{0j})| = O\left(\frac{1}{|\log \varepsilon|}\right); \quad (3.28)$$

moreover

$$|R \nabla(\Delta(Z_{0j} - \hat{Z}_{0j}))| \text{ and } |\Delta^2(Z_{0j} - \hat{Z}_{0j})| = O\left(\frac{1}{R^2 |\log \varepsilon|}\right), \quad (3.29)$$

Thus, using (3.12) and the fact that, in Ω_1 , $|D^\alpha \eta_{1j}| \leq CR^{-|\alpha|}$, for any multi-index $|\alpha| \leq 4$,

$$F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right).$$

On the other hand,

$$\mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right), \quad (3.30)$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = O(\varepsilon R) + O\left(\frac{1}{R^4 |\log \varepsilon|}\right). \quad (3.31)$$

In conclusion, if $y \in \Omega_1$,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j})(y) = O\left(\frac{1}{R^4 |\log \varepsilon|}\right). \quad (3.32)$$

In Ω_2 ,

$$\begin{aligned} W(1 - a_{0j}G(\varepsilon y, \xi_j)) &= O\left(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^4} \left\{ H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8 \log \frac{r_j}{R} \right\}\right) \\ &= O\left(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^{7/2}} \frac{\log r_j}{(\mu_j^2 + r_j^2)^{1/2}}\right) \\ &= O\left(\frac{1}{|\log \varepsilon|} \frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}}\right), \end{aligned}$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right).$$

Thus, in this region

$$\mathcal{L}(\tilde{Z}_{0j}) = O\left(\frac{\mu_j^4 |\log \varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}\right) \quad (3.33)$$

In Ω_3 , thanks to (3.18), $|\hat{Z}_{0j}| = O\left(\frac{1}{|\log \varepsilon|}\right)$, $|\nabla \hat{Z}_{0j}| = O\left(\frac{\varepsilon}{|\log \varepsilon|}\right)$,
 $|\Delta \hat{Z}_{0j}| = O\left(\frac{\varepsilon^2}{|\log \varepsilon|}\right)$, $|\nabla(\Delta \hat{Z}_{0j})| = O\left(\frac{\varepsilon^3}{|\log \varepsilon|}\right)$ and $|\Delta^2 \hat{Z}_{0j}| = O\left(\frac{\varepsilon^4}{|\log \varepsilon|}\right)$. Thus,

$$F(\eta_{2j}, \hat{Z}_{0j}) = O\left(\frac{\varepsilon^4}{|\log \varepsilon|}\right).$$

Finally,

$$\begin{aligned} \mathcal{L}_\varepsilon(\hat{Z}_{0j}) &= \mathcal{L}_\varepsilon(Z_{0j}) + W a_{0j} \left(H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8 \log \frac{r_j}{R} \right) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) + O\left(\frac{\mu_j^4}{(\mu_j^2 + r_j^2)^4}\right) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) \end{aligned}$$

and then, combining (3.32), (3.33) and the previous estimate, we can again write the estimate (3.27):

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right), & \text{in } \Omega_0 \\ O\left(\frac{1}{|\log \varepsilon|}\right), & \text{in } \Omega_1 \\ O\left(\frac{\mu_j^4 |\log \varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}\right), & \text{in } \Omega_2 \\ O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right), & \text{in } \Omega_3. \end{cases} \quad (3.34)$$

In conclusion,

$$\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* = O\left(\frac{1}{|\log \varepsilon|}\right). \quad (3.35)$$

Finally, we prove the bounds of d_j . Testing equation (3.23) against \tilde{Z}_{0j} and using relations (3.24) and the above estimate, we get

$$\begin{aligned} |d_j| \left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| &= \left| \int_{\Omega_\varepsilon} h \tilde{Z}_{0j} + \int_{\Omega_\varepsilon} \tilde{\psi} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \right| \\ &\leq C \|h\|_* + C \|\tilde{\psi}\|_\infty \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \\ &\leq C \|h\|_* \left\{ 1 + \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \right\} + C \sum_{k=1}^m |d_k| \|\mathcal{L}_\varepsilon(\tilde{Z}_{0k})\|_* \|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\|_* \end{aligned}$$

where we have used that

$$\int_{\Omega_\varepsilon} \frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}} \leq C \quad \text{for all } j.$$

But estimate (3.35) imply

$$|d_j| \left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| \leq C \|h\|_* + C \sum_{k=1}^m \frac{|d_k|}{|\log \varepsilon|^2}. \quad (3.36)$$

It only remains to estimate the integral term of the left side. For this purpose, we have the following

Claim 2. *If R is sufficiently large,*

$$\left| \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \right| = \frac{E}{|\log \varepsilon|} (1 + o(1)), \quad (3.37)$$

where E is a positive constant independent of ε and R .

Assume for the moment the validity of this claim. We replace (3.37) in (3.36), we get

$$|d_j| \leq C |\log \varepsilon| \|h\|_* + C \sum_{k=1}^m \frac{|d_k|}{|\log \varepsilon|}, \quad (3.38)$$

and then,

$$|d_j| \leq C |\log \varepsilon| \|h\|_*.$$

Claim 1 is thus proven. Let us proof Claim 2. We decompose

$$\begin{aligned} \int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} &= O(\varepsilon) + \int_{\Omega_1} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} + \int_{\Omega_2} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} + \int_{\Omega_3} \mathcal{L}_\varepsilon(\tilde{Z}_{0j}) \tilde{Z}_{0j} \\ &\equiv O(\varepsilon) + I_1 + I_2 + I_3. \end{aligned}$$

First we estimate I_2 . From (3.34),

$$\begin{aligned} I_2 &= O \left(\frac{1}{|\log \varepsilon|} \int_{\Omega_2} \frac{\mu_j^4 \hat{Z}_{0j}}{(\mu_j^2 + r_j^2)^{7/2}} \right) \\ &= O \left(\frac{1}{R^3 |\log \varepsilon|} \right). \end{aligned}$$

Now we estimate I_3 . From the estimates in Ω_3 , $|I_3| = O\left(\frac{\varepsilon^4}{|\log \varepsilon|}\right)$. On the other hand, since (3.32) holds true and $\hat{Z}_{0j} = Z_{0j}\left(1 + O\left(\frac{1}{R|\log \varepsilon|}\right)\right)$, we conclude

$$\begin{aligned} |I_1| &= \frac{1}{R^4|\log \varepsilon|} \int_{R < r_j \leq R+1} \hat{Z}_{0j}(y) dy \\ &= \frac{1}{R^4|\log \varepsilon|} \int_{R < r_j \leq R+1} \left\{ O\left(\frac{1}{R|\log \varepsilon|}\right) + \hat{Z}_{0j}(y) \right\} dy \\ &= \frac{1}{R^5|\log \varepsilon|^2} + \frac{|S^3|}{R^4|\log \varepsilon|} \int_R^{R+1} r^3 \left(\frac{r^2 - \mu_j^2}{\mu_j^2 + r^2}\right) (1 + o(1)) dr \\ &= \frac{E}{|\log \varepsilon|} (1 + o(1)), \end{aligned}$$

where E is a positive constant independent of ε and R . Thus, for fixed R large and ε small, we obtain (3.37). \square

Now we can try with the original linear problem (3.5).

Proof of Proposition 3.1. We first establish the validity of the a priori estimate (3.7) for solutions ψ of problem (3.5), with $h \in L^\infty(\Omega_\varepsilon)$ and $\|h\|_* < \infty$. Lemma 3.3 implies

$$\|\psi\|_{**} \leq C|\log \varepsilon| \left\{ \|h\|_* + \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}| \|\chi_j Z_{ij}\|_* \right\}. \quad (3.39)$$

On the other hand,

$$\|\chi_j Z_{ij}\|_* \leq C,$$

then, it is sufficient to estimate the values of the constants c_{ij} . To this end, we multiply the first equation in (3.5) by $Z_{ij}\eta_{2j}$, with η_{2j} the cut-off function introduced in (3.20), and integrate by parts to find

$$\int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(Z_{ij}\eta_{2j}) = \int_{\Omega_\varepsilon} h Z_{ij}\eta_{2j} + c_{ij} \int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2, \quad (3.40)$$

It is easy to see that $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij} h = O(\|h\|_*)$ and $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2 = C > 0$. On the other hand we have

$$\begin{aligned} \mathcal{L}_\varepsilon(\eta_{2j} Z_{ij}) &= \eta_{2j} \mathcal{L}_\varepsilon(Z_{ij}) + F(\eta_{2j}, Z_{ij}) \\ &= O\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) \eta_{2j} |Z_{ij}| + F(\eta_{2j}, Z_{ij}). \end{aligned}$$

Directly from (3.12) we get

$$\begin{aligned} F(\eta_{2j}, Z_{ij}) &= O\left(\frac{\varepsilon^4}{(\mu_j^2 + r_j^2)^{1/2}}\right) + O\left(\frac{\varepsilon^3}{\mu_j^2 + r_j^2}\right) \\ &\quad + O\left(\frac{\varepsilon^2}{(\mu_j^2 + r_j^2)^{3/2}}\right) + O\left(\frac{\varepsilon}{(\mu_j^2 + r_j^2)^2}\right), \end{aligned}$$

in the region $\frac{\delta_0}{3\varepsilon} \leq r_j \leq \frac{\delta_0}{2\varepsilon}$. Thus

$$\|\mathcal{L}_\varepsilon(\eta_{2j} Z_{ij})\|_* = O(\varepsilon) \quad \text{and} \quad (3.41)$$

$$\left| \int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(\eta_{2j} Z_{ij}) \right| \leq C\varepsilon |\log \varepsilon| \|\psi\|_\infty \leq C\varepsilon |\log \varepsilon| \|\psi\|_{**}.$$

Using the above estimates in (3.40), we obtain

$$|c_{ij}| \leq C\{\varepsilon |\log \varepsilon| \|\psi\|_{**} + \|h\|_*\}, \quad (3.42)$$

and then

$$|c_{ij}| \leq C\left\{ (1 + \varepsilon |\log \varepsilon|^2) \|h\|_* + \varepsilon |\log \varepsilon|^2 \sum_{l,k} |c_{lk}| \right\}.$$

Then $|c_{ij}| \leq C\|h\|_*$ and putting this estimate in (3.39), we conclude the validity of (3.17).

We now prove the solvability assertion. To this purpose we consider the space

$$\mathcal{H} = \left\{ \psi \in H^3(\Omega_\varepsilon) : \psi = \Delta\psi = 0 \text{ on } \partial\Omega_\varepsilon, \text{ and such that} \right. \\ \left. \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, \quad \text{for all } i = 1, \dots, 4; j = 1, \dots, m \right\},$$

endowed with the usual inner product $(\psi, \varphi) = \int_{\Omega_\varepsilon} \Delta\psi \Delta\varphi$. Problem (3.16) expressed in a weak form is equivalent to that of finding a $\psi \in \mathcal{H}$, such that

$$(\psi, \varphi) = \int_{\Omega_\varepsilon} \{h + W\psi\} \varphi, \quad \text{for all } \varphi \in \mathcal{H}.$$

With the aid of Riesz's representation Theorem, this equation can be rewritten in \mathcal{H} in the operator form $\psi = K(W\psi + h)$, where K is a compact operator in \mathcal{H} . Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\psi = K(W\psi)$ has only the zero solution in \mathcal{H} . This last equation is equivalent to (3.16) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (3.17). This concludes the proof. \square

The result of Proposition 3.1 implies that the unique solution $\psi = T(h)$ of (3.5) defines a continuous linear map from the Banach space \mathcal{C}_* of all functions $h \in L^\infty(\Omega_\varepsilon)$ with $\|h\|_* < +\infty$, into $W^{3,\infty}(\Omega_\varepsilon)$, with norm bounded uniformly in ε .

Remark 3.1. *The operator T is differentiable with respect to the variables ξ^l . In fact, computations similar to those used in [14] yield the estimate*

$$\|\partial_{\xi^l} T(h)\|_{**} \leq C |\log \varepsilon|^2 \|h\|_*, \quad \text{for all } l = 1, 2; k = 1, \dots, m. \quad (3.43)$$

4. THE INTERMEDIATE NONLINEAR PROBLEM

In order to solve Problem (2.20) we consider first the intermediate nonlinear problem.

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = -R + N(\psi) + \sum_{i=1}^4 \sum_{j=1}^m c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \psi = 0, & \text{for all } i = 1, \dots, 4 \ j = 1, \dots, m. \end{cases} \quad (4.1)$$

For this problem we will prove

Proposition 4.1. *Let $\xi \in \mathcal{O}$. Then, there exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the nonlinear problem (4.1) has a unique solution ψ which satisfies*

$$\|\psi\|_{**} \leq C\varepsilon|\log \varepsilon|. \quad (4.2)$$

Moreover, if we consider the map $\xi' \in \mathcal{O} \rightarrow \psi \in \mathcal{C}^{4,\alpha}(\overline{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\psi$ exists and defines a continuous map of ξ' . Besides

$$\|D_{\xi'}\psi\|_{**} \leq C\varepsilon|\log \varepsilon|^2. \quad (4.3)$$

Proof. In terms of the operator T defined in Proposition 3.1, Problem (4.1) becomes

$$\psi = \mathcal{B}(\psi) \equiv T(N(\psi) - R).$$

Let us consider the region

$$\mathcal{F} \equiv \{\psi \in \mathcal{C}^{4,\alpha}(\overline{\Omega}_\varepsilon) : \|\psi\|_{**} \leq \varepsilon|\log \varepsilon|\}.$$

>From Proposition 3.1,

$$\|\mathcal{B}(\psi)\|_{**} \leq C|\log \varepsilon| \{\|N(\psi)\|_* + \|R\|_*\},$$

and Lemma 2.2 implies

$$\|R\|_* \leq C\varepsilon.$$

Also, from Lemma 2.4

$$\|N(\psi)\|_* \leq C\|\psi\|_\infty^2 \leq C\|\psi\|_{**}^2.$$

Hence, if $\psi \in \mathcal{F}$, $\|\mathcal{B}(\psi)\|_{**} \leq C\varepsilon|\log \varepsilon|$. Along the same way we obtain $\|N(\psi_1) - N(\psi_2)\|_* \leq C \max_{i=1,2} \|\psi_i\|_\infty \|\psi_1 - \psi_2\|_\infty \leq C \max_{i=1,2} \|\psi_i\|_{**} \|\psi_1 - \psi_2\|_{**}$, for any $\psi_1, \psi_2 \in \mathcal{F}$. Then, we conclude

$$\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\|_{**} \leq C|\log \varepsilon| \|N(\psi_1) - N(\psi_2)\|_* \leq C\varepsilon|\log \varepsilon|^2 \|\psi_1 - \psi_2\|_{**}$$

It follows that for all ε small enough \mathcal{B} is a contraction mapping of \mathcal{F} , and therefore a unique fixed point of \mathcal{B} exists in this region. The proof of (4.3) is similar to one included in [14] and we thus omit it. \square

5. VARIATIONAL REDUCTION

We have solved the nonlinear problem (4.1). In order to find a solution to the original problem (2.20) we need to find ξ such that

$$c_{ij} = c_{ij}(\xi') = 0, \quad \text{for all } i, j. \quad (5.1)$$

where $c_{ij}(\xi')$ are the constants in (4.1). Problem (5.1) is indeed variational: it is equivalent to finding critical points of a function of ξ' . In fact, we define the function for $\xi \in \mathcal{O}$

$$\mathcal{F}_\varepsilon(\xi) \equiv J_\rho[U(\xi) + \hat{\psi}_\xi] \quad (5.2)$$

where J_ρ is defined in (2.16), ρ is given by (2.4), $U = U(\xi)$ is our approximate solution from (2.5) and $\hat{\psi}_\xi = \psi\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)$, $x \in \Omega$, with $\psi = \psi_{\xi'}$ the unique solution to problem (4.1) given by Proposition 4.1. Then we obtain that critical points of \mathcal{F} correspond to solutions of (5.1) for small ε . That is,

Lemma 5.1. $\mathcal{F}_\varepsilon : \mathcal{O} \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 . Moreover, for all ε small enough, if $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$ then ξ satisfies (5.1).

Proof. We define

$$I_\varepsilon[v] \equiv \frac{1}{2} \int_{\Omega_\varepsilon} (\Delta v)^2 - \int_{\Omega_\varepsilon} k(\varepsilon y) e^v.$$

Let us differentiate the function \mathcal{F}_ε with respect to ξ . Since $J_\rho[U(\xi) + \hat{\psi}_\xi] = I_\varepsilon[V(\xi') + \psi_{\xi'}]$, we can differentiate directly under the integral sign, so that

$$\begin{aligned} \partial_{(\xi'_k)_l} \mathcal{F}_\varepsilon(\xi) &= \varepsilon^{-1} D I_\varepsilon[V + \psi] \left(\partial_{(\xi'_k)_l} V + \partial_{(\xi'_k)_l} \psi \right) \\ &= \varepsilon^{-1} \sum_{i=1}^4 \sum_{j=1}^m \int_{\Omega_\varepsilon} c_{ij} \chi_j Z_{ij} \left(\partial_{(\xi'_k)_l} V + \partial_{(\xi'_k)_l} \psi \right). \end{aligned}$$

From the results of the previous section, this expression defines a continuous function of ξ' , and hence of ξ . Let us assume that $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$. Then

$$\sum_{i=1}^4 \sum_{j=1}^m \int_{\Omega_\varepsilon} c_{ij} \chi_j Z_{ij} \left(\partial_{(\xi'_k)_l} V + \partial_{(\xi'_k)_l} \psi \right) = 0, \quad \text{for } k = 1, 2, 3, 4; \quad l = 1, \dots, m.$$

Since $\|D_{\xi'} \psi_{\xi'}\| \leq C\varepsilon |\log \varepsilon|^2$, we have

$$\partial_{(\xi'_k)_l} V + \partial_{(\xi'_k)_l} \psi = Z_{kl} + o(1),$$

where $o(1)$ is uniformly small as $\varepsilon \rightarrow 0$. Thus, we have the following linear system of equation

$$\sum_{i=1}^4 \sum_{j=1}^m c_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{ij} (Z_{kl} + o(1)) = 0, \quad \text{for } k = 1, 2, 3, 4; \quad l = 1, \dots, m.$$

This system is dominant diagonal, thus $c_{ij} = 0$ for all i, j . This concludes the proof. \square

We also have the validity of the following Lemma

Lemma 5.2. *Let ρ be given by (2.4). For points $\xi \in \mathcal{O}$ the following expansion holds*

$$\mathcal{F}_\varepsilon(\xi) = J_\rho[U(\xi)] + \theta_\varepsilon(\xi), \quad (5.3)$$

where $|\theta_\varepsilon| + |\nabla \theta_\varepsilon| = o(1)$, uniformly on $\xi \in \mathcal{O}$ as $\varepsilon \rightarrow 0$.

Proof. The proof follows directly from an application of Taylor expansion for \mathcal{F}_ε in the expanded domain Ω_ε and from the estimates for the solution $\psi_{\xi'}$ to Problem (4.1) obtained in Proposition 4.1. \square

6. PROOF OF THE THEOREMS

In this section we carry out the proofs of our main results.

6.1. Proof of Theorem 1. Taking into account the result of Lemma 5.1, a solution to Problem (1.1) exists if we prove the existence of a critical point of \mathcal{F}_ε , which automatically implies that $c_{ij} = 0$ in (2.20) for all i, j . The qualitative properties of the solution found follow from the ansatz.

Finding critical points of $\mathcal{F}_\varepsilon(\xi)$ is equivalent to finding critical points of

$$\tilde{\mathcal{F}}_\varepsilon(\xi) = \mathcal{F}_\varepsilon(\xi) - 256 \pi^2 m |\log \varepsilon|. \quad (6.1)$$

On the other hand, if $\xi \in \mathcal{O}$, from Lemmas 2.3 and 5.2 we get the existence of constants $\alpha > 0$ and β such that

$$\alpha \tilde{\mathcal{F}}_\varepsilon(\xi) + \beta = \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \quad (6.2)$$

with Θ_ε and $\nabla_\xi \Theta_\varepsilon$ uniformly bounded in the considered region as $\varepsilon \rightarrow 0$.

We shall prove that, under the assumptions of Theorems 1 and 2, $\tilde{\mathcal{F}}_\varepsilon$ has a critical point in \mathcal{O} for ε small enough. We start with a topological lemma. We denote by D the diagonal

$$D := \{\xi \in \Omega^m : \xi_i = \xi_j \text{ for some } i \neq j\},$$

and we write $H^* := H^*(\cdot; \mathbb{K})$ for singular cohomology with coefficients in a field \mathbb{K} .

Lemma 6.1. *If $H^d(\Omega) \neq 0$ for some $d \geq 1$, and $H^j(\Omega) = 0$ for $j > d$, then the homomorphism*

$$H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m),$$

induced by the inclusion of pairs $(\Omega^m, \emptyset) \hookrightarrow (\Omega^m, D)$, is an epimorphism. In particular, $H^{md}(\Omega^m, D) \neq 0$.

Proof. Let us prove first that $H^j(D) = 0$ if $j > (m-1)d$. For this purpose we write

$$D = \bigcup_{1 \leq i < j \leq m} X_{i,j}, \quad \text{where } X_{i,j} := \{(x_1, \dots, x_m) \in \Omega^m : x_i = x_j\},$$

and consider the sets $\mathcal{F}_0 := \{\Omega^m\}$, $\mathcal{F}_1 := \{X_{i,j} : 1 \leq i < j \leq m\}$, and

$$\mathcal{F}_{k+1} := \{Z \cap Z' : Z, Z' \in \mathcal{F}_k \text{ and } Z \neq Z'\}, \quad k = 1, \dots, m-2.$$

Note that

$$Z \cong \Omega^{m-k'} \text{ for some } k \leq k' \leq m-1 \text{ if } Z \in \mathcal{F}_k, \quad k = 0, \dots, m-1,$$

where \cong means that the sets are homeomorphic. Künneth's formula

$$H^j(\Omega^{m-k}) = \bigoplus_{p+q=j} (H^p(\Omega) \otimes H^q(\Omega^{m-k-1})) \quad (6.3)$$

(see, for example, [17, Proposition 8.18]) yields inductively that, for $0 \leq k \leq m-1$,

$$H^j(Z) = 0 \quad \text{if } Z \in \mathcal{F}_k \text{ and } j > (m-k)d. \quad (6.4)$$

We claim that, for each $0 \leq k \leq m-1$, one has that

$$H^j(Z_1 \cup \dots \cup Z_\ell) = 0 \quad \text{if } Z_1, \dots, Z_\ell \in \mathcal{F}_k \text{ and } j > (m-k)d. \quad (6.5)$$

Let us prove this claim. Since \mathcal{F}_{m-1} has only one element and (6.4) holds, we have that the claim is true for $k = m-1$. Assume that the claim is true for $k+1$ with $k+1 \leq m-1$ and let us then prove it for k . We do this by induction on ℓ . If $\ell = 1$ the assertion reduces to (6.4). Now assume that the assertion is true for every union of at most $\ell-1$ sets in \mathcal{F}_k , and let $Z_1, \dots, Z_\ell \in \mathcal{F}_k$ be pairwise distinct sets. Consider the Mayer-Vietoris sequence

$$\dots \rightarrow H^{j-1} \left(\bigcup_{i=1}^{\ell-1} (Z_i \cap Z_\ell) \right) \rightarrow H^j(Z_1 \cup \dots \cup Z_\ell) \rightarrow H^j(Z_1 \cup \dots \cup Z_{\ell-1}) \oplus H^j(Z_\ell) \rightarrow \dots \quad (6.6)$$

Our induction hypothesis on ℓ yields that $H^j(Z_1 \cup \dots \cup Z_{\ell-1}) = 0$ and $H^j(Z_\ell) = 0$ if $j > (m-k)d$. Since Z_1, \dots, Z_ℓ are pairwise distinct, we have that $Z_i \cap Z_\ell \in \mathcal{F}_{k+1}$

for each $i = 1, \dots, \ell - 1$ and, since we are assuming that the claim is true for $k + 1$ we have that

$$H^{j-1} \left(\bigcup_{i=1}^{\ell-1} (Z_i \cap Z_\ell) \right) = 0 \quad \text{if } j - 1 > (m - (k + 1))d.$$

Note that $j > (m - k)d$ implies $j - 1 > (m - (k + 1))d$. This proves that both ends of the exact sequence (6.6) are zero if $j > (m - k)d$, hence the middle term is also zero in this case. This concludes the proof of claim (6.5).

Now, since $D = \bigcup_{Y \in \mathcal{F}_1} Y$, assertion (6.5) with $k = 1$ yields that $H^j(D) = 0$ if $j > (m - 1)d$. So the exact cohomology sequence

$$H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m) \longrightarrow H^{md}(D) = 0$$

gives that $H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m)$ is an epimorphism. But (6.3) implies that $H^{md}(\Omega^m) \neq 0$. Therefore, $H^{md}(\Omega^m, D) \neq 0$, as claimed. \square

For each positive number δ define

$$\begin{aligned} \Omega_\delta &:= \{\xi \in \Omega : \text{dist}(\xi, \partial\Omega) > \delta\} \\ \mathfrak{D}_\delta &:= \{\xi = (\xi_1, \dots, \xi_m) \in \Omega^m : \xi_j \in \Omega_\delta\}. \end{aligned}$$

Lemma 6.2. *Given $K > 0$ there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$, the following holds: For every $\xi = (\xi_1, \dots, \xi_m) \in \partial\mathfrak{D}_\delta$ with $|\varphi_m(\xi)| \leq K$ there exists an $i \in \{1, \dots, m\}$ such that*

$$\begin{aligned} \nabla_{\xi_i} \varphi_m(\xi) &\neq 0 && \text{if } \xi_i \in \Omega_\delta \\ \nabla_{\xi_i} \varphi_m(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial\Omega_\delta) && \text{if } \xi_i \in \partial\Omega_\delta \end{aligned}$$

where $T_{\xi_i}(\partial\Omega_\delta)$ denotes the tangent space to $\partial\Omega_\delta$ at the point ξ_i .

Proof. We first need to establish some facts related to the regular part of the Green function on the half hyperplane

$$\mathcal{H} := \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \geq 0\}.$$

It is well known that the regular part of the Green function on \mathcal{H} is given by

$$H(x, y) = 8 \log|x - \bar{y}|, \quad \bar{y} = (y_1, y_2, y_3, -y_4),$$

for $x, y \in \mathcal{H}$ and the Green function is

$$G(x, y) = -8 \log|x - y| + 8 \log|x - \bar{y}|.$$

Consider the function of $k \geq 2$ distinct points of \mathcal{H}

$$\Psi_k(x_1, \dots, x_k) := -8 \sum_{i \neq j} \log|x_i - x_j|,$$

and denote by I_+ and I_0 the set of indices i for which $(x_i)_4 > 0$ and $(x_i)_4 = 0$, respectively. Define also

$$\varphi_{k, \mathcal{H}}(x_1, \dots, x_k) = -8 \sum_{j=1}^k \log|x_j - \bar{x}_j| + 8 \sum_{i \neq j} \log \frac{|x_i - x_j|}{|x_i - \bar{x}_j|}.$$

Claim 3. *We have the following alternative: Either*

$$\nabla_{x_i} \Psi_k(x_1, \dots, x_k) \neq 0 \quad \text{for some } i \in I_+,$$

or

$$\partial_{(x_i)_j} \Psi_k(x_1, \dots, x_k) \neq 0 \quad \text{for some } i \in I_0 \text{ and } j \in \{1, 2, 3\},$$

where $\partial_{(x_i)_j} \equiv \frac{\partial}{\partial(x_i)_j}$.

Proof. We have that

$$\frac{\partial}{\partial\lambda} \Psi_k(\lambda x_1, \dots, \lambda x_k) \Big|_{\lambda=1} = \sum_{i \in I_+} \nabla_{x_i} \Psi_k(x_1, \dots, x_k) \cdot x_i + \sum_{i \in I_0} \nabla_{x_i} \Psi_k(x_1, \dots, x_k) \cdot x_i.$$

On the other hand

$$\frac{\partial}{\partial\lambda} \Psi_k(\lambda x_1, \dots, \lambda x_k) \Big|_{\lambda=1} = -8k(k-1) \neq 0,$$

and Claim 3 follows.

Claim 4. For any k distinct points $x_i \in \text{Int } \mathcal{H}$ we have $\nabla \varphi_{k, \mathcal{H}}(x_1, \dots, x_k) \neq 0$.

Proof. We have that

$$\frac{\partial}{\partial\lambda} \varphi_{k, \mathcal{H}}(\lambda x_1, \dots, \lambda x_k) \Big|_{\lambda=1} = \sum_{i=1}^k \nabla_{x_i} \varphi_{k, \mathcal{H}}(x_1, \dots, x_k) \cdot x_i,$$

On the other hand

$$\frac{\partial}{\partial\lambda} \varphi_{k, \mathcal{H}}(\lambda x_1, \dots, \lambda x_k) \Big|_{\lambda=1} = -8k(k-1) \neq 0,$$

and Claim 4 follows.

Now we will need an estimate for the regular part $H(x, y)$ of the Green's function for points x, y close to $\partial\Omega$.

Claim 5. There exists $C_1, C_2 > 0$ constants such that for any $x, y \in \Omega$

$$|\nabla_x H(x, y)| + |\nabla_y H(x, y)| \leq C_1 \min \left\{ \frac{1}{|x-y|}, \frac{1}{\text{dist}(y, \partial\Omega)} \right\} + C_2.$$

Proof. For $y \in \Omega$ a point close to $\partial\Omega$ we denote by \bar{y} its uniquely determined reflection with respect to $\partial\Omega$. Define $\psi(x, y) = H(x, y) + 8 \log \frac{1}{|x-\bar{y}|}$. It is straightforward to see that ψ is bounded in $\bar{\Omega} \times \bar{\Omega}$ and that $|\nabla_x \psi(x, y)| + |\nabla_y \psi(x, y)| \leq C$ for some positive constant C . Claim 5 follows.

We have now all elements to prove Lemma 6.2. Assume, by contradiction, that for some sequence $\delta_n \rightarrow 0$ there are points $\xi^n \in \partial\mathcal{D}_{\delta_n}$, such that $|\varphi_m(\xi^n)| \leq K$ and, for every $i \in \{1, \dots, m\}$,

$$\nabla_{\xi_i^n} \varphi_m(\xi^n) = 0 \quad \text{if } \xi_i^n \in \Omega_{\delta_n}, \quad (6.7)$$

and

$$\nabla_{\xi_i^n} \varphi_m(\xi^n) \cdot \tau = 0 \quad \text{if } \xi_i^n \in \partial\Omega_{\delta_n}, \quad (6.8)$$

for any vector τ tangent to $\partial\Omega_{\delta_n}$ at ξ_i^n . It follows that there exists a point $\xi_l^n \in \partial\Omega_{\delta_n}$ such that $H(\xi_l^n, \xi_l^n) \rightarrow -\infty$ as $n \rightarrow \infty$. Since $|\varphi_m(\xi^n)| \leq K$, there are necessarily two distinct points ξ_i^n and ξ_j^n coming closer to each other, that is,

$$\rho_n := \inf_{i \neq j} |\xi_i^n - \xi_j^n| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Without loss of generality we can assume $\rho_n = |\xi_1^n - \xi_2^n|$. We define $x_j^n := \frac{\xi_j^n - \xi_1^n}{\rho_n}$.

Thus, up to a subsequence, there exists a k , $2 \leq k \leq m$, such that

$$\lim_{n \rightarrow \infty} |x_j^n| < +\infty, \quad j = 1, \dots, k, \quad \text{and} \quad \lim_{n \rightarrow \infty} |x_j^n| = +\infty, \quad j > k.$$

For $j \leq k$ we set

$$\bar{x}_j = \lim_{n \rightarrow \infty} x_j^n.$$

We consider two cases:

(1) Either

$$\frac{\text{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \rightarrow +\infty,$$

(2) or there exists $C_0 < +\infty$ such that for almost all n we have

$$\frac{\text{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} < C_0.$$

Case 1. It is easy to see that in this case we actually have

$$\frac{\text{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \rightarrow +\infty, \quad j = 1, \dots, k.$$

Furthermore, the points ξ_1^n, \dots, ξ_k^n are all in the interior of Ω_{δ_n} , hence (6.7) is satisfied for all partial derivatives ∇_{ξ_j} , $j \leq k$. Define

$$\tilde{\varphi}_m(x_1, \dots, x_m) := \varphi_m(\xi_1^n + \rho_n x_1, \xi_1^n + \rho_n x_2, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m),$$

and $x = (x_1, \dots, x_m)$. We have that, for all $l = 1, 2$, $j = 1, \dots, k$,

$$\partial_{(x_j)_l} \tilde{\varphi}_m(x) = \rho_n \partial_{(\xi_j)_l} \varphi_m(\xi_1^n + \rho_n x_1, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m).$$

Then at $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)$ we have

$$\partial_{(x_j)_l} \tilde{\varphi}_m(\bar{x}) = 0.$$

On the other hand, using Claim 5 and letting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \rho_n \partial_{(\xi_j)_l} \varphi_m(\xi_1^n + \rho_n \tilde{x}_1, \dots, \xi_m^n + \rho_n \tilde{x}_m) = 8 \sum_{i \neq j, i \leq k} \partial_{(x_j)_i} \log|\bar{x}_i - \bar{x}_j| = 0,$$

a contradiction with Claim 3.

Case 2. In this case we actually have

$$\frac{\text{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} < C_1, \quad j = 1, \dots, m,$$

for some constant $C_1 > 0$ and for almost all n . If the points ξ_j^n are all interior to Ω_{δ_n} , we argue as in Case 1 above to reach a contradiction to Claim 4.

Therefore, we assume that for some j^* we have $\xi_{j^*}^n \in \partial\Omega_{\delta_n}$. Assume first that there exists a constant C such that $\delta_n \leq C\rho_n$. Consider the following sum

$$s_n := \sum_{i \neq j} G(\xi_j^n, \xi_i^n).$$

In this case it is not difficult to see that $s_n = O(1)$ as $n \rightarrow +\infty$. On the other hand

$$\sum_j H(\xi_j^n, \xi_j^n) \leq H(\xi_{j^*}^n, \xi_{j^*}^n) + C \leq 8 \log|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| + C,$$

where $\bar{\xi}_{j^*}^n$ is the reflection of the point $\xi_{j^*}^n$ with respect to $\partial\Omega$. Since $|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| \leq 2\delta_n$ we have that

$$\sum_j H(\xi_j^n, \xi_j^n) \rightarrow -\infty, \quad \text{as } n \rightarrow \infty.$$

But $|\varphi_m(\xi^n)| \leq K$, a contradiction.

Finally assume that $\rho_n = o(\delta_n)$. In this case after scaling with ρ_n around $\xi_{j^*}^n$, and arguing similarly as in the Case 1 we get a contradiction with Claim 3 since those points ξ_j^n which lie on $\partial\Omega_{\delta_n}$, after passing to the limit, give rise to points that lie on the same straight line. Thus this case cannot occur. \square

We shall now show that we can perturb the gradient vector field of φ_m near $\partial\mathcal{D}_\delta$ to obtain a new vector field with the same stationary points, such that φ_m is a Lyapunov function for the associated flow and $\mathcal{D}_\delta \cap \varphi_m^{-1}[-K, K]$ is positively invariant.

We consider the following more general situation. Let U be a bounded open subset of \mathbb{R}^N with smooth boundary, and let $m \in \mathbb{N}$. We consider a decomposition of \bar{U}^m as follows. Let S be the set of all functions $\sigma : \{1, \dots, m\} \rightarrow \{U, \partial U\}$, and define

$$\mathcal{Y}_\sigma := \sigma(1) \times \dots \times \sigma(m) \subset \mathbb{R}^{mN}.$$

Then

$$\bar{U}^m = \bigcup_{\sigma \in S} \mathcal{Y}_\sigma, \quad \partial(U^m) = \bigcup_{\sigma \in S \setminus \sigma_U} \mathcal{Y}_\sigma, \quad \text{and} \quad \mathcal{Y}_\sigma \cap \mathcal{Y}_\zeta = \emptyset \text{ if } \sigma \neq \zeta,$$

where σ_U stands for the constant function $\sigma_U(i) = U$. Note that \mathcal{Y}_σ is a manifold of dimension $\leq mN$. We denote by $T_\xi(\mathcal{Y}_\sigma)$ the tangent space to \mathcal{Y}_σ at the point $\xi \in \mathcal{Y}_\sigma$. The following holds.

Lemma 6.3. *Let \mathcal{F} be a function of class \mathcal{C}^1 in a neighborhood of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$. Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ with } \sigma \neq \sigma_U, \quad (6.9)$$

where $\nabla_\sigma \mathcal{F}(\xi)$ is the projection of $\nabla \mathcal{F}(\xi)$ onto the tangent space $T_\xi(\mathcal{Y}_\sigma)$. Then there exists a locally Lipschitz continuous vector field $\chi : \mathcal{U} \rightarrow \mathbb{R}^N$, defined in an open neighborhood \mathcal{U} of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$, with the following properties: For $\xi \in \mathcal{U}$,

- (i) $\chi(\xi) = 0$ if and only if $\nabla \mathcal{F}(\xi) = 0$,
- (ii) $\chi(\xi) \cdot \nabla \mathcal{F}(\xi) > 0$ if $\nabla \mathcal{F}(\xi) \neq 0$,
- (iii) $\chi(\xi) \in T_\xi(\mathcal{Y}_\sigma)$ if $\xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c]$.

Proof. Let $\mathcal{N}_\alpha := \{x \in \mathbb{R}^N : \text{dist}(x, \partial U) < \alpha\}$. Fix $\alpha > 0$ small enough so that there exists a smooth retraction $r : \mathcal{N}_\alpha \rightarrow \partial U$. For every $\sigma \in S$, let $\hat{\sigma} : \{1, \dots, m\} \rightarrow \{U, \partial \mathcal{N}_\alpha\}$ be the function $\hat{\sigma}(i) = \sigma(i)$ if $\sigma(i) = U$ and $\hat{\sigma}(i) = \mathcal{N}_\alpha$ if $\sigma(i) = \partial U$. Set

$$\mathcal{U}_\sigma := \hat{\sigma}(1) \times \dots \times \hat{\sigma}(m).$$

Then \mathcal{U}_σ is an open neighborhood of \mathcal{Y}_σ . Let $r_\sigma : \mathcal{U}_\sigma \rightarrow \mathcal{Y}_\sigma$ be the obvious retraction. Assumption (6.9) implies that \mathcal{F} has no critical points on $\partial(U^m) \cap \mathcal{F}^{-1}[b, c]$ and, moreover, that

$$\nabla_\sigma \mathcal{F}(\xi) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

So taking α even smaller if necessary, we may assume that \mathcal{F} has no critical points in $\mathcal{U}_\sigma \cap \mathcal{F}^{-1}[b, c]$ if $\sigma \neq \sigma_U$, and that

$$\nabla_\sigma \mathcal{F}(r_\sigma(\xi)) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha) \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

Let $\{\pi_\sigma : \sigma \in S\}$ be a locally Lipschitz partition of unity subordinated to the open cover $\{\mathcal{U}_\sigma : \sigma \in S\}$. Define

$$\chi(\xi) := \sum_{\sigma \in S} \pi_\sigma(\xi) \nabla_\sigma \mathcal{F}(r_\sigma(\xi)), \quad \xi \in \mathcal{U} := \bigcup_{\sigma \in S} \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha).$$

One can easily verify that χ has the desired properties. \square

As usual, set $\mathcal{F}^c := \{\xi \in \text{dom}\mathcal{F} : \mathcal{F}(\xi) \leq c\}$.

Lemma 6.4 (Deformation lemma). *Let \mathcal{F} be a function of class \mathcal{C}^1 in a neighborhood of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$. Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \quad \text{with } \sigma \neq \sigma_U.$$

If \mathcal{F} has no critical points in $U^m \cap \mathcal{F}^{-1}[b, c]$, then there exists a continuous deformation $\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \mathcal{F}^c) \rightarrow \bar{U}^m \cap \mathcal{F}^c$ such that

$$\begin{aligned} \tilde{\eta}(0, \xi) &= \xi \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c, \\ \tilde{\eta}(s, \xi) &= \xi \quad \text{for all } (s, \xi) \in [0, 1] \times (\bar{U}^m \cap \mathcal{F}^b), \\ \tilde{\eta}(1, \xi) &\in \bar{U}^m \cap \mathcal{F}^b \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c. \end{aligned}$$

Proof. Let $\chi : \mathcal{U} \rightarrow \mathbb{R}^N$ be as in Lemma 6.3 and consider the flow η defined by

$$\begin{cases} \frac{\partial}{\partial t} \eta(t, \xi) = -\chi(\eta(t, \xi)), \\ \eta(0, \xi) = \xi, \end{cases} \quad (6.10)$$

for $\xi \in \mathcal{U}$ and $t \in [0, t^+(\xi))$, where $t^+(\xi)$ is the maximal existence time of the trajectory $t \mapsto \eta(t, \xi)$ in \mathcal{U} . For each $\xi \in \mathcal{U}$, let

$$t_b(\xi) := \inf\{t \geq 0 : \mathcal{F}(\eta(t, \xi)) \leq b\} \in [0, \infty]$$

be the entrance time into the sublevel set \mathcal{F}^b . Property (ii) in Lemma 6.3 implies that

$$\frac{d}{dt} \mathcal{F}(\eta(t, \xi)) = -\nabla \mathcal{F}(\eta(t, \xi)) \cdot \chi(\eta(t, \xi)) \leq 0,$$

therefore $\mathcal{F}(\eta(t, \xi))$ is nonincreasing in t . This, together with (iii) in Lemma 6.3 yields

$$\eta(t, \xi) \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c] \quad \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c] \text{ and } t \in [0, t_b(\xi)].$$

Since \mathcal{F} has no critical points in $U^m \cap \mathcal{F}^{-1}[b, c]$, we have that $t_b(\xi) < \infty$ for every $\xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b, c]$, and the entrance time map $t_b : \bar{U}^m \cap \mathcal{F}^c \cap \mathcal{U} \rightarrow [0, \infty)$ is continuous. It follows that the map

$$\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \mathcal{F}^c) \rightarrow \bar{U}^m \cap \mathcal{F}^c$$

given by

$$\tilde{\eta}(s, \xi) := \begin{cases} \eta(st_b(\xi), \xi) & \text{if } \xi \in (\bar{U}^m \cap \mathcal{F}^c) \cap \mathcal{U} \\ \xi & \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^b \end{cases}$$

is a continuous deformation of $\bar{U}^m \cap \mathcal{F}^c$ into $\bar{U}^m \cap \mathcal{F}^b$ which leaves $\bar{U}^m \cap \mathcal{F}^b$ fixed, as claimed. \square

Proof of Theorem 1. Fix δ_1 small enough so that the inclusions

$$\mathfrak{D}_{\delta_1} \hookrightarrow \Omega^m \quad \text{and} \quad \mathfrak{D}_{\delta_1} \cap D \hookrightarrow B_{\delta_1}(D) := \{x \in \Omega^m : \text{dist}(x, D) \leq \delta_1\} \quad (6.11)$$

are homotopy equivalences, where $D := \{\xi \in \Omega^m : \xi_i = \xi_j \text{ for some } i \neq j\}$. Since φ_m is bounded above on \mathfrak{D}_{δ_1} and bounded below on $\Omega^m \setminus B_{\delta_1}(D)$, we may choose $b_0, c_0 > 0$ such that

$$\mathfrak{D}_{\delta_1} \subset \varphi_m^{c_0} \quad \text{and} \quad \varphi_m^{b_0} \subset B_{\delta_1}(D).$$

Fix $K > \max\{-b_0, c_0\}$ and, for this K , fix $\delta \in (0, \delta_1)$ as in Lemma 6.2. By property (6.2), for each ε small enough, there exist $b < c$ such that

$$\varphi_m^{c_0} \subset \tilde{\mathcal{F}}_\varepsilon^c \subset \varphi_m^K, \quad \varphi_m^{-K} \subset \tilde{\mathcal{F}}_\varepsilon^b \subset \varphi_m^{b_0},$$

and such that, for every $\xi = (\xi_1, \dots, \xi_m) \in \partial\mathfrak{D}_\delta$ with $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [b, c]$ there is an $i \in \{1, \dots, m\}$ with

$$\begin{aligned} \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) &\neq 0 && \text{if } \xi_i \in \Omega_\delta \\ \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial\Omega_\delta) && \text{if } \xi_i \in \partial\Omega_\delta. \end{aligned}$$

We wish to prove that $\tilde{\mathcal{F}}_\varepsilon$ has a critical point in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$. We argue by contradiction: Assume that $\tilde{\mathcal{F}}_\varepsilon$ has no critical points in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$. Then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0, 1] \times (\overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c) \rightarrow \overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$$

of $\overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$ into $\overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$ which keeps $\overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$ fixed. Our choices of b and c imply that $\mathfrak{D}_{\delta_1} \subset \overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$ and $\tilde{\eta}$ induces a deformation of \mathfrak{D}_{δ_1} into $\overline{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b \subset B_{\delta_1}(D)$, which keeps the diagonal D fixed. Consequently, the homomorphism

$$\iota^* : H^*(\Omega^m, B_{\delta_1}(D)) \rightarrow H^*(\mathfrak{D}_{\delta_1}, \mathfrak{D}_{\delta_1} \cap D),$$

induced by the inclusion map $\iota : \mathfrak{D}_{\delta_1} \hookrightarrow \Omega^m$, factors through $H^*(B_{\delta_1}(D), B_{\delta_1}(D)) = 0$. Hence, ι^* is the zero homomorphism. On the other hand, our choice (6.11) of δ_1 implies that ι^* is an isomorphism. Therefore, $H^*(\Omega^m, B_{\delta_1}(D)) = H^*(\Omega^m, D) = 0$. But, by assumption, $H^d(\Omega) \neq 0$ for some $d \geq 1$. If we choose d so that $H^j(\Omega) = 0$ for $j > d$, then Lemma 6.1 asserts that $H^{md}(\Omega^m, D) \neq 0$. This is a contradiction. Consequently, $\tilde{\mathcal{F}}_\varepsilon$ must have critical point in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b, c]$, as claimed. \square

6.2. Proof of Theorem 2. Assume that there exist an open subset U of Ω with smooth boundary, compactly contained in Ω , and two closed subsets $B_0 \subset B$ of U^m , which satisfy conditions **P1**) and **P2**) stated in Section 1. By property (6.2), for ε small enough, $\tilde{\mathcal{F}}_\varepsilon$ satisfies those conditions too, that is,

$$b_\varepsilon := \sup_{\xi \in B_0} \tilde{\mathcal{F}}_\varepsilon(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \tilde{\mathcal{F}}_\varepsilon(\gamma(\xi)) =: c_\varepsilon,$$

where $\Gamma := \{\gamma \in \mathcal{C}(B, \overline{U}^m) : \gamma(\xi) = \xi \text{ for every } \xi \in B_0\}$ and, for every $\xi = (\xi_1, \dots, \xi_m) \in \partial U^m$ with $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [c_\varepsilon - \alpha, c_\varepsilon + \alpha]$, $\alpha \in (0, c_\varepsilon - b_\varepsilon)$ small enough, one has that

$$\begin{aligned} \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) &\neq 0 && \text{if } \xi_i \in U \\ \nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau &\neq 0 \text{ for some } \tau \in T_{\xi_i}(\partial U) && \text{if } \xi_i \in \partial U, \end{aligned}$$

for some $i \in \{1, \dots, m\}$. If $\tilde{\mathcal{F}}_\varepsilon$ has no critical points in $U^m \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[c_\varepsilon - \alpha, c_\varepsilon + \alpha]$, then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0, 1] \times (\overline{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}) \rightarrow \overline{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$$

of $\overline{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$ into $\overline{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$ which keeps $\overline{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$ fixed. Let $\gamma \in \Gamma$ be such that $\tilde{\mathcal{F}}_\varepsilon(\gamma(\xi)) \leq c_\varepsilon + \alpha$ for every $\xi \in B$. Since $b_\varepsilon < c_\varepsilon - \alpha$, the map $\tilde{\gamma}(\xi) := \tilde{\eta}(1, \gamma(\xi))$ belongs to Γ . But $\tilde{\mathcal{F}}_\varepsilon(\tilde{\gamma}(\xi)) \leq c_\varepsilon - \alpha$ for every $\xi \in B$, contradicting the definition of c_ε . Therefore, c_ε is a critical value of $\tilde{\mathcal{F}}_\varepsilon$. \square

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