Minimal nodal solutions of the pure critical exponent problem on a symmetric domain

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Abstract

We establish existence of nodal solutions to the pure critical exponent problem $-\Delta u = |u|^{2^*-2} u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω a bounded smooth domain which is invariant under an orthogonal involution of $\mathbb{R}^N$. We extend previous results for positive solutions due to Coron, Dancer, Ding, and Passaseo to existence and multiplicity results for solutions which change sign exactly once.

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1 Introduction

Consider the problem

$$(\wp) \qquad \begin{cases} -\Delta u = | u |^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

In the last twenty years there has been a great amount of activity in the study of this problem and much progress has been made concerning the

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existence of positive solutions. It is known that there is an effect of the domain topology on the existence of positive solutions. A first result in this direction is due to Coron [10], who showed the existence of a positive solution in a domain with a small hole. The most remarkable result was obtained by Bahri and Coron [3], who showed that $(\wp)$ has a positive solution if Ω has nontrivial topology. On the other hand, Dancer [11], Ding [13], and Passaseo [17, 18], among others, have shown the existence of positive solutions in some contractible domains.

In contrast to the achievements on positive solutions, very little progress has been made concerning the existence of sign changing solutions. Besides the difficulty posed by the lack of compactness, an additional difficulty shows up in this last case: Unlike what occurs for positive solutions, there is no natural regular constraint for sign changing solutions of problem $(\wp)$. We shall present an approach which allows us overcome this difficulty and to obtain existence and multiplicity results for sign changing solutions in domains with small holes and in contractible domains, similar to those obtained in [10, 11, 13, 17, 18] for positive solutions.

We assume that Ω is invariant under a nontrivial orthogonal involution τ of $\mathbb{R}^N$, that is, τ is an orthogonal linear transformation of $\mathbb{R}^N$ such that $\tau \neq I$ and $\tau^2 = I$ (I being the identity of $\mathbb{R}^N$), and Ω satisfies $\tau x \in \Omega$ iff $x \in \Omega$. A solution u of $(\wp)$ will be said to be *τ-symmetric* if $u \circ \tau = u$, and it will be called *τ-antisymmetric* if $u \circ \tau = -u$. Note that nontrivial τ-antisymmetric solutions are nodal solutions. A nodal solution u is said to change sign exactly once if $\Omega \backslash u^{-1}(0)$ has precisely two connected components and u is positive in one of them and negative in the other. Nodal solutions which change sign exactly once will be called *minimal nodal solutions.*

Before stating our results we recall some notions and fix some notation.

Given a domain U of $\mathbb{R}^N$ and a subset K of U the *capacity of K with respect to U* is defined as

$$cap_U K = \inf \left\{ \int_U |\nabla u|^2 : u \in D_0^{1,2}(U) \text{ and } u \geq 1 \text{ on } K \right\},$$

where, as usual, $D_0^{1,2}(U)$ denotes the completion of the space $C_0^\infty(U)$ with respect to the norm $\|u\|_{D^{1,2}} = \left(\int_U |\nabla u|^2 \right)^{1/2}$. If the closed convex set $\left\{ u \in D_0^{1,2}(U) : u \geq 1 \text{ on } K \right\}$ is nonempty, $cap_U K$ is uniquely achieved at a $\psi \in D_0^{1,2}(U)$ which satisfies $\psi = 1$ on K [18]. Thus, if K and U are τ-invariant, then ψ is τ-symmetric.

Let $\widetilde{\Omega}, \Omega, M, M_0$ be τ-invariant subsets of $\mathbb{R}^N$ such that $M_0 \subset M \cap \Omega$ and $M \cup \Omega \subset \widetilde{\Omega}$. We say that M is *τ-deformable into Ω in $\widetilde{\Omega}$ rel. M_0* if there is a homotopy

$$\Theta : M \times [0,1] \to \widetilde{\Omega}$$

such that $\Theta(\tau y, t) = \tau\Theta(y, t)$, $\Theta(y, 0) = y$, $\Theta(y, 1) \in \Omega$, and $\Theta(z, t) = z$ for every $y \in M$, $z \in M_0$, $t \in [0, 1]$.

Given $R > 0$ and a nonempty subset X of $\mathbb{R}^N$ we write

$$X_R = \{x \in \mathbb{R}^N : dist(x, X) \leq R\}.$$

For the empty subset we set $\emptyset_R := \emptyset$. If X is τ-invariant, we denote by

$$X^\tau = \{x \in X : \tau x = x\}$$

the τ-fixed point set of X.

We shall prove the following results.

Theorem 1. *Let $\widetilde{\Omega}$ be a τ-invariant bounded smooth domain in $\mathbb{R}^N$ and let M be a compact τ-invariant submanifold of $\widetilde{\Omega}$ (with or without boundary) such that $M^\tau = \emptyset$. Let $R > 0$. Then there is an $\varepsilon = \varepsilon(\widetilde{\Omega}, M, R) > 0$ such that, for every τ-invariant bounded smooth domain Ω which satisfies*

a) $(\partial M)_R \subset \Omega \subset \widetilde{\Omega}$ and $cap_{\mathbb{R}^N \setminus (\partial M)_R}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$,

b) M is not τ-deformable into Ω in $\widetilde{\Omega}$ rel. ∂M,

problem $(\wp)$ has at least one pair of τ-antisymmetric minimal nodal solutions.

A similar result for positive solutions (with $\partial M = \emptyset$) was proved by Passaseo in [18].

Theorem 2. *Let $\widetilde{\Omega}$ be a τ-invariant bounded smooth domain in $\mathbb{R}^N$, $x_+, x_- \in \widetilde{\Omega}^\tau$ (not necessarily distinct) and $R > 0$. Let $C(x_+)$ and $C(x_-)$ denote the closures of the connected components of $\widetilde{\Omega}^\tau$ which contain x_+ and x_-, respectively. Then there is an $\varepsilon = \varepsilon(\widetilde{\Omega}, R) > 0$ such that, for every τ-invariant bounded smooth domain $\Omega \subset \widetilde{\Omega}$ which satisfies*

a) $cap_{\mathbb{R}^N}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$,

b) $C(x_+) \cup C(x_-)$ does not intersect $\overline{\Omega}$,

problem $(\wp)$ has at least one pair $\pm u$ of τ-symmetric minimal nodal solutions such that the barycenter of u^+ lies in $C(x_+)_R$ and the barycenter of u^- lies in $C(x_-)_R$.

Here we set as usual $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Let us consider some examples. We denote by $B(x, r)$ is the open ball of radius r and center x in $\mathbb{R}^N$.

Example 3. *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ which is symmetric with respect to the origin (i.e. $x \in \widetilde{\Omega}$ iff $-x \in \widetilde{\Omega}$), and let $0 \neq \xi \in \widetilde{\Omega}$. Then, for ε small enough, problem $(\wp)$ has a pair of minimal nodal odd solutions on $\Omega^\varepsilon = \widetilde{\Omega} \setminus (\overline{B(\xi, \varepsilon)} \cup \overline{B(-\xi, \varepsilon)})$.*

Indeed, if $0 < d < dist(\xi, \partial\widetilde{\Omega} \cup \{0\})$, then $M = \overline{B(\xi, d)} \cup \overline{B(-\xi, d)}$, $R < d$ and $\tau = -I$ satisfy the assumptions of Theorem 1.

Example 4. *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ which is symmetric with respect to the origin such that $0 \in \widetilde{\Omega}$. Then, for ε small enough, problem $(\wp)$ has a pair of minimal nodal even solutions on $\Omega^\varepsilon = \widetilde{\Omega} \setminus \overline{B(0, \varepsilon)}$.*

This follows from Theorem 2 with $\tau = -I$ and $x_+ = x_- = 0$.

Example 5. *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ which is symmetric with respect to the origin and let $0 \neq \xi \in \widetilde{\Omega}$. Then, for ε small enough, problem $(\wp)$ has a pair of minimal nodal odd solutions on $\Omega^\varepsilon = \widetilde{\Omega} \setminus \{x \in \mathbb{R}^N : dist(x, \mathbb{R}\xi) \leq \varepsilon\}$, where $\mathbb{R}\xi$ is the subspace generated by ξ. If $0 \in \widetilde{\Omega}$ then Ω^ε also contains a pair of minimal even solutions.*

The first assertion follows from Theorem 1 with $\tau = -I$, $M = \{x \pm \xi \in \mathbb{R}^N : x \cdot \xi = 0, |x| \leq d\}$, $0 < 2R < 2d < dist(\xi, \partial\widetilde{\Omega})$. The second one follows from Theorem 2 with $\tau = -I$ and $x_+ = x_- = 0$.

Example 6. *Let $\widetilde{\Omega} = B(0, 2) \setminus \overline{B(0, \frac{1}{2})}$. Then, for ε small enough, problem $(\wp)$ has a pair of minimal nodal τ-symmetric solutions on $\Omega^\varepsilon = \widetilde{\Omega} \setminus \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| \leq \varepsilon, x_N > 0\}$ where $\tau(x', x_N) = (-x', x_N)$.*

This follows from Theorem 2 with $x_+ = x_- = e_N := (0, ..., 0, 1)$. Observe that Ω^ε has τ-fixed points (but they are not in the component of $\overline{\widetilde{\Omega}}^\tau$ which contains e_N!) and that Ω^ε is τ-contractible.

Example 7. *Let $k \in \mathbb{N}$, let $\widetilde{\Omega} = B(0, k) \setminus \bigcup_{m=0}^{k-1} \overline{B(me_N, \frac{1}{4})}$. Then, for ε small enough, problem $(\wp)$ has $\frac{k(k+1)}{2}$ pairs of minimal nodal τ-symmetric solutions $\pm u_{lm}$, $1 \leq l \leq m \leq k$ on $\Omega^\varepsilon = \widetilde{\Omega} \setminus \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| \leq \varepsilon,\ x_N > 0\}$ where τ is as in Example 6. The barycenter of u^+_{lm} lies in $\mathbb{R}^{N-1} \times (l-1, l)$, and the barycenter of u^-_{lm} lies in $\mathbb{R}^{N-1} \times (m-1, m)$.*

This follows by applying Theorem 2 with $x_+ = (l - \frac{1}{2})e_N$, $x_- = (m - \frac{1}{2})e_N$ and $R < \frac{1}{4}$ for each pair (l, m). Again observe that Ω^ε is τ-contractible.

In all examples above the existence of one or more positive solutions is known (see [10, 11, 13, 17, 18]). Marchi and Pacella [16] claimed the existence of a nodal solution for Example 6 but there is a gap in their proof. This gap is a common in the literature (see e.g. [5, 6, 12, 21, 23] and [8, Chapter 8]). Basically, it consists in assuming that the natural constraint

$$\mathcal{E}(\Omega) = \{u \in H_0^1(\Omega) : u^\pm \neq 0, \ \int_\Omega \left|\nabla u^\pm\right|^2 = \int_\Omega \left|u^\pm\right|^{2^*}\}$$

for sign changing solutions of $(\wp)$ is a differentiable manifold. Unfortunately this is not true, since the functionals $u \mapsto \int_\Omega |\nabla u^\pm|^2$ are not differentiable on $H_0^1(\Omega)$. However, if the domain is τ-invariant there is a natural constraint for τ-antisymmetric solutions which is indeed a differentiable manifold (see [7]). This fact will be used in the following section to prove Theorem 1. For τ-symmetric nodal solutions there is no natural regular constraint. Theorem 2 will be proved via a careful linking argument combined with a degree argument which allows us to keep track of the barycenter of the solutions. This will be done in section 3. We believe that our methods may be applied to prove the results for nodal solutions in [5, 6, 23] and [8, Chapter 8].

We would like to make the reader aware of the fact that $\mathcal{E}(\Omega) \cap H^2(\Omega)$ is a differentiable manifold, even though $\mathcal{E}(\Omega)$ is not. This was shown in [4] and applied to obtain information on the Morse index of minimal nodal solutions. However, this fact seems to be of no help for finding critical points.

Finally we mention some related work. If the domain is invariant under an additional group action which provides compactness, Hebey and Vaugon [14, Proposition 3] showed the existence of infinitely many τ-antisymmetric solutions.

Recently Bahri and Chanillo [2] studied nodal solutions of the pure Yamabe problem on the standard three-sphere. Their main results involve a description of the difference in topology near the critical points at infinity when the solutions are close to the space generated by two fundamental masses.

2 τ-antisymmetric solutions

The solutions of problem ($\wp$) are the critical points of the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}.$$

defined on $H_0^1(\Omega)$. We write

$$\|u\|^2 = \int_\Omega |\nabla u|^2 \quad \text{and} \quad |u|_{2^*}^{2^*} = \int_\Omega |u|^{2^*}.$$

The nontrivial critical points of E lie on the Nehari manifold

$$\mathcal{N}(\Omega) = \{u \in H_0^1(\Omega) : u \neq 0,\ DE(u)u = 0\} \tag{1}$$

where DE is the derivative of E. This is a manifold of class $C^{1,1}$ which is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)$ [22, Lemma 4.1]. We denote by $\rho : H_0^1(\Omega)\backslash\{0\} \to \mathcal{N}(\Omega)$ the radial projection, thus,

$$\rho(u) = \left(\frac{\|u\|^{(N-2)/2}}{|u|_{2^*}^{N/2}}\right) u.$$

The involution τ of Ω induces two orthogonal involutions of $H_0^1(\Omega)$ as follows: For each $u \in H_0^1(\Omega)$ we define $\tau_s u, \tau_a u \in H_0^1(\Omega)$ by

$$(\tau_s u)(x) = u(\tau x) \quad \text{and} \quad (\tau_a u)(x) = -u(\tau x).$$

The τ-symmetric solutions of problem ($\wp$) are the critical points of E which lie in the τ_s-fixed point space

$$\mathcal{S}H_0^1(\Omega) = \{u \in H_0^1(\Omega) : \tau_s u = u\}, \tag{2}$$

and the τ-antisymmetric solutions are those which lie in the τ_a-fixed point space

$$\mathcal{A}H_0^1(\Omega) = \{u \in H_0^1(\Omega) : \tau_a u = u\}. \tag{3}$$

Notice that $\nabla E(u) \in \mathcal{S}H_0^1(\Omega)$ if $u \in \mathcal{S}H_0^1(\Omega)$, and that $\nabla E(u) \in \mathcal{A}H_0^1(\Omega)$ if $u \in \mathcal{A}H_0^1(\Omega)$.

The rest of this section will be devoted to the existence of τ-antisymmetric solutions. The nontrivial τ-antisymmetric solutions of ($\wp$) are the critical points of the restriction of E to the τ-antisymmetric Nehari manifold

$$\mathcal{A}\mathcal{N}(\Omega) = \{u \in \mathcal{N}(\Omega) : \tau_a u = u\} = \mathcal{N}(\Omega) \cap \mathcal{A}H_0^1(\Omega).$$

Notice that every nontrivial τ-antisymmetric solution is a nodal solution. It is not hard to see that

$$\inf_{\mathcal{AN}(\Omega)} E = \frac{2}{N} S^{N/2}$$

where S is the best Sobolev constant. This infimum does not depend on Ω and therefore it is not achieved on a bounded domain. Moreover, the restriction of E to $\mathcal{AN}(\Omega)$ satisfies the Palais-Smale condition at every level $c \in (\frac{2}{N}S^{N/2}, \frac{4}{N}S^{N/2})$, and a solution $u \in \mathcal{AN}(\Omega)$ such that $E(u) < \frac{4}{N}S^{N/2}$ changes sign exactly once (see e.g. [7]).

Proof of Theorem 1. Without loss of generality we may assume that the given $R > 0$ is such that $(\partial M)_R \cap M$ is a τ-invariant tubular neighborhood of ∂M in M parametrized by a homeomorphism $h : \partial M \times [0,2] \cong (\partial M)_R \cap M$ such that $h(z,0) = z$ and $h(\tau z, s) = \tau h(z,s)$ for $z \in \partial M,\ s \in [0,2]$. We fix $r > 0$ such that $dist(M, \partial\widetilde{\Omega}) > r$, $|x - \tau x| > 2r$ for every $x \in M$, $dist(h(\partial M \times [0,1]), \mathbb{R}^N \setminus (\partial M)_R) > r$, and the inclusion $\widetilde{\Omega} \hookrightarrow \widetilde{\Omega}_r = \{x \in \mathbb{R}^N : dist(x, \widetilde{\Omega}) \leq r\}$ is a τ-homotopy equivalence. Consider the barycenter map $\beta : H_0^1(\widetilde{\Omega})\backslash\{0\} \to \mathbb{R}^N$,

$$\beta(u) = \frac{\int_{\mathbb{R}^N} x\, |u(x)|^{2^*}\, dx}{\int_{\mathbb{R}^N} |u(x)|^{2^*}\, dx}, \tag{4}$$

and choose $0 < \kappa < \frac{1}{2N}S^{N/2}$ such that

$$\beta(u^+) \in \widetilde{\Omega}_r \quad \text{for every } u \in \mathcal{AN}(\widetilde{\Omega}) \text{ with } E(u) < \frac{2}{N} S^{\frac{N}{2}} + 4\kappa$$

(see [7, Lemma 14]). Note that $\beta((-u)^+) = \tau\beta(u^+)$ if $u \in \mathcal{AN}(\widetilde{\Omega})$. Fix a radially symmetric cut-off function $\varphi \in C_c^\infty(B(0,r))$ and choose $\lambda > 0$ such that

$$w_\lambda := \rho\,(U_\lambda \varphi) \in \mathcal{N}(B(0,r))$$

satisfies $E(w_\lambda) < \frac{1}{N}S^{N/2} + \kappa$, where $U_\lambda \in D^{1,2}(\mathbb{R}^N)$ is the Aubin-Talenti instanton [1, 20] defined by

$$U_\lambda(x) = \left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{\frac{N-2}{2}} \tag{5}$$

and ρ is the radial projection onto the Nehari manifold. We fix $\delta > 0$ with the property that

$$E(\rho(u)) < \frac{1}{N} S^{N/2} + 2\kappa$$

for every $u \in H_0^1(\widetilde{\Omega})$ with $\inf\{\|u - w_\lambda(\cdot - y)\| : y \in M\} < \delta$.

Let Ω be a bounded smooth domain with $(\partial M)_R \subset \Omega \subset \widetilde{\Omega}$, and let ψ_Ω be the unique element in $D_0^{1,2}(\mathbb{R}^N \setminus (\partial M)_R)$ such that $\psi_\Omega = 1$ on $\widetilde{\Omega} \setminus \Omega$ and

$$\|\psi_\Omega\|^2 = cap_{\mathbb{R}^N \setminus (\partial M)_R}(\widetilde{\Omega} \setminus \Omega).$$

Then there is an $\varepsilon > 0$ such that

$$\sup_{y \in M} \|\psi_\Omega w_\lambda(\cdot - y)\| < \delta \quad \text{ if } \; cap_{\mathbb{R}^N \setminus (\partial M)_R}(\widetilde{\Omega} \setminus \Omega) < \varepsilon.$$

We fix such an ε and assume that Ω satisfies $(\partial M)_R \subset \Omega \subset \widetilde{\Omega}$ and $cap_{\mathbb{R}^N \setminus (\partial M)_R}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$. For every $y \in M \setminus \partial M$, $t \in [0, 1]$, define

$$\vartheta(y,t) = \begin{cases} (1 - t\psi_\Omega)[w_\lambda(\cdot - y) - w_\lambda(\cdot - \tau y)] & \text{if } y \in M \setminus h(\partial M \times [0,1]) \\ w_{\lambda s}(\cdot - y) - w_{\lambda s}(\cdot - \tau y) & \text{if } y = h(z,s) \in h(\partial M \times (0,1]) \end{cases}$$

Observe that $w_{\lambda s}(\cdot - y)$ concentrates at y as $s \to 0$. Since

$$\sup_{y \in M} \|(1 - t\psi_\Omega) w_\lambda(\cdot - y) - w_\lambda(\cdot - y)\| \le \sup_{y \in M} \|\psi_\Omega w_\lambda(\cdot - y)\| < \delta,$$

we conclude that

$$E(\rho\vartheta(y,t)) < \frac{2}{N} S^{N/2} + 4\kappa < \frac{4}{N} S^{N/2}.$$

Notice that $\vartheta(y,t) \in \mathcal{A}H_0^1(\widetilde{\Omega})$, $\vartheta(\tau y, t) = -\vartheta(y,t)$, and $\vartheta(y,1) \in \mathcal{A}H_0^1(\Omega)$ for every $y \in M \setminus \partial M$, $t \in [0,1]$.

Now we argue by contradiction. Recall that the restriction of E to $\mathcal{AN}(\Omega)$ satisfies the Palais-Smale condition at every level $c \in (\frac{2}{N} S^{N/2}, \frac{4}{N} S^{N/2})$. Thus, if E has no critical point $u \in \mathcal{AN}(\Omega)$ with $E(u) < \frac{4}{N} S^{\frac{N}{2}}$, then we may use the negative gradient flow to obtain a deformation

$$\eta : (\mathcal{AN}(\Omega) \cap E^{\frac{2}{N} S^{N/2} + 4\kappa}) \times [0,1] \to \mathcal{AN}(\Omega) \cap E^{\frac{2}{N} S^{N/2} + 4\kappa}$$

such that $\eta(-u,t) = -\eta(u,t)$, $\eta(u,0) = u$, $\eta(u,1) \in E^{\frac{2}{N} S^{N/2} + \theta}$, and $\eta(u,t) = u$ if $E(u) \le \frac{2}{N} S^{N/2} + \frac{1}{2}\theta$, for any choice of $\theta > 0$. Here E^γ denotes, as usual, the sublevel set $E^\gamma = \{u \in H_0^1(\Omega) : E(u) \le \gamma\}$. We fix $d > 0$ such that

$\Omega \hookrightarrow \Omega_d$ is a τ-homotopy equivalence and choose θ such that $\beta(u^+) \in \Omega_d$ for every $u \in \mathcal{AN}(\Omega)$ with $E(u) \leq \frac{2}{N} S^{\frac{N}{2}} + \theta$. Then $\Theta : M \times [0,1] \to \widetilde{\Omega}_r$,

$$\Theta(y,t) = \begin{cases} \beta([\rho\vartheta(y,2t)]^+) & y \in M \setminus \partial M, \ 0 \leq t \leq \frac{1}{2} \\ \beta([\eta(\rho\vartheta(y,1), 2t-1)]^+) & y \in M \setminus \partial M, \ \frac{1}{2} \leq t \leq 1 \\ y & y \in \partial M \end{cases}$$

is continuous and it is a τ-deformation of M into Ω_d in $\widetilde{\Omega}_r$ rel.∂M, contradicting assumption *b)*. Therefore, E must have a critical point $\overline{u} \in \mathcal{AN}(\Omega)$ with $E(\overline{u}) < \frac{4}{N} S^{\frac{N}{2}}$. Such a $\overline{u}$ changes sign precisely once. ■

3 τ-symmetric solutions

In this section we prove Theorem 2. A critical point u of E in $H_0^1(\Omega)$ satisfies

$$DE(u)u^{\pm} = \left\|u^{\pm}\right\|^2 - \left|u^{\pm}\right|_{2^*}^{2^*} = 0.$$

Therefore nodal solutions of problem $(\wp)$ lie in the set

$$\mathcal{E}(\Omega) = \{u \in \mathcal{N}(\Omega) : u^+ \in \mathcal{N}(\Omega) \text{ and } u^- \in \mathcal{N}(\Omega)\},$$

where $\mathcal{N}(\Omega)$ is the Nehari manifold, see (1). The τ-symmetric solutions of $(\wp)$ lie in the space $\mathcal{S}H_0^1(\Omega)$ defined in (2). Observe that, if $u \in \mathcal{S}H_0^1(\Omega)$, then $u^+, u^- \in \mathcal{S}H_0^1(\Omega)$. So, if we write

$$\begin{aligned} \mathcal{SN}(\Omega) &= \{u \in \mathcal{N}(\Omega) : u \circ \tau = u\} = \mathcal{N}(\Omega) \cap \mathcal{S}H_0^1(\Omega) \\ \mathcal{SE}(\Omega) &= \{u \in \mathcal{SN}(\Omega) : u^+, u^- \in \mathcal{SN}(\Omega)\} = \mathcal{E}(\Omega) \cap \mathcal{S}H_0^1(\Omega), \end{aligned}$$

then nodal τ-symmetric solutions of problem $(\wp)$ lie in $\mathcal{SE}(\Omega)$. Neither $\mathcal{E}(\Omega)$ nor $\mathcal{SE}(\Omega)$ are differentiable manifolds, so they are not suitable restriction sets for our problem. We shall prove Theorem 2 via a careful linking argument on the whole of $\mathcal{S}H_0^1(\Omega)$ which allows us to keep track of the barycenter of the solutions.

Let $\widetilde{\Omega}$ be a τ-invariant domain, and let $\beta : H_0^1(\widetilde{\Omega}) \setminus \{0\} \to \mathbb{R}^N$ be the barycenter map defined in (4). We start with some lemmas.

Lemma 8. *Let (v_n) be a sequence in $\mathcal{E}(\widetilde{\Omega})$ and let (u_n) be a sequence in $H_0^1(\widetilde{\Omega})$ such that $E(v_n) \to \frac{2}{N} S^{N/2}$ and $\|v_n - u_n\| \to 0$. Then $|\beta(v_n^{\pm}) - \beta(u_n^{\pm})| \to 0$.*

Proof. Since $E(v_n) \to \frac{2}{N}S^{N/2}$ and $v_n^{\pm} \in \mathcal{N}(\widetilde{\Omega})$ it follows that $E(v_n^{\pm}) \to \frac{1}{N}S^{N/2}$. By a standard concentration compactness argument [15] there are sequences $y_n \in \widetilde{\Omega}$, $y_n \to y$, $\theta_n \in (0,\infty)$, $\theta_n \to 0$, such that the sequence $(\widetilde{v}_n)$ in $L^{2^*}(\mathbb{R}^N)$ given by

$$\widetilde{v}_n(x) = \theta_n^{(N-2)/2} v_n^+(\theta_n x - y_n)$$

converges to a ground state solution U of the limiting problem in $L^{2^*}(\mathbb{R}^N)$. Consider the sequence $(\widetilde{u}_n)$ in $L^{2^*}(\mathbb{R}^N)$ given by

$$\widetilde{u}_n(x) = \theta_n^{(N-2)/2} u_n^+(\theta_n x - y_n).$$

By translation and dilation invariance,

$$|\widetilde{v}_n - \widetilde{u}_n|_{2^*} = \left|v_n^+ - u_n^+\right|_{2^*} \leq |v_n - u_n|_{2^*} \to 0.$$

Hence $(\widetilde{u}_n)$ converges also to U in $L^{2^*}(\mathbb{R}^N)$. We conclude that $\beta(v_n^+) \to y$ and $\beta(u_n^+) \to y$, thus $|\beta(v_n^+) - \beta(u_n^+)| \to 0$. A similar argument gives $|\beta(v_n^-) - \beta(u_n^-)| \to 0$. □

Let $x_+, x_- \in \widetilde{\Omega}^\tau$ and $R > 0$ be as in Theorem 2. By taking a smaller R if necessary we may assume that

$$C(x_\pm)_R \cap \overline{\widetilde{\Omega}}^\tau \subset C(x_\pm). \tag{6}$$

The following is an immediate consequence of Lemma 8.

Corollary 9. *There exists $\kappa_1 > \frac{2}{N}S^{N/2}$ and $\alpha > 0$ such that, for every $v \in \mathcal{E}(\widetilde{\Omega})$ with $E(v) \leq \kappa_1$ and every $u \in H_0^1(\widetilde{\Omega})$ such that $\|v - u\| \leq \alpha$, there holds $|\beta(v^\pm) - \beta(u^\pm)| < \frac{R}{2}$.*

We put $A_\pm = C(x_\pm)_R \setminus C(x_\pm)_{\frac{R}{2}}$ and

$$\mathcal{S}_0 = \{v \in \mathcal{S}\mathcal{E}(\widetilde{\Omega}) : \text{either } \ \beta(v^+) \in A_+ \ \text{ or } \ \beta(v^-) \in A_-\}.$$

Then the following holds.

Lemma 10. $\kappa_2 := \inf\limits_{\mathcal{S}_0} E > \frac{2}{N}S^{N/2}$.

Proof. Assume there is a sequence $v_n \in \mathcal{S}_0$ such that $E(v_n) \to \frac{2}{N} S^{N/2}$. Then $v_n^{\pm} \in \mathcal{SN}(\widetilde{\Omega})$ and $E(v_n^{\pm}) \to \frac{1}{N} S^{N/2}$. By concentration compactness, passing to a subsequence, we have that $\beta(v_n^+) \to y_1$ and $\beta(v_n^-) \to y_2$ with $y_1, y_2 \in \overline{\widetilde{\Omega}}$. Without loss of generality we may assume that $\beta(v_n^+) \in A_+$ for all n. Notice that the barycenter of a τ-symmetric function is a fixed point of τ. Thus $y_1 \in \overline{A_+}^{\tau}$. This is a contradiction because $\overline{A_+}^{\tau}$ is empty by our choice of R. □

We set

$$\kappa = \min\{\kappa_1, \kappa_2, \frac{3}{N} S^{N/2}\}.$$

For each τ-invariant domain $\Omega \subset \widetilde{\Omega}$ let

$$\begin{aligned} \mathcal{S}_1(\Omega) &= \{v \in \mathcal{SE}(\Omega) : \beta(v^+) \in C(x_+)_{\frac{R}{2}}, \beta(v^-) \in C(x_-)_{\frac{R}{2}}\} \\ c(\Omega) &= \inf_{\mathcal{S}_1(\Omega)} E. \end{aligned}$$

Lemma 11. *There exists $\varepsilon > 0$ such that, for every τ-invariant subdomain $\Omega \subset \widetilde{\Omega}$ with $cap_{\mathbb{R}^N}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$, there holds $c(\Omega) < \kappa$.*

Proof. The case $C(x_+) \cap C(x_-) = \varnothing$ is easy. We restrict our attention to the more subtle case where $x_+ = x_- =: x_0$. Let $\mu = \kappa - \frac{2}{N} S^{N/2} > 0$. Choose $r_0 > 0$ such that $B(x_0, r_0) \subset \widetilde{\Omega}$. Fix a radially symmetric cut-off function $\varphi \in C_c^\infty(B(0, r_0))$ and $\lambda > 0$ such that

$$v_1 = \rho\,(U_\lambda \varphi) \in \mathcal{N}(B(0, r_0))$$

satisfies $E(v_1) < \frac{1}{N} S^{N/2} + \frac{\mu}{4}$, where U_λ is the Aubin-Talenti instanton defined in (5), and ρ is the radial projection onto the Nehari manifold. For $r \in (0, r_0)$ let $\psi_r \in H_0^1(B(0, r_0))$ be the unique function with $\psi_r = 1$ on $B(0, r)$ and

$$\|\psi_r\|^2 = cap_{B(0,r_0)}(B(0, r)).$$

Notice that ψ_r is radially symmetric and $\psi_r \to 0$ in $H_0^1(B(0, r_0))$ as $r \to 0$. Thus $\rho((1 - \psi_r)v_1) \to v_1$ in $H_0^1(B(0, r_0))$, and we may fix $r \in (0, r_0)$ such that

$$E(\rho((1 - \psi_r)v_1)) < \frac{1}{N} S^{N/2} + \frac{\mu}{3}.$$

Next consider $v_2 \in H_0^1(B(0, r_0))$ given by

$$v_2(x) = \left[\frac{r_0}{r}\right]^{(N-2)/2} v_1\left(\frac{r_0}{r} x\right).$$

By dilation invariance we have that $v_2 \in \mathcal{N}(B(0,r))$ and $E(v_2) = E(v_1) < \frac{1}{N}S^{N/2} + \frac{\mu}{4}$. Define $w_1, w_2 \in \mathcal{N}(\widetilde{\Omega})$ by

$$w_1(x) = \rho((1-\psi_r)v_1)(x - x_0), \qquad w_2(x) = v_2(x - x_0).$$

Then $w_i \in \mathcal{SN}(\widetilde{\Omega})$ and $E(w_i) < \frac{1}{N}S^{N/2} + \frac{\mu}{3}$ for $i = 1, 2$. Notice that w_1 and w_2 have disjoint supports. We now fix $\delta > 0$ such that

$$E(\rho(u)) < \frac{1}{N}S^{N/2} + \frac{\mu}{2} \quad \text{and} \quad \beta(u) \in C(x_0)_{\frac{R}{2}}$$

for every $u \in H_0^1(\widetilde{\Omega})$ with $\min\{\|u - w_1\|, \|u - w_2\|\} < \delta$.

Let $\Omega \subset \widetilde{\Omega}$ be a τ-invariant subdomain with $cap_{\mathbb{R}^N}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$, and let ψ_Ω be the unique element in $D^{1,2}(\mathbb{R}^N)$ with $\psi_\Omega = 1$ on $\widetilde{\Omega} \setminus \Omega$ and

$$\|\psi_\Omega\|^2 = cap_{\mathbb{R}^N}(\widetilde{\Omega} \setminus \Omega).$$

Note that ψ_Ω is τ-symmetric because $\widetilde{\Omega} \setminus \Omega$ is τ-invariant. For $\varepsilon > 0$ small enough,

$$\|\psi_\Omega w_i\| < \delta, \quad i = 1, 2.$$

Thus $u_i = \rho((1-\psi_\Omega)w_i) \in \mathcal{SN}(\Omega)$ satisfies $E(u_i) < \frac{1}{N}S^{N/2} + \frac{\mu}{2}$ and $\beta(u_i) \in C(x_0)_{\frac{R}{2}}$. Setting $u_0 = u_1 - u_2$ we observe that $u_0^+ = u_1$ and $u_0^- = -u_2$. Hence $u_0 \in \mathcal{S}_1(\Omega)$ and

$$c(\Omega) \le E(u_0) = E(u_1) + E(u_2) < \frac{2}{N}S^{N/2} + \mu = \kappa$$

as claimed. □

We now fix $\varepsilon > 0$ such that Lemma 11 holds. We also fix a smooth τ-invariant subdomain $\Omega \subset \widetilde{\Omega}$ with $cap_{\mathbb{R}^N}(\widetilde{\Omega} \setminus \Omega) < \varepsilon$. Then, under the assumptions of Theorem 2, the value $c(\Omega)$ has the following localized compactness property.

Proposition 12. *If $C(x_\pm) \cap \overline{\Omega} = \emptyset$, then every sequence (u_n) in $\mathcal{S}H_0^1(\Omega)$ such that $dist(u_n, \mathcal{E}(\Omega)) \to 0$, $E(u_n) \to c(\Omega)$, $\|DE(u_n)\| \to 0$ and $\beta(u_n^\pm) \in C(x_\pm)_R$ has a convergent subsequence.*

Proof. Asume that (u_n) has no convergent subsequence. Then, since $c(\Omega) < \frac{3}{N}S^{N/2}$, the symmetric compactness result in [9] implies that one of the following two cases must occur:

i) There are sequences $y_n \in \Omega$, $y_n \to y \in \overline{\Omega}^\tau$, $\theta_n \in (0,\infty)$, $\theta_n \to 0$, and a positive solution $\widehat{u}$ of problem $(\wp)$, such that

$$\left\| u_n \pm \left[\widehat{u} - \theta_n^{\frac{2-N}{2}} U\left(\theta_n^{-1}(\ \cdot \ - y_n)\right)\right]\right\| \longrightarrow 0 \quad \text{in } D^{1,2}(\mathbb{R}^N).$$

ii) There are sequences $y_{1,n}, y_{2,n} \in \Omega$ and $\theta_{1,n}, \theta_{2,n} \in (0,\infty)$ with $y_{i,n} \to y_i \in \overline{\Omega}^\tau$ and $\theta_{i,n} \to 0$ for $i = 1,2$, such that

$$\left\| u_n - \left[\theta_{1,n}^{\frac{2-N}{2}} U\left(\theta_{1,n}^{-1}(\ \cdot \ - y_{1,n})\right) - \theta_{2,n}^{\frac{2-N}{2}} U\left(\theta_{2,n}^{-1}(\ \cdot \ - y_{2,n})\right)\right]\right\| \longrightarrow 0 \ \text{ in } D^{1,2}(\mathbb{R}^N).$$

In both cases either $(\beta(u_n^+))$ or $(\beta(u_n^-))$ (or each one of them) converges to a point $y \in \overline{\Omega}^\tau \cap [C(x_+)_R \cup C(x_-)_R]$. However, (6) implies that

$$\overline{\Omega}^\tau \cap [C(x_+)_R \cup C(x_-)_R] = \overline{\Omega} \cap \overline{\widetilde{\Omega}}^\tau \cap [C(x_+)_R \cup C(x_-)_R] \subset \overline{\Omega} \cap [C(x_+) \cup C(x_-)]$$

which is empty by assumption. This is a contradiction. □

The main step for proving Theorem 2 is to show the existence of a sequence (u_n) as in Proposition 12. We start with the following technical lemma.

Lemma 13. *Let $v_0 \in \mathcal{E}(\Omega)$. There is a map $h : H_0^1(\Omega) \to \mathbb{R}^2$ such that*
i) $h(s_1 v_0^+ + s_2 v_0^-) = (s_1, s_2)$ for $s_1, s_2 \geq 0$,
ii) $h(u) = (1,1)$ if and only if $u \in \mathcal{E}(\Omega)$.

Proof. We first define $\sigma : H_0^1(\Omega) \to [0,\infty)$ by

$$\sigma(u) = \begin{cases} \frac{|u|_{2^*}^{2^*}}{\|u\|^2} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

Notice that σ is continuous and that $\sigma(u) = 1$ if and only if $u \in \mathcal{N}(\Omega)$. Note also that

$$s \longmapsto \xi_+(s) := \sigma(s v_0^+) \quad \text{and} \quad s \longmapsto \xi_-(s) := \sigma(s v_0^-)$$

are strictly increasing on $[0,\infty)$, $\xi_\pm(1) = 1$ and $\xi_\pm(s) \to \infty$ as $s \to \infty$. Hence $\xi_\pm^{-1} : [0,\infty) \to [0,\infty)$ are continuous. Now we define

$$h(u) = (\xi_+^{-1}\sigma(u^+), \xi_-^{-1}\sigma(u^-)).$$

It is straightforward to check that h has the desired properties. □

Proposition 14. *For every $\delta \in (0, \kappa - c(\Omega))$ and every $\alpha' \in (0, \alpha)$ there is a $u_0 \in \mathcal{S}H_0^1(\Omega)$ such that*

a) $dist(u_0, \mathcal{E}(\Omega)) \leq \alpha'$,
b) $E(u_0) \in [c(\Omega), c(\Omega) + \delta)$,
c) $\|DE(u_0)\| \leq \max\{\sqrt{\delta}, \frac{\delta}{\alpha'}\}$, and
d) $\beta(u_0^{\pm}) \in C(x_{\pm})_R$.

Proof. Fix $v_0 \in \mathcal{S}_1(\Omega)$ such that $E(v_0) < c(\Omega) + \delta$, and fix $\ell > 1$ such that $E(\ell v_0^{\pm}) \leq 0$. Let $h : H_0^1(\Omega) \to \mathbb{R}^2$ be as in Lemma 13. We put $K = [0, \ell] \times [0, \ell]$ and define

$$j : K \to \mathcal{S}H_0^1(\Omega), \quad j(s_1, s_2) = s_1 v_0^+ + s_2 v_0^-.$$

Then $h \circ j = id : K \to K$, in particular

$$\deg(h \circ j, K, (1, 1)) = 1. \tag{7}$$

Notice also that

$$E(j(s_1, s_2)) \leq E(v_0) < c(\Omega) + \delta \quad \text{for every } (s_1, s_2) \in K. \tag{8}$$

We now choose a Lipschitz continuous function $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi(s) = 1$ for $s \geq 0$ and $\chi(s) = 0$ for $s \leq -1$. Then, since $\nabla E(u) \in \mathcal{S}H_0^1(\Omega)$ for every $u \in \mathcal{S}H_0^1(\Omega)$, there is a global semiflow $\varphi : [0, \infty) \times \mathcal{S}H_0^1(\Omega) \to \mathcal{S}H_0^1(\Omega)$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t, u) = -\chi(E(\varphi(t, u)))\nabla E(\varphi(t, u)) \\ \varphi(0, u) = u \end{cases}$$

We will frequently write φ^t in place of $\varphi(t, \cdot)$. Since

$$E(v_0^{\pm}) < c(\Omega) + \delta - \frac{1}{N} S^{N/2} < \frac{2}{N} S^{N/2}$$

and $E(\ell v_0^{\pm}) \leq 0$, it follows that

$$\sup E(j(\partial K)) < \frac{2}{N} S^{N/2}.$$

Hence

$$(\varphi^t \circ j)(\partial K) \cap \mathcal{S}\mathcal{E}(\Omega) = \emptyset \quad \text{for every } t \geq 0$$

and, by Lemma 13, this implies

$$(h \circ \varphi^t \circ j)(y) \neq (1, 1) \quad \text{for every } y \in \partial K, \ t \geq 0.$$

Equality (7) and the global continuation principle of Leray-Schauder (see e.g. [24, p. 629]) imply that there exists a connected subset $Z \subset K \times [0,1]$ such that

$$\begin{aligned}&(1,1,0) \in Z\\ &\varphi^t(j(s_1,s_2)) \in \mathcal{SE}(\Omega) \quad \text{for every } (s_1,s_2,t) \in Z\\ &Z \cap (K \times \{1\}) \neq \emptyset.\end{aligned}$$

We put

$$\widetilde{Z} = \{\varphi^t(j(s_1,s_2)) \in \mathcal{SE}(\Omega) : (s_1,s_2,t) \in Z\}.$$

By inequality (8),

$$\sup E(\widetilde{Z}) < c(\Omega) + \delta < \kappa_2 = \inf E(\mathcal{S}_0).$$

So, since Z is connected, we obtain that $\widetilde{Z} \subset \mathcal{S}_1(\Omega)$. We now pick $(s_1,s_2,1) \in Z \cap (K \times \{1\})$ and write

$$v_1 := j(s_1,s_2), \qquad v_2 := \varphi^1(v_1).$$

Then $v_2 \in \widetilde{Z} \subset \mathcal{S}_1(\Omega)$. We distinguish two cases.

Case 1: $\|\varphi^t(v_1) - v_2\| \leq \alpha'$ for all $t \in [0,1]$. Then $\beta((\varphi^t(v_1))^\pm) \in C(x_\pm)_R$ for every $t \in [0,1]$ by Corollary 9. We choose $t_0 \in [0,1]$ with

$$\left\|\nabla E(\varphi^{t_0}(v_1))\right\| = \min_{0\leq t\leq 1} \left\|\nabla E(\varphi^t(v_1))\right\|$$

and put $u_0 = \varphi^{t_0}(v_1)$. Then

$$\delta \geq E(v_1) - E(v_2) = -\int_0^1 \frac{\partial}{\partial t} E(\varphi^t(v_1))dt = \int_0^1 \left\|\nabla E(\varphi^t(v_1))\right\|^2 dt \geq \|\nabla E(u_0)\|^2 .$$

Hence u_0 has the desired properties.

Case 2: There exists $t \in [0,1]$ such that $\|\varphi^t(v_1) - v_2\| > \alpha'$. Then let

$$t_1 = \sup\{s \geq t : \|\varphi^s(v_1) - v_2\| > \alpha'\}.$$

By Corollary 9, $\beta((\varphi^t(v_1))^\pm) \in C(x_\pm)_R$ for every $t \in [t_1,1]$. We choose $t_0 \in [0,1]$ with

$$\left\|\nabla E(\varphi^{t_0}(v_1))\right\| = \min_{t_1\leq t\leq 1} \left\|\nabla E(\varphi^t(v_1))\right\|$$

and put $u_0 = \varphi^{t_0}(v_1)$. Then

$$\alpha' \leq \int_{t_1}^{1} \left\| \frac{\partial}{\partial t} \varphi^t(v_1) \right\| dt \leq \int_{t_1}^{1} \left\| \nabla E(\varphi^t(v_1)) \right\| dt$$

and

$$\delta \geq E(\varphi^{t_1}(v_1)) - E(v_2) = \int_{t_1}^{1} \left\| \nabla E(\varphi^t(v_1)) \right\|^2 dt \geq \|\nabla E(u_0)\| \int_{t_1}^{1} \left\| \nabla E(\varphi^t(v_1)) \right\| dt.$$

We conclude that $\|\nabla E(u_0)\| \leq \frac{\delta}{\alpha'}$. Thus u_0 has the desired properties. $\square$

Proof of Theorem 2. For $\delta_n = \frac{1}{n^2}$ and $\alpha'_n = \frac{1}{n}$, n large enough, Proposition 14 yields a $u_n \in \mathcal{S}H_0^1(\Omega)$ such that the sequence (u_n) satisfies the hypotheses of Proposition 12. Therefore a subsequence converges in $H_0^1(\Omega)$ to a critical point $\overline{u} \in \mathcal{S}\mathcal{E}(\Omega)$ of E with $E(\overline{u}) = c(\Omega) < \frac{3}{N} S^{N/2}$. Hence, by standard arguments similar to those in [7], $\overline{u}$ has precisely two nodal domains. Moreover, $\beta(\overline{u}^{\pm}) \in C(x_{\pm})_R$, as required. $\blacksquare$

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