

MULTIPLE SOLUTIONS TO THE PURE CRITICAL EXPONENT PROBLEM IN DOMAINS WITH A HOLE OF ARBITRARY SIZE

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ABSTRACT. We consider the problem

$$-\Delta u = |u|^{2^*-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $N \geq 3$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. We assume that Ω is annular shaped, i.e. there are constants $R_2 > R_1 > 0$ such that $\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega$ and $0 \notin \Omega$. We also assume that Ω is invariant under a group Γ of orthogonal transformations of \mathbb{R}^N without fixed points. We establish the existence of multiple sign changing solutions if, either Γ is arbitrary and R_1/R_2 is small enough, or R_1/R_2 is arbitrary and the minimal Γ -orbit of Ω is large enough. We believe this is the first existence result for sign changing solutions in domains with holes of arbitrary size. The proof takes advantage of the invariance of this problem under the group of Möbius transformations.

1. INTRODUCTION

We consider the problem

$$(\varphi_\Omega) \quad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $N \geq 3$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

Equations of this type arise in fundamental questions in Differential Geometry like Yamabe's problem or the prescribed scalar curvature problem. Problem (φ_Ω) has a rich geometric structure: it is invariant under the group of Möbius transformations (see Section 2 for the definition). This fact turns it into an interesting and challenging problem because, though it can be expressed as a variational problem, the usual variational methods cannot be applied to it in a straightforward manner due to the lack of compactness caused by the Möbius invariance.

It is well known that the existence of a solution depends on the domain Ω . Indeed, Pohožaev [17] showed that (φ_Ω) has no nontrivial solution if Ω is strictly starshaped. On the other hand, if the domain is an annulus

$$A = A_{R_1, R_2} := \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\}$$

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and one looks for radial solutions, then compactness is restored, and usual methods yield infinitely many radial solutions to (φ_A) . This was pointed out by Kazdan and Warner in [11].

The first nontrivial existence result is due to Coron [8]. He showed that, if Ω is annular-shaped, that is, if

$$0 \notin \Omega \quad \text{and} \quad \Omega \supset A_{R_1, R_2},$$

and R_1/R_2 is small enough, then problem (φ_Ω) has at least one positive solution. Due to the Möbius invariance, this is equivalent to saying that Ω has a small enough hole, that is, that $0 \notin \Omega$, $\Omega \supset A_{R,1}$, and R is small enough. Later a remarkable result was obtained by Bahri and Coron [1] who showed that the same holds for any R . In fact, they showed that (φ_Ω) has at least one positive solution in every domain Ω having nontrivial homology with $\mathbb{Z}/2$ -coefficients. This last condition is, however, not necessary: examples of contractible domains Ω for which problem (φ_Ω) has a solution have been given, for example, in [9, 10, 15, 16, 5].

Recently Clapp and Weth [7] extended Coron's result. They showed that, if Ω has a small enough hole, then (φ_Ω) has at least two solutions. But nothing can be said about the sign of the second one. Existence of sign changing solutions is only known for domains which have some symmetries. The first such result was pointed out by Marchi and Pacella [13] for domains with thin tunnels. For symmetric domains with a small hole existence of a sign changing solution was shown by Clapp and Weth in [5] and their method applies also to get the result of [13]. Recently Musso and Pistoia [14] proved that, if the domain has certain symmetries and a small hole whose boundary is a sphere, then the number of sign changing solutions becomes arbitrarily large as the radius of the sphere goes to zero.

Spheres play an important role in this problem because they are mapped onto spheres by Möbius transformations. Indeed, the next proposition states a result which is a consequence of the Möbius invariance and is worth to be singled out.

Proposition 1.1. *If the boundary of Ω is the union of two disjoint spheres, then (φ_Ω) has infinitely many sign changing solutions.*

Next we assume that Ω is annular-shaped and invariant under the action of a closed subgroup Γ of orthogonal transformations of \mathbb{R}^N . We denote by $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ the Γ -orbit of $x \in \mathbb{R}^N$, by $\#\Gamma x$ its cardinality, and by

$$\ell = \ell(\Gamma) := \min\{\#\Gamma x : x \in \mathbb{R}^N \setminus \{0\}\}.$$

Recall that Ω is said to be Γ -invariant if $\Gamma x \subset \Omega$ for every $x \in \Omega$, and a function $u : \Omega \rightarrow \mathbb{R}$ is Γ -invariant if it is constant on every Γ -orbit.

The aim of our paper is to obtain multiplicity results by using the Möbius invariance, both in domains with a small hole and in domains with a hole of arbitrary size. We believe that Theorem 1.3 below is the first multiplicity result which does not require any hypothesis on the size of the hole. Hence it can be seen as an extension of the Bahri-Coron result [1] to the case of sign changing solutions. Actually we also prove the existence of a positive solution as in [1], but only for symmetric domains.

We start with the result in domains with a small hole. As usual, S will denote the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ denotes the set of functions in $L^{2^*}(\mathbb{R}^N)$ whose gradient is in $L^2(\mathbb{R}^N)$.

Theorem 1.2. *Given $\delta > 0$ there exists R_δ with the following property: for every closed subgroup Γ of $O(N)$ with $\ell = \ell(\Gamma) \geq 2$ and every Γ -invariant domain Ω such that*

$$0 \notin \Omega, \quad \Omega \supset \{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \quad \text{and} \quad 0 < R_1/R_2 < R_\delta,$$

problem (φ_Ω) has at least one positive Γ -invariant solution u_1 and ℓ pairs of Γ -invariant sign changing solutions $\pm u_2, \dots, \pm u_{\ell+1}$ which satisfy

$$\int_{\Omega} |\nabla u_j|^2 \leq jS^{N/2} + \delta, \quad j = 1, \dots, \ell + 1.$$

Some information on the shape of the solutions provided by Theorem 1.2 will be given in Section 4.

To state the result for domains with a hole of arbitrary size let us define

$$c(R_1, R_2) := \frac{1}{N} \min \left\{ \int_{\Omega} |\nabla u|^2 : u \text{ is a radial solution of } (\varphi_{A_{R_1, R_2}}) \right\}.$$

Theorem 1.3. *Given $R_2 > R_1 > 0$ and $m \in \mathbb{N}$, there exists a positive integer ℓ_0 , depending only on m and R_1/R_2 , with the following property: for every closed subgroup Γ of $O(N)$ with $\ell(\Gamma) > \ell_0$ and every Γ -invariant domain Ω with*

$$0 \notin \Omega \quad \text{and} \quad \{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega,$$

problem (φ_Ω) has at least one positive Γ -invariant solution u_1 and $m - 1$ pairs of Γ -invariant sign changing solutions $\pm u_2, \dots, \pm u_m$. These solutions satisfy

$$\int_{\Omega} |\nabla u_j|^2 \leq jNc(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}}), \quad j = 1, \dots, m.$$

This last theorem requires $\ell(\Gamma)$ to be large if the hole is large. If N is even, there are subgroups Γ of $O(N)$ with arbitrarily large $\ell(\Gamma)$. Indeed, the group of rotations $\Gamma_n = \{e^{2\pi ik/n} : k = 0, \dots, n-1\}$ acting by multiplication on each complex coordinate of $\mathbb{C}^{N/2} \cong \mathbb{R}^N$ satisfies $\ell(\Gamma_n) = n$.

The proof of these results uses standard variational methods combined with the symmetry assumptions, which allow to increase the energy interval in which the Palais-Smale condition holds for the functional associated to the problem. However a new simple idea is introduced to get some model functions generating low energy finite dimensional spaces, which allow to produce multiple solutions by a minimax procedure. These model functions are given in Lemma 2.1 where the Möbius invariance of the equation is again exploited as we apply the Kelvin transform to positive radial solutions in annuli.

Finally, the invariance of the problem under inversions in spheres suggests that the solutions of (φ_Ω) should inherit this kind of symmetry if the domain is also invariant (see Remark 2.2). Anyway, solutions of this type should exist in suitable domains. A result in this direction is proved in Theorem 5.1.

The paper is organized as follows. In Section 2 we give preliminaries about Möbius transformations and prove Proposition 1.1 and Lemma 2.1. In Section 3 we prove a variational principle for sign changing solutions, while Section 4 is devoted to the proofs of Theorem 1.2 and Theorem 1.3. Finally Section 5 contains the result about the existence of Kelvin invariant solutions.

2. MÖBIUS INVARIANCE

Let $a \in \mathbb{R}^N$, $r > 0$. The inversion in the sphere $S_r(a) := \{x \in \mathbb{R}^N : |x - a| = r\}$ is the map $\phi : \mathbb{R}^N \cup \{\infty\} \rightarrow \mathbb{R}^N \cup \{\infty\}$ defined by

$$\phi(x) := a + \left(\frac{r}{|x - a|} \right)^2 (x - a) \quad \text{if } x \neq a,$$

$\phi(a) := \infty$, $\phi(\infty) := a$. We also consider the reflection in the plane $P_r(a) := \{x \in \mathbb{R}^N : x \cdot a = r\} \cup \{\infty\}$ defined by

$$\phi(x) := x - 2 \left(\frac{x \cdot a - r}{|a|^2} \right) a \quad \text{if } x \in \mathbb{R}^N,$$

$\phi(\infty) := \infty$. A *Möbius transformation* $\phi : \mathbb{R}^N \cup \{\infty\} \rightarrow \mathbb{R}^N \cup \{\infty\}$ is a finite composition of inversions in spheres and reflections in planes. Since euclidean isometries of \mathbb{R}^N are compositions of reflections in planes, they are Möbius transformations. Dilations $x \mapsto kx$, $k > 0$, are also Möbius transformations (see, for example, [2]).

Let Ω be an open subset of \mathbb{R}^N . Given a Möbius transformation $\phi : \mathbb{R}^N \cup \{\infty\} \rightarrow \mathbb{R}^N \cup \{\infty\}$ we define $\phi(\Omega) := \{\phi(x) : x \in \Omega\} \setminus \{\infty\}$ and, for $u \in D_0^{1,2}(\phi(\Omega))$, we define $u_\phi \in D_0^{1,2}(\Omega)$ by

$$u_\phi(x) := (\det D\phi)^{\frac{N-2}{2N}} u(\phi(x)),$$

where $\det D\phi$ is the Jacobian determinant of ϕ . The map $u \mapsto u_\phi$ is a linear isometry of $D_0^{1,2}(\phi(\Omega)) \cong D_0^{1,2}(\Omega)$ and of $L^{2^*}(\phi(\Omega)) \cong L^{2^*}(\Omega)$, that is,

$$(2.1) \quad \int_{\Omega} \nabla u_\phi \cdot \nabla v_\phi = \int_{\phi(\Omega)} \nabla u \cdot \nabla v \quad \text{and} \quad \int_{\Omega} |u_\phi|^{2^*} = \int_{\phi(\Omega)} |u|^{2^*}.$$

We write

$$\|u\|^2 := \int_{\Omega} |\nabla u|^2 \quad \text{and} \quad |u|_{2^*}^{2^*} := \int_{\Omega} |u|^{2^*}.$$

The solutions of problem

$$(\varphi_\Omega) \quad -\Delta u = |u|^{2^*-2} u, \quad u \in D_0^{1,2}(\Omega),$$

are the critical points of the functional $J_\Omega : D_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_\Omega(u) := \frac{1}{2} \|u\|^2 - \frac{1}{2^*} |u|_{2^*}^{2^*}.$$

It follows from (2.1) that $J_\Omega(u_\phi) = J_{\phi(\Omega)}(u)$. Moreover, problem (φ_Ω) is *Möbius invariant*, that is, u_ϕ is a solution of (φ_Ω) if and only if u is a solution of $(\varphi_{\phi(\Omega)})$. This property allows us to prove Proposition 1.1. From now on, we assume that Ω is bounded.

Proof of Proposition 1.1. After a suitable translation, we may assume that $\partial\Omega = S_r(a) \cup S_t(0)$ with $r + |a| < t$. If $a = 0$ then Ω is an annulus and it is well known that problem (φ_Ω) has infinitely many sign changing (radial) solutions [11]. Now assume that $a \neq 0$. We claim that there exists an inversion ϕ in a sphere, such that $\phi(\Omega)$ is an annulus. If this is true then, by Möbius invariance, problem (φ_Ω) has infinitely many sign changing solutions.

For each $\lambda > \frac{t}{|a|}$, let $R_\lambda^2 := |\lambda a|^2 - t^2$. The sphere $S_{R_\lambda}(\lambda a)$ is orthogonal to the sphere $S_t(0)$ and, therefore, the inversion ϕ_λ in $S_{R_\lambda}(\lambda a)$ maps $S_t(0)$ onto itself.

On the other hand, ϕ_λ maps the sphere $S_r(a)$ onto the sphere $S_{r_\lambda}(a_\lambda)$ defined as follows: for each $x \in S_r(a)$ one has that

$$(2.2) \quad |\phi_\lambda(x) - \lambda a| |x - \lambda a| = R_\lambda^2 = |\lambda a|^2 - t^2.$$

Let \hat{x} be the second point of intersection of the line through λa and x with the sphere $S_r(a)$. Then, for each $x \in S_r(a)$,

$$(2.3) \quad |\hat{x} - \lambda a| |x - \lambda a| = |\lambda a - a|^2 - r^2.$$

Let a_λ be the point where the line through $\phi(x)$ parallel to $\hat{x} - a$ intersects the line through a and the origin. Dividing (2.2) by (2.3), and by triangle similarity we obtain

$$(2.4) \quad \frac{|\lambda a|^2 - t^2}{|\lambda a - a|^2 - r^2} = \frac{|\phi_\lambda(x) - \lambda a|}{|\hat{x} - \lambda a|} = \frac{|\phi_\lambda(x) - a_\lambda|}{|\hat{x} - a|} = \frac{|a_\lambda - \lambda a|}{|a - \lambda a|}.$$

In particular, a_λ does not depend on x and

$$|\phi_\lambda(x) - a_\lambda| = \frac{(|\lambda a|^2 - t^2)r}{|\lambda a - a|^2 - r^2} =: r_\lambda.$$

Now we claim that there exists a $\lambda > \frac{t}{|a|}$ such that $a_\lambda = 0$, that is, such that ϕ_λ maps Ω onto the annulus $\{x \in \mathbb{R}^N : r_\lambda < |x| < t\}$. Indeed, since $r + |a| < t$, the number

$$\lambda := \frac{1}{2} \left[1 + \frac{t^2 - r^2}{|a|^2} + \sqrt{\left(1 + \frac{t^2 - r^2}{|a|^2} \right)^2 - \frac{4t^2}{|a|^2}} \right]$$

is well defined, and it satisfies $\lambda > \frac{t}{|a|}$, and

$$\frac{|\lambda a|^2 - t^2}{|\lambda a - a|^2 - r^2} = \frac{|\lambda a|}{|a - \lambda a|}.$$

Hence, by (2.4), $a_\lambda = 0$. This concludes the proof. \square

Let G be a closed subgroup of the group $GM(\mathbb{R}^N \cup \{\infty\})$ of Möbius transformations of $\mathbb{R}^N \cup \{\infty\}$, and assume that Ω is bounded and G -invariant, that is, $\phi(\Omega) = \Omega$ for every $\phi \in G$. Then $\phi u := u_{\phi^{-1}}$ defines an orthogonal action of G on $H_0^1(\Omega)$, and the functional $J := J_\Omega$ is G -invariant. By the principle of symmetric criticality [20, Theorem 1.28], the critical points of the restriction of J to the space

$$\begin{aligned} H_0^1(\Omega)^G : &= \{u \in H_0^1(\Omega) : \phi u = u \text{ for all } \phi \in G\} \\ &= \{u \in H_0^1(\Omega) : (\det D\phi)^{\frac{N-2}{2N}} u(\phi(x)) = u(x) \text{ for all } \phi \in G, x \in \Omega\} \end{aligned}$$

are solutions to problem (φ_Ω) . The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}(\Omega)^G := \{u \in H_0^1(\Omega)^G : u \neq 0, \|u\|^2 = |u|_{2^*}^{2^*}\},$$

which is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^G$. If $G = \{1\}$ is the trivial group, then $H_0^1(\Omega)^G = H_0^1(\Omega)$ and $\mathcal{N}(\Omega)^G = \mathcal{N}(\Omega)$ is the usual Nehari manifold. It is well known that

$$\inf\{J(u) : u \in \mathcal{N}(\Omega)\} = \frac{1}{N} S^{N/2} =: c_\infty,$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. So this infimum does not depend on Ω , and it is never attained. But if G is nontrivial the infimum of J on $\mathcal{N}(\Omega)^G$ might be attained. For example, if Ω is an annulus

$$\{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\} =: A_{R_1, R_2}$$

and $G = O(N)$, then $H_0^1(A_{R_1, R_2})^{O(N)}$ is the space of radial functions and the infimum

$$c(R_1, R_2) := \inf\{J(u) : u \in \mathcal{N}(A_{R_1, R_2})^{O(N)}\}$$

is always attained. Due to the Möbius invariance, $c(R_1, R_2) = c(R_1/R_2, 1)$, and it is easy to see that

$$(2.5) \quad c(R, 1) \rightarrow c_\infty \quad \text{as } R \rightarrow 0.$$

The following lemma will play an important role in the proof of our main results.

Lemma 2.1. *Given $0 < R_1 < R_2$ and $m \in \mathbb{N}$, there exist $R_2 =: P_0 > P_1 > \dots > P_m =: R_1$ and positive radial functions $\omega_1, \dots, \omega_m \in \mathcal{N}(A_{R_1, R_2})$ such that*

$$\text{supp}(\omega_i) \subset A_{P_i, P_{i-1}} \quad \text{and} \quad J(\omega_i) = c(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}})_i, \quad i = 1, \dots, m.$$

Proof. Let $R_2 =: P_0 > P_1 > \dots > P_m =: R_1$ be defined by $P_{i+1}P_{i-1} = P_i^2$, $i = 1, \dots, m-1$. Let ϕ_i be the inversion in the sphere $S_{P_i}(0)$, that is,

$$\phi_i(x) = \frac{P_i^2 x}{|x|^2}.$$

Fix a positive radial minimizer ω_1 of J on $\mathcal{N}(A_{P_1, P_0})^{O(N)}$ and define, inductively, $\omega_{i+1} := (\omega_i)_{\phi_i}$, that is,

$$\omega_{i+1}(x) = \frac{P_i^{2N-4}}{|x|^{N-2}} \omega_i \left(\frac{P_i^2 x}{|x|^2} \right)$$

is the Kelvin transform of ω_i with respect to the inversion ϕ_i .

Since $\phi_i(A_{P_i, P_{i-1}}) = A_{P_{i+1}, P_i}$, the Möbius invariance yields that ω_{i+1} is a positive radial minimizer of J on $\mathcal{N}(A_{P_{i+1}, P_i})^{O(N)}$, with $J(\omega_{i+1}) = J(\omega_1) = c(P_1, P_0)$. An easy computation shows that $P_1 = R_2^{\frac{m-1}{m}} R_1^{\frac{1}{m}}$. Hence, $c(P_1, P_0) = c(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}})$, as claimed. \square

Remark 2.2. Note that, by regularity, $\omega_i \in C^2(A_{P_i, P_{i-1}})$ is a classical solution of

$$-\Delta \omega_i = |\omega_i|^{2^*-2} \omega_i \quad \text{in } A_{P_i, P_{i-1}}, \quad \omega_i = 0 \quad \text{on } \partial A_{P_i, P_{i-1}}.$$

Moreover, by construction, the normal derivatives of ω_i and $-\omega_{i+1}$ coincide at each point $x \in S_{P_i}(0)$. Therefore, the function

$$\hat{\omega}_m(x) := \sum_{i=1}^m (-1)^i \omega_i$$

is of class C^2 on A_{R_1, R_2} and it is a radial solution of

$$-\Delta \hat{\omega}_m = |\hat{\omega}_m|^{2^*-2} \hat{\omega}_m \quad \text{in } A_{R_1, R_2}, \quad \hat{\omega}_m = 0 \quad \text{on } \partial A_{R_1, R_2},$$

with precisely m nodal domains. We conjecture that $\pm \hat{\omega}_m$ are the only radial solutions.

3. A VARIATIONAL PRINCIPLE FOR SIGN CHANGING SOLUTIONS

Let G be a closed subgroup of the group $GM(\mathbb{R}^N \cup \{\infty\})$ of Möbius transformations, and let Ω be a G -invariant bounded domain in \mathbb{R}^N . The sign changing solutions $u \in \mathcal{N}(\Omega)^G$ of (\wp_Ω) lie on the set

$$\mathcal{E}^G := \{u \in \mathcal{N}(\Omega)^G : u^+, u^- \in \mathcal{N}(\Omega)^G\}$$

(see, for example, [3]). Here, as usual, $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Consider the negative gradient flow $\varphi : \mathcal{G} \rightarrow H_0^1(\Omega)^G$ of J , defined by

$$\begin{cases} \frac{\partial}{\partial t} \varphi(t, u) = -\nabla J(\varphi(t, u)) \\ \varphi(0, u) = u \end{cases}$$

where $\mathcal{G} := \{(t, u) : u \in H_0^1(\Omega)^G, 0 \leq t < T(u)\}$ and $T(u) \in (0, \infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t, u)$. A subset \mathcal{D} of $H_0^1(\Omega)^G$ is said to be strictly positively invariant under φ if

$$\varphi(t, u) \in \text{int}(\mathcal{D}) \quad \text{for every } u \in \mathcal{D} \text{ and every } t \in (0, T(u)),$$

where $\text{int}(\mathcal{D})$ denotes the interior of \mathcal{D} in $H_0^1(\Omega)^G$. If \mathcal{D} is strictly positively invariant under φ , then the set

$$\mathcal{A}(\mathcal{D}) := \{u \in H_0^1(\Omega)^G : \varphi(t, u) \in \mathcal{D} \text{ for some } t \in (0, T(u))\}$$

is open in $H_0^1(\Omega)^G$, and the entrance time map $e_{\mathcal{D}} : \mathcal{A}(\mathcal{D}) \rightarrow \mathbb{R}$ defined by

$$e_{\mathcal{D}}(u) := \inf\{t \geq 0 : \varphi(t, u) \in \mathcal{D}\}$$

is continuous. We write $\mathcal{P}^G := \{u \in H_0^1(\Omega)^G : u \geq 0\}$ for the convex cone of positive functions in $H_0^1(\Omega)^G$ and, for $\alpha > 0$, we set

$$B_\alpha(\mathcal{P}^G) := \{u \in H_0^1(\Omega)^G : \text{dist}(u, \mathcal{P}^G) \leq \alpha\},$$

where $\text{dist}(u, \mathcal{A}) := \inf_{v \in \mathcal{A}} \|u - v\|$.

Lemma 3.1. *There exists $\alpha > 0$ such that*

- (a) $[B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G)] \cap \mathcal{E}^G = \emptyset$, and
- (b) $B_\alpha(\mathcal{P}^G)$ and $B_\alpha(-\mathcal{P}^G)$ are strictly positively invariant under φ .

Proof. For every $u \in H_0^1(\Omega)^G$,

$$|u^-|_{2^*} = \min_{v \in \mathcal{P}^G} |u - v|_{2^*} \leq S^{-1/2} \min_{v \in \mathcal{P}^G} \|u - v\| = S^{-1/2} \text{dist}(u, \mathcal{P}^G),$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Therefore, since $\inf_{\mathcal{N}(\Omega)^G} J > 0$, there exists $\alpha > 0$ such that $\text{dist}(u, \mathcal{P}^G) > \alpha$ for every $u \in \mathcal{E}^G$. And, since \mathcal{E}^G is symmetric with respect to the origin, $\text{dist}(u, -\mathcal{P}^G) = \text{dist}(-u, \mathcal{P}^G) > \alpha$. This proves (a). The proof of (b) is completely analogous to that of [6, Lemma 2]. \square

We fix $\alpha > 0$ as in the Lemma 3.1. Then J has no sign changing critical points in $B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G)$. Let $J^d := \{u \in H_0^1(\Omega)^G : J(u) \leq d\}$. The following holds.

Corollary 3.2. *If J has no sign changing critical point $u \in H_0^1(\Omega)^G$ with $J(u) = d$, then the set*

$$\mathcal{D}_d^G := B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G) \cup J^d$$

is strictly positively invariant under φ , and the map

$$\varrho_d : \mathcal{A}(\mathcal{D}_d^G) \rightarrow \mathcal{D}_d^G, \quad \varrho_d(u) := \varphi(e_{\mathcal{D}_d^G}(u), u)$$

is odd and continuous, and satisfies $\varrho_d(u) = u$ for every $u \in \mathcal{D}_d^G$.

A subset \mathcal{Y} of $H_0^1(\Omega)^G$ will be called symmetric if $-u \in \mathcal{Y}$ for every $u \in \mathcal{Y}$.

Definition 3.3. Let \mathcal{D} and \mathcal{Y} be symmetric subsets of $H_0^1(\Omega)^G$. The *genus of \mathcal{Y} relative to \mathcal{D}* , denoted $\mathfrak{g}(\mathcal{Y}, \mathcal{D})$, is the smallest number m such that \mathcal{Y} can be covered by $m + 1$ open symmetric subsets $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_m$ of $H_0^1(\Omega)^G$ with the following two properties:

(i) $\mathcal{Y} \cap \mathcal{D} \subset \mathcal{U}_0$ and there exists an odd continuous map $\vartheta_0 : \mathcal{U}_0 \rightarrow \mathcal{D}$ such that $\vartheta_0(u) = u$ for $u \in \mathcal{Y} \cap \mathcal{D}$,

(ii) there exist odd continuous maps $\vartheta_j : \mathcal{U}_j \rightarrow \{1, -1\}$ for every $j = 1, \dots, m$.

If no such cover exists, we define $\mathfrak{g}(\mathcal{Y}, \mathcal{D}) := \infty$.

If $\mathcal{D} = \emptyset$ we write $\mathfrak{g}(\mathcal{Y}) := \mathfrak{g}(\mathcal{Y}, \emptyset)$. This is the usual Krasnoselskii genus. \mathcal{D} is called a symmetric neighborhood retract if there exist a symmetric neighborhood \mathcal{U} of \mathcal{D} in $H_0^1(\Omega)^G$ and an odd continuous map $\varrho : \mathcal{U} \rightarrow \mathcal{D}$ such that $\varrho(u) = u$ for every $u \in \mathcal{D}$.

Lemma 3.4. Let \mathcal{D} , \mathcal{Y} , and \mathcal{Z} be closed symmetric subsets of $H_0^1(\Omega)^G$. The genus relative to \mathcal{D} has the following properties:

Monotonicity: If \mathcal{D} is a symmetric neighborhood retract and if there exists an odd continuous map $f : \mathcal{Z} \rightarrow \mathcal{Y}$ such that $f(u) = u$ for every $u \in \mathcal{D}$, then

$$\mathfrak{g}(\mathcal{Z}, \mathcal{D}) \leq \mathfrak{g}(\mathcal{Y}, \mathcal{D}).$$

Subadditivity: If $\mathcal{Z} \cap \mathcal{D} = \emptyset$, then

$$\mathfrak{g}(\mathcal{Y} \cup \mathcal{Z}, \mathcal{D}) \leq \mathfrak{g}(\mathcal{Y}, \mathcal{D}) + \mathfrak{g}(\mathcal{Z}).$$

Proof. Subadditivity is straightforward. We prove monotonicity: Let $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_m$ be open symmetric subsets of $H_0^1(\Omega)^G$ which cover \mathcal{Y} and let $\vartheta_0 : \mathcal{U}_0 \rightarrow \mathcal{D}$ and $\vartheta_j : \mathcal{U}_j \rightarrow \{1, -1\}$ be as in Definition 3.3. Then $f^{-1}(\mathcal{U}_j)$ is symmetric and relatively open in \mathcal{Z} . By Tietze's extension theorem, there exist open symmetric subsets $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_m$ with $\mathcal{V}_j \cap \mathcal{Y} = f^{-1}(\mathcal{U}_j)$, and odd maps $\eta_0 : \mathcal{V}_0 \rightarrow H_0^1(\Omega)^G$, $\eta_j : \mathcal{V}_j \rightarrow \mathbb{R}$, $j = 1, \dots, m$, such that $\eta_j(u) = \vartheta_j(f(u))$ for all $u \in f^{-1}(\mathcal{U}_j)$ and $j = 0, 1, \dots, m$. Let $\varrho : \mathcal{U} \rightarrow \mathcal{D}$ be an odd continuous map on an open symmetric neighborhood \mathcal{U} of \mathcal{D} , such that $\varrho(u) = u$ for every $u \in \mathcal{D}$. Set $\mathcal{W}_0 := \eta_0^{-1}(\mathcal{U})$, $\mathcal{W}_j := \eta_j^{-1}(\mathbb{R} \setminus \{0\})$, and define $\psi_0 : \mathcal{W}_0 \rightarrow \mathcal{D}$ by $\psi_0 := \varrho \circ \eta_0$, and $\psi_j : \mathcal{W}_j \rightarrow \{1, -1\}$ by $\psi_j(u) = 1$ if $\eta_j(u) > 0$ and $\psi_j(u) = -1$ if $\eta_j(u) < 0$. These are odd continuous maps. Note that $\mathcal{W}_0 \supset \mathcal{Z} \cap \mathcal{D}$ and that $\psi_0(u) = u$ for every $u \in \mathcal{Z} \cap \mathcal{D}$. Hence $\mathfrak{g}(\mathcal{Z}, \mathcal{D}) \leq m$. \square

Definition 3.5. Let $\mathcal{D} \subset \mathcal{H}$ be subsets of $H_0^1(\Omega)$. We shall say that J satisfies $(PS)_c$ relative to \mathcal{D} in \mathcal{H} , if every sequence (u_n) in \mathcal{H} such that

$$u_n \notin \mathcal{D}, \quad J(u_n) \rightarrow c, \quad \nabla J(u_n) \rightarrow 0,$$

has a convergent subsequence. If $\mathcal{D} = \emptyset$ we simply say that J satisfies $(PS)_c$ in \mathcal{H} .

Set $\mathcal{D}_c^G := B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G) \cup J^c$, and define

$$c_j := \inf\{c \in \mathbb{R} : \mathfrak{g}(\mathcal{D}_c^G, \mathcal{D}_0^G) \geq j\}.$$

Proposition 3.6. Assume that J satisfies $(PS)_{c_j}$ relative to \mathcal{D}_0^G in $H_0^1(\Omega)^G$. Then, the following holds:

- (a) There exists a sign changing critical point $u \in H_0^1(\Omega)^G$ of J with $J(u) = c_j$.
- (b) If $c_j = c_{j+1}$, then J has infinitely many sign changing critical points $u \in$

$H_0^1(\Omega)^G$ with $J(u) = c_j$.

Consequently, if J satisfies $(PS)_c$ relative to \mathcal{D}_0^G in $H_0^1(\Omega)^G$ for every $c \leq d$, then J has at least $\mathfrak{g}(\mathcal{D}_d^G, \mathcal{D}_0^G)$ pairs of sign changing critical points u in $H_0^1(\Omega)^G$ with $J(u) \leq d$.

Proof. (a) We argue by contradiction. If J has no sign changing critical point u with $J(u) = c_j$ then, since J satisfies $(PS)_{c_j}$ relative to \mathcal{D}_0^G , there exist $\varepsilon \in (0, c_j)$ and $\beta > 0$ such that $\|\nabla J(u)\| \geq \beta$ for every $u \in J^{-1}[c_j - \varepsilon, c_j + \varepsilon] \setminus \mathcal{D}_0^G$. It follows that $\mathcal{D}_{c_j+\varepsilon}^G \subset \mathcal{A}(\mathcal{D}_{c_j-\varepsilon}^G)$. By Corollary 3.2, \mathcal{D}_0^G is a symmetric neighborhood retract, the map $\varrho_{c_j-\varepsilon} : \mathcal{D}_{c_j+\varepsilon}^G \rightarrow \mathcal{D}_{c_j-\varepsilon}^G$ is odd and continuous, and satisfies $\varrho_{c_j-\varepsilon}(u) = u$ for every $u \in \mathcal{D}_0^G$. The monotonicity of the genus yields

$$j \leq \mathfrak{g}(\mathcal{D}_{c_j+\varepsilon}^G, \mathcal{D}_0^G) = \mathfrak{g}(\mathcal{D}_{c_j-\varepsilon}^G, \mathcal{D}_0^G) < j,$$

a contradiction. Therefore, (a) must hold.

(b) Let $c := c_j = c_{j+1}$. Arguing again by contradiction, we assume J has finitely many sign changing critical points $\pm u_1, \dots, \pm u_m$ with $J(u_i) = c$. Fix $\delta > 0$ smaller than half the distance between each two of these points and smaller than the distance of each of them to \mathcal{D}_0^G . Denote by $\mathcal{U}_r := B_r(\{\pm u_1, \dots, \pm u_m\})$. Then $\mathfrak{g}(\mathcal{U}_\delta) = 1$ and $\mathcal{U}_\delta \cap \mathcal{D}_0^G = \emptyset$. Since J satisfies $(PS)_c$ relative to \mathcal{D}_0^G , there exist $\varepsilon \in (0, c)$ and $\beta > 0$ such that $\|\nabla J(u)\| \geq \beta$ for every $u \in J^{-1}[c_j - \varepsilon, c_j + \varepsilon] \setminus (\mathcal{D}_0^G \cup \mathcal{U}_{\delta/2})$. Let $u \in \mathcal{D}_{c+\varepsilon}^G \setminus \text{int}(\mathcal{U}_\delta)$. If $\varphi(t, u) \notin \mathcal{U}_{\delta/2}$ for all $t \in (0, T(u))$ then $u \in \mathcal{A}(\mathcal{D}_{c-\varepsilon}^G)$. If $\varphi(t, u) \in \mathcal{U}_{\delta/2}$ for some t , and $t_0 := \min\{t \in (0, T(u)) : \varphi(t, u) \in \mathcal{U}_{\delta/2}\}$, then

$$\frac{\delta}{2} \leq \int_0^{t_0} \left\| \frac{\partial \varphi(t, u)}{\partial t} \right\| dt \leq \frac{1}{\beta} \int_0^{t_0} \|\nabla J(\varphi(t, u))\|^2 dt \leq \frac{1}{\beta} [J(u) - J(\varphi(t_0, u))]$$

and, taking $\varepsilon < 2(\beta\delta)^{-1}$, we obtain that $J(\varphi(t_0, u)) \leq c - \varepsilon$. It follows that $\mathcal{D}_{c+\varepsilon}^G \setminus \text{int}(\mathcal{U}_\delta) \subset \mathcal{A}(\mathcal{D}_{c-\varepsilon}^G)$ and, by Corollary 3.2 and the monotonicity property, $\mathfrak{g}(\mathcal{D}_{c+\varepsilon}^G \setminus \text{int}(\mathcal{U}_\delta), \mathcal{D}_0^G) \leq \mathfrak{g}(\mathcal{D}_{c-\varepsilon}^G, \mathcal{D}_0^G)$. Subadditivity and monotonicity yield

$$\begin{aligned} j+1 &\leq \mathfrak{g}(\mathcal{D}_{c+\varepsilon}^G, \mathcal{D}_0^G) \\ &\leq \mathfrak{g}(\mathcal{D}_{c+\varepsilon}^G \setminus \text{int}(\mathcal{U}_{\delta/2}), \mathcal{D}_0^G) + \mathfrak{g}(\mathcal{U}_\delta) \\ &\leq \mathfrak{g}(\mathcal{D}_{c-\varepsilon}^G, \mathcal{D}_0^G) + 1 < j+1, \end{aligned}$$

a contradiction. Therefore, (b) must hold. \square

We now prove a mountain pass result for sign changing solutions.

Theorem 3.7. *Let W be a finite dimensional subspace of $H_0^1(\Omega)^G$ and let $d := \sup_W J$. If J satisfies $(PS)_c$ relative to \mathcal{D}_0^G in $H_0^1(\Omega)^G$ for every $c \leq d$, then J has at least $\dim(W) - 1$ pairs of sign changing critical points $u \in \mathcal{H}_0^1(\Omega)^G$ with $J(u) \leq d$.*

Proof. By Proposition 3.6, we need only to show that $m := \mathfrak{g}(\mathcal{D}_d^G, \mathcal{D}_0^G) \geq \dim(W) - 1$. Let $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_m$ be open symmetric subsets of $H_0^1(\Omega)^G$ which cover \mathcal{D}_d^G with $\mathcal{U}_0 \supset \mathcal{D}_0^G$, and let $\vartheta_0 : \mathcal{U}_0 \rightarrow \mathcal{D}_0^G$ and $\vartheta_j : \mathcal{U}_j \rightarrow \{1, -1\}$, $j = 1, \dots, m$, be odd continuous maps, such that $\vartheta_0(u) = u$ for $u \in \mathcal{Y} \cap \mathcal{D}_0^G$. Taking \mathcal{U}_0 a little smaller if necessary, we may assume that ϑ_0 is defined on $\overline{\mathcal{U}_0}$, and use Tietze's extension theorem to extend ϑ_0 to an odd map $\tilde{\vartheta}_0 : H_0^1(\Omega)^G \rightarrow H_0^1(\Omega)^G$. Let \mathcal{B} be the connected component of the complement of the Nehari manifold $\mathcal{N}^G := \mathcal{N}(\Omega)^G$ in $H_0^1(\Omega)^G$ which contains 0, and set $\mathcal{O} := \{u \in W : \tilde{\vartheta}_0(u) \in \mathcal{B}\}$. Then \mathcal{O} is an open

symmetric neighborhood of 0 in W . Since W is finite dimensional, there exists $s > 0$ such that $J(u) \leq 0$ for every $u \in W$ with $\|u\| \geq s$. But if $J(u) \leq 0$ and $u \neq 0$, we have that $\tilde{\vartheta}_0(u) = \vartheta_0(u) = u \notin \mathcal{B}$. Therefore $\mathcal{O} \subset B_s(0)$.

Let $\mathcal{V}_j := \mathcal{U}_j \cap \partial\mathcal{O}$. Then $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_m$ are symmetric, open in $\partial\mathcal{O}$, and they cover $\partial\mathcal{O}$. Moreover, by Lemma 3.1,

$$\vartheta_0(\mathcal{V}_0) \subset \mathcal{D}_0^G \cap \mathcal{N}^G \subset \mathcal{N}^G \setminus \mathcal{E}^G.$$

It is well known that $\mathcal{N}^G \setminus \mathcal{E}^G$ consists of two connected components of the form \mathcal{U} , $-\mathcal{U}$ (see, for example, [3]), hence, there exists an odd map $\eta : \mathcal{N}^G \setminus \mathcal{E}^G \rightarrow \{1, -1\}$. Let $\eta_j : \mathcal{V}_j \rightarrow \{1, -1\}$ be the restrictions of the maps $\eta_0 := \eta \circ \vartheta_0$, $\eta_j := \vartheta_j$, $j = 1, \dots, m$. Take a partition of unity $\{\pi_j : \partial\mathcal{O} \rightarrow [0, 1] : j = 0, 1, \dots, m\}$ subordinated to the cover $\{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_m\}$ consisting of even functions, and let $\{e_1, \dots, e_{m+1}\}$ be the canonical basis of \mathbb{R}^{m+1} . The map $\psi : \partial\mathcal{O} \rightarrow \mathbb{R}^{m+1}$ given by

$$\psi(u) = \sum_{j=0}^m \eta_j(u) \pi_j(u) e_{j+1}$$

is odd and continuous, and satisfies $\psi(u) \neq 0$ for every $u \in \partial\mathcal{O}$. By the Borsuk-Ulam theorem, $\dim(W) \leq m + 1$, as claimed. \square

4. MULTIPLE SOLUTIONS IN ANNULAR DOMAINS

Let Γ be a closed subgroup of $O(N)$, and let

$$\ell := \min\{\#\Gamma x : x \in \mathbb{R}^N \setminus \{0\}\}.$$

The following compactness lemma is needed to prove Theorems 1.2 and 1.3.

Lemma 4.1. (i) J satisfies $(PS)_c$ in $H_0^1(\Omega)^\Gamma$ for every $c < \ell c_\infty$.

(ii) If $\ell \geq 2$ then there exists $\varepsilon_0 > 0$ such that J satisfies $(PS)_c$ relative to \mathcal{D}_0^Γ in $H_0^1(\Omega)^\Gamma$ for every $c < (\ell + 1)c_\infty + \varepsilon_0$.

Proof. (i) follows from Struwe's representation theorem [18, 19] taking the symmetries into account (see also [12]). We now prove (ii). Let $\varepsilon_0 \in (0, c_\infty]$ be such that (φ_Ω) has no solution u with $J(u) < c_\infty + \varepsilon_0$. Then, $2\ell \geq \ell + 1 + \varepsilon_0/c_\infty$. Let (u_n) be a sequence in $H_0^1(\Omega)^\Gamma$ such that

$$u_n \in \mathcal{D}_0^\Gamma, \quad J(u_n) \rightarrow c < (\ell + 1)c_\infty + \varepsilon_0, \quad \nabla J(u_n) \rightarrow 0.$$

Assume, by contradiction, that (u_n) has no convergent subsequence. Then, Struwe's representation theorem [18, 19] (see also [4]) implies that $\ell < \infty$, and that there exist sequences (y_n) in Ω and (ε_n) in $(0, \infty)$, and a $\nu \in \{1, -1\}$, such that $\#\Gamma y_n = \ell$ and

$$\left\| u_n - \nu \sum_{z \in \Gamma y_n} U_{\varepsilon_n, z} \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $U_{\varepsilon, z}$ is the instanton

$$(4.1) \quad U_{\varepsilon, z}(x) := a_N \left(\frac{\varepsilon}{\varepsilon^2 + |x - z|^2} \right)^{\frac{N-2}{2}}, \quad a_N := [N(N-2)]^{\frac{N-2}{4}},$$

which is a solution of problem $(\varphi_{\mathbb{R}^N})$ whose energy is precisely

$$J_{\mathbb{R}^N}(U_{\varepsilon, z}) = \frac{1}{N} S^{N/2} =: c_\infty.$$

But this implies that $\text{dist}(u_n, \mathcal{P}^\Gamma \cup -\mathcal{P}^\Gamma) \rightarrow 0$ as $n \rightarrow \infty$, contradicting the fact that $u_n \notin \mathcal{D}_0^\Gamma$. This proves (ii). \square

We are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $\varepsilon_0 \in (0, c_\infty)$ be as in Lemma 4.1. Without loss of generality we may assume that $\delta < \varepsilon_0$. By (2.5) there is a $R_\delta > 0$ such that

$$c(R_1^{\frac{1}{\ell+1}}, R_2^{\frac{1}{\ell+1}}) < c_\infty + \frac{\delta}{\ell+1} \quad \text{if } R_1/R_2 < R_\delta.$$

Let $\omega_1, \dots, \omega_{\ell+1}$ be positive radial functions as in Lemma 2.1. Then, $\omega_1, \dots, \omega_{\ell+1} \in \mathcal{N}(\Omega)^\Gamma$. Let $W_k := \text{span}\{\omega_1, \dots, \omega_k\}$ be the vector space generated by $\omega_1, \dots, \omega_k$. Since ω_i and ω_j have disjoint supports if $i \neq j$ (up to a set of measure zero), they are orthogonal in $H_0^1(\Omega)^\Gamma$. Thus, $\dim(W_k) = k$. Moreover, for each $k = 1, \dots, \ell+1$,

$$\max_{W_k} J \leq \sum_{i=1}^k \max_{t>0} J(t\omega_i) = kc(R_1^{\frac{1}{\ell+1}}, R_2^{\frac{1}{\ell+1}}) \leq kc_\infty + \delta < (\ell+1)c_\infty + \varepsilon_0.$$

Since $\ell \geq 2$, we have that $J(\omega_1) \leq c_\infty + \delta < \ell c_\infty$. So, by Lemma 4.1, $\inf_{\mathcal{N}(\Omega)^\Gamma} J$ is attained at a positive solution $u_1 \in \mathcal{N}(\Omega)^\Gamma$ with $J(u_1) \leq c_\infty + \delta$.

On the other hand, Lemma 4.1 and Theorem 3.7 yield the existence of ℓ pairs of sign changing critical points $\pm u_2, \dots, \pm u_{\ell+1} \in \mathcal{N}(\Omega)^\Gamma$ of J with

$$J(u_k) \leq kc_\infty + \delta, \quad k = 2, \dots, \ell+1,$$

as claimed. \square

Proof of Theorem 1.3. Let now $R_2 > R_1 > 0$ and $m \in \mathbb{N}$ be given. Define $\ell_0 := \frac{1}{c_\infty} mc(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}})$. Let $\omega_1, \dots, \omega_m \in \mathcal{N}(\Omega)^\Gamma$ be positive radial functions as in Lemma 2.1. Let $W_k := \text{span}\{\omega_1, \dots, \omega_k\}$ be the vector space generated by $\omega_1, \dots, \omega_k$. Thus, $\dim(W_k) = k$ and, for each $k = 1, \dots, m$, one has that

$$\max_{W_k} J \leq \sum_{i=1}^k \max_{t>0} J(t\omega_i) = kc(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}}) \leq \ell_0 c_\infty.$$

Assume that $\ell > \ell_0$. Then $\max_{W_k} J < \ell c_\infty$ and hence Lemma 4.1 guarantees that $\inf_{\mathcal{N}(\Omega)^\Gamma} J$ is attained at a positive solution $u_1 \in \mathcal{N}(\Omega)^\Gamma$ with $J(u_1) \leq c(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}})$. Moreover, Theorem 3.7 and Lemma 4.1 yield the existence of $m-1$ pairs of sign changing critical points $\pm u_2, \dots, \pm u_m \in \mathcal{N}(\Omega)^\Gamma$ of J with

$$J(u_k) \leq kc(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}}), \quad k = 2, \dots, m,$$

as claimed. \square

Recently Musso and Pistoia [14] obtained a multiplicity result in domains with a small spherical hole, which are invariant under the reflections $\tau_i(x_1, \dots, x_i, \dots, x_N) = (x_1, \dots, -x_i, \dots, x_N)$. They showed that, as the radius of the spherical hole goes to 0, the number of solutions goes to ∞ . Theorem 1.2 does not require the hole to be spherical but the number of solutions obtained depends on the symmetry group Γ , and not only on the size of the hole.

Musso and Pistoia gave also a precise description of their solutions. They showed that they are *bubble-towers*, that is, as the radius of the hole goes to 0, they approach an alternating sums of instantons (4.1) with appropriate dilation coefficients. The

following proposition gives some information on the shape of the solutions provided by Theorem 1.2, as the diameter of the hole goes to 0.

Let Γ be a closed subgroup of $O(N)$, $\ell := \ell(\Gamma)$, and let $\tilde{\Omega} \subset \mathbb{R}^N$ be a Γ -invariant domain such that $\{x \in \mathbb{R}^N : |x| \leq 1\} \subset \tilde{\Omega}$. Let Θ_n be a closed Γ -invariant neighborhood of the origin in \mathbb{R}^N . Set $\Omega_n := \tilde{\Omega} \setminus \Theta_n$, and denote by d_n be the diameter of Ω_n .

Proposition 4.2. *Assume that $\ell := \ell(\Gamma) \geq 2$, that $\tilde{\Omega}$ is strictly starshaped with respect to the origin and that $d_n \rightarrow 0$. Let $m_0 < \ell + 1$, and let $u_n \in H_0^1(\Omega_n)^\Gamma$ be a sign changing solution of (\wp_{Ω_n}) with $J(u_n) \leq m_0 c_\infty + \frac{1}{n}$. Then, for some subsequence of (u_n) still denoted by (u_n) , there exists a positive integer $m \leq m_0$, nontrivial Γ -invariant solutions W_1, \dots, W_m of problem $(\wp_{\mathbb{R}^N})$, and sequences of positive numbers $(\varepsilon_{j,n})$, $j = 1, \dots, m$, such that*

$$(4.2) \quad \varepsilon_{j,n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$$(4.3) \quad \left\| u_n - \sum_{j=1}^m \varepsilon_{j,n}^{\frac{2-N}{2}} W_j \left(\frac{\cdot}{\varepsilon_{j,n}} \right) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$(4.4) \quad \sum_{j=1}^m J_{\mathbb{R}^N}(W_j) \leq m_0 c_\infty.$$

In particular, if $m_0 = 2$, then W_1 and $-W_2$ are instantons.

Proof. Since the space of test functions $C_c^\infty(\tilde{\Omega} \setminus \{0\})$ with compact support in $\tilde{\Omega} \setminus \{0\}$ is dense in $H_0^1(\tilde{\Omega})$, the sequence (u_n) is a Palais-Smale sequence for $J_{\tilde{\Omega}}$ in $\mathcal{N}(\tilde{\Omega})^\Gamma$ with $J_{\tilde{\Omega}}(u_n) \rightarrow c \leq m_0 c_\infty < (\ell + 1)c_\infty$. Since $(\wp_{\tilde{\Omega}})$ has no nontrivial solution, the representation theorem in [4] asserts that there exist closed subgroups $\Gamma_1, \dots, \Gamma_m$ of finite index in Γ and, for each $j = 1, \dots, m$, a sequence of positive numbers $(\varepsilon_{j,n})$, a sequence of points $(y_{j,n})$ in $\tilde{\Omega}$, and a Γ_j -invariant solution W_j of problem $(\wp_{\mathbb{R}^N})$, such that $\#\Gamma y_{j,n} = \#(\Gamma/\Gamma_j)$,

$$(4.5) \quad \varepsilon_{j,n}^{-1} \text{dist}(y_{j,n}, \partial\tilde{\Omega}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$$(4.6) \quad \left\| u_n - \sum_{j=1}^m \sum_{[\gamma] \in \Gamma/\Gamma_j} \varepsilon_{j,n}^{\frac{2-N}{2}} W_j \left(\gamma^{-1} \left(\frac{\cdot - \gamma y_{j,n}}{\varepsilon_{j,n}} \right) \right) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$(4.7) \quad \lim_{n \rightarrow \infty} J_{\tilde{\Omega}}(u_n) = \sum_{j=1}^m \#(\Gamma/\Gamma_j) J_{\mathbb{R}^N}(W_j).$$

If $\Gamma_j \neq \Gamma$ for some j then, since u_n is sign changing and since $J_{\mathbb{R}^N}(W) > 2c_\infty$ for every sign changing solution of $(\wp_{\mathbb{R}^N})$, we would have that

$$\sum_{j=1}^m \#(\Gamma/\Gamma_j) J_{\mathbb{R}^N}(W_j) \geq (\ell + 1)c_\infty,$$

a contradiction. Therefore, $\Gamma_j = \Gamma$ for all j and, since $\#\Gamma y \geq \ell \geq 2$ for every $y \in \tilde{\Omega} \setminus \{0\}$, we have $y_{j,n} = 0$ for all n, j . Assertions (4.2), (4.3), and (4.4), and our claim that $m \leq m_0$ follow immediately from (4.5), (4.6), and (4.7). \square

We believe that the solutions are bubble-towers, that is, that W_j is always an instanton (up to sign), but Proposition 4.2 yields this only for $m_0 = 2$.

5. KELVIN INVARIANT SOLUTIONS

Let $r^2 := R_1 R_2$, and consider the inversion

$$\phi_r : \mathbb{R}^N \cup \{\infty\} \rightarrow \mathbb{R}^N \cup \{\infty\}, \quad \phi_r x := \frac{r^2 x}{|x|^2},$$

in the sphere $S_r(0)$. We shall say that Ω is ϕ_r -invariant if $\phi_r x \in \Omega$ for every $x \in \Omega$. Note that the annulus A_{R_1, R_2} is ϕ_r -invariant. A function $u : \Omega \rightarrow \mathbb{R}$ will be said to be ϕ_r -symmetric if it coincides with its Kelvin transform, i.e.

$$u(x) = \frac{r^{2N-4}}{|x|^{N-2}} u(\phi_r x) \quad \text{for every } x \in \Omega,$$

and we shall say that it is ϕ_r -antisymmetric if

$$u(x) = -\frac{r^{2N-4}}{|x|^{N-2}} u(\phi_r x) \quad \text{for every } x \in \Omega.$$

Arguing in a similar way as we did to prove Theorem 1.2 we obtain the following.

Theorem 5.1. *Given $\delta > 0$ there exists R_δ with the following property: for every closed subgroup Γ of $O(N)$ with $\ell = \ell(\Gamma) \geq 2$ and every Γ -invariant domain Ω such that*

$$0 \notin \Omega, \quad \Omega \supset \{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \quad \text{and} \quad 0 < R_1/R_2 < R_\delta,$$

which is also ϕ_r -invariant with $r^2 := R_1 R_2$, problem (\wp_Ω) has at least

- (a) one positive Γ -invariant ϕ_r -symmetric solution v_1 if $\ell \geq 3$,
- (b) $\lceil \frac{\ell+1}{2} \rceil - 1$ pairs of sign changing Γ -invariant ϕ_r -symmetric solutions $\pm v_2, \dots, \pm v_{\lceil \frac{\ell+1}{2} \rceil}$, and
- (c) $\ell + 1$ pairs of Γ -invariant ϕ_r -antisymmetric solutions $\pm \hat{u}_1, \dots, \pm \hat{u}_{\ell+1}$, which satisfy

$$\begin{aligned} \int |\nabla v_j|^2 &\leq 2j S^{N/2} + \delta, \quad j = 1, \dots, \left\lceil \frac{\ell+1}{2} \right\rceil, \\ \int |\nabla \hat{u}_j|^2 &\leq 2j S^{N/2} + \delta, \quad j = 1, \dots, \ell + 1. \end{aligned}$$

Proof of Theorem 5.1. Let $\ell \geq 3$. We assume that $\delta < \varepsilon_0$, with $\varepsilon_0 \in (0, c_\infty)$ as in Lemma 4.1. Let $m := \lceil \frac{\ell+1}{2} \rceil$. By (2.5) there is a $R_\delta > 0$ such that

$$c(r^{\frac{1}{m}}, R_2^{\frac{1}{m}}) < c_\infty + \frac{\delta}{2m} \quad \text{if } r/R_2 < R_\delta.$$

By Lemma 2.1, there are positive radial functions $\omega_1, \dots, \omega_m \in \mathcal{N}(A_{r, R_2})^\Gamma$ with pairwise disjoint supports, such that $J(\omega_j) = c(r^{\frac{1}{m}}, R_2^{\frac{1}{m}})$. Define

$$\tilde{\omega}_k(x) := \begin{cases} \omega_k(x) & \text{if } |x| \geq r \\ \frac{r^{2N-4}}{|x|^{N-2}} \omega_k(\phi_r x) & \text{if } |x| \leq r. \end{cases}$$

Then $\tilde{\omega}_k \in \mathcal{N}(A_{R_1, R_2})^G$, where G is the subgroup of $GM(\mathbb{R}^N \cup \{\infty\})$ generated by $\Gamma \cup \{\phi_r\}$. Let $\tilde{W}_k := \text{span}\{\tilde{\omega}_1, \dots, \tilde{\omega}_k\}$. Then, $\dim(\tilde{W}_k) = k$ and, for each $k = 1, \dots, m$,

$$\max_{\tilde{W}_m} J \leq \sum_{i=1}^k \max_{t>0} J(t\tilde{\omega}_i) = 2kc(R_1^{\frac{1}{\ell+1}}, R_2^{\frac{1}{\ell+1}}) \leq 2kc_\infty + \delta < (\ell+1)c_\infty + \varepsilon_0.$$

Since $\ell \geq 3$, we have that $J(\tilde{\omega}_1) \leq 2c_\infty + \delta < 3c_\infty$. So, by Lemma 4.1, $\inf_{\mathcal{N}(\Omega)^G} J$ is attained at a positive solution $v_1 \in \mathcal{N}(\Omega)^G$ with $J(v_1) \leq 2c_\infty + \delta$. This proves (a). Furthermore, Lemma 4.1 and Theorem 3.7 yield the existence of $m-1$ pairs of sign changing critical points $\pm v_2, \dots, \pm v_m \in \mathcal{N}(\Omega)^G$ with

$$J(v_k) \leq 2kc_\infty + \delta, \quad k = 2, \dots, m := \left\lfloor \frac{\ell+1}{2} \right\rfloor,$$

as claimed in (b).

To prove (c) we recall that our problem is Möbius invariant, that is, u is a Γ -invariant solution in $\{x \in \Omega : |x| > r\}$ if and only if u_{ϕ_r} is a Γ -invariant solution in $\{x \in \Omega : |x| < r\}$, where

$$u_{\phi_r}(x) := \frac{r^{2N-4}}{|x|^{N-2}} u(\phi_r x).$$

Since the normal derivatives of u and $-u_{\phi_r}$ coincide at each point $x \in S_r(0)$, the function

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } |x| \geq r \\ -u_{\phi_r}(x) & \text{if } |x| \leq r \end{cases}$$

is a Γ -invariant ϕ_r -antisymmetric solution of (\varnothing_Ω) . Claim (c) now follows from Theorem 1.2 if $\ell \geq 2$. For $\ell = 1$ it follows from the main theorem in [7] which asserts that there are at least two pairs of solutions in $\{x \in \Omega : |x| > r\}$ if r/R_2 is small enough. \square

A similar result can be obtained for domains with large holes, using the same arguments as for Theorem 1.3. We omit the details.

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