Multiple solutions of nonlinear scalar field equations

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Abstract

We establish existence of multiple solutions of the semilinear elliptic equation

$$(\wp) \qquad \begin{cases} -\Delta u + a(x)u = q(x)\,|u|^{p-2}\,u, & x \in \mathbb{R}^N, \\ u(x) \to 0 \ \text{ as } \ |x| \to \infty \end{cases}$$

where $N \geq 3$, $2 < p < \frac{2N}{N-2}$, and $a(x)$ and $q(x)$ tend to some positive limits a_∞ and q_∞ respectively as $|x| \to \infty$. Assuming some weak one-sided asymptotic estimates for $a(x)$ and $q(x)$, we prove that $(\wp)$ has at least $\frac{N}{2} + 1$ pairs $\pm u$ of nontrivial weak solutions.

1 Introduction and statement of results

We consider the superlinear elliptic problem

$$(\wp) \qquad \begin{cases} -\Delta u + a(x)u = q(x)\,|u|^{p-2}\,u \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

where $N \geq 3$ and $2 < p < 2^* := \frac{2N}{N-2}$. Equations of this form arise in various branches of mathematical physics and in some problems in biology as well,

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see for example [8, 14, 22, 23]. Solutions of ($\wp$) correspond, in particular, to standing waves of the nonlinear Schrödinger equation

$$i\frac{\partial \phi}{\partial t} - \Delta\phi + (a(x) + \omega)\phi - q(x)\,|\phi|^{p-2}\,\phi = 0,$$

and of the nonlinear Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi + (a(x) + \omega^2)\phi - q(x)\,|\phi|^{p-2}\,\phi = 0,$$

that is, to solutions of the form $\phi(x, t) = e^{i\omega t}u(x)$ which vanish at infinity.

We assume that a and q satisfy the following assumptions:

(A_1) $a, q \in L^\infty(\mathbb{R}^N)$ are positive functions, and $\inf_{x\in\mathbb{R}^N} a(x) > 0$.

(A_2) $\lim_{|x|\to\infty} a(x) = a_\infty$ and $\lim_{|x|\to\infty} q(x) = q_\infty$ for some $a_\infty, q_\infty > 0$.

The existence of solutions to ($\wp$) for this type of nonlinearities has been extensively studied during the last twenty years (for a detailed account we refer the reader to [3]). The main difficulty in dealing with this problem arises from the lack of compactness, which becomes clear when one looks at the special case

$$(\wp_\infty) \qquad \begin{cases} -\Delta u + a_\infty u = q_\infty\,|u|^{p-2}\,u \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

The set of solutions of ($\wp_\infty$) is invariant under translations and, therefore, it is not compact. If both a and q are radially symmetric, then compactness can be restored by looking only for radially symmetric solutions. In fact, the existence of infinitely many radially symmetric solutions has been established for quite general radially symmetric nonlinearities [6, 8, 11, 21].

The breakthrough in understanding the lack of compactness for this problem was achieved by P.-L. Lions [19]. His concentration compactness method lead to the existence of a ground state solution of ($\wp_\infty$), that is, a nontrivial solution u_∞ for which the action functional

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + a_\infty u^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} q_\infty|u|^p dx$$

is minimal, and provided a criterion for the existence of a ground state solution of $(\wp)$. Ding and Ni [13] established the existence of a ground state solution when $a \equiv 1$ and the infimum of q over $\mathbb{R}^N$ coincides with its limit at infinity.

The lack of compactness of $(\wp)$ is by now quite well understood. Assumption (A_2) allows to classify the Palais-Smale sequences of the action functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} q(x)|u|^p dx$$

associated to $(\wp)$ completely in terms of critical points of its limit functional J_∞ [19, 9, 3]. This fact was used by Bahri and Li [2] to show existence of a positive solution of $(\wp)$, with $a \equiv 1$, under additional assumptions on q for which there is no ground state solution. It has also been applied to show existence of a positive solution in domains other than $\mathbb{R}^N$, for example, by Bahri and Lions [3] who settled the case of exterior domains, under appropriate asymptotic conditions on q.

In contrast to the mere solvability of $(\wp)$ little is known about multiple solutions when a and q are not radially symmetric (or have no other appropriate symmetries as in [5]). For $N \geq 5$ and $a \equiv 1$ Zhu [25] showed the existence of two pairs of nontrivial solutions $\pm u_0, \pm u_1$ when, besides (A_1) and (A_2),

$$q(x) \geq (1 + C\,|x|^{-m})q_\infty \qquad \text{for } |x| \geq S_0$$

holds for some constants $m, C, S_0 > 0$. Some abstract conditions for the existence of multiple solutions were given by Li in [18]. Recently Hirano [16] applied a perturbative approach to treat the case where $a \equiv 1$ and

$$\sup_{\mathbb{R}^N} |q(x) - q_\infty| \quad \text{is sufficiently small.} \tag{1}$$

He proved that $(\wp)$ has at least three pairs of solutions provided that, in addition,

$$q(x) \geq (1 + Ce^{-\alpha|x|})q_\infty \qquad \text{for } |x| \geq S_0$$

holds for some constants $C, S_0 > 0$, $\alpha \in (0, 1)$.

Before stating our result we would like to mention that there are many multiplicity results for the singularly perturbed problem

$$-\varepsilon\Delta u + a(x)u = q(x)|u|^{p-2}u, \qquad x \in \mathbb{R}^N$$

for small $\varepsilon > 0$, see for example [1] and references therein. As $\varepsilon \to 0$ the solutions to this problem concentrate at some points whose location, as well as the number of solutions, depend on local geometric properties of a and q. A similar thing happens when $p \to 2^*$ [20]. In contrast to these phenomena, in the nonperturbative situation one cannot expect that local properties of a and q have an impact on the number of solutions of ($\wp$).

We now introduce our final assumption.

(A_3) There exist $S_0 > 0$, $C_1, C_2 \geq 0$ with $C_1 + C_2 > 0$, and $\alpha \in (0,2)$ such that

$$a(x) \leq a_\infty - C_1 e^{-\alpha\sqrt{a_\infty}|x|}, \qquad q(x) \geq q_\infty + C_2 e^{-\alpha\sqrt{a_\infty}|x|}, \qquad \text{for } |x| \geq S_0.$$

We write $c_\infty := J_\infty(u_\infty)$ for the minimal value of the action on the set of nontrivial solutions of ($\wp_\infty$). We shall prove the following.

Theorem 1 *Assume that* (A_1), (A_2) *and* (A_3) *hold. Then problem* ($\wp$) *has at least* $\frac{N}{2} + 1$ *pairs* $\pm u$ *of nontrivial solutions which satisfy*

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx < \frac{4p}{p-2} c_\infty. \tag{2}$$

In particular we get at least three pairs of solutions for $N = 3, 4$ and at least four pairs of solutions for $N = 5, 6$.

We like to point out that Theorem 1 not only yields more solutions than those obtained by Zhu [25] and Hirano [16, Theorem 4], it is also valid in the *nonperturbative* situation where (1) is not assumed. We also note that the asymptotic condition (A_3) is weaker than those imposed by Zhu and Hirano. However, in contrast to the perturbative case handled by Hirano [16], our argument gives no precise information whether there are sign changing solutions or not, see Remark 12.

The proof of Theorem 1 combines variational methods with ideas from the theory of dynamical systems. The key step is the construction of a symmetric subset Λ of $H^1(\mathbb{R}^N)$ on which J is bounded below, which consists of sign changing functions, and which is invariant under the negative gradient flow of J. We then apply minimax methods. Namely, we consider the values c_k of J at which the Krasnoselski genus of its sublevel sets in Λ changes. Since Λ is invariant under the negative gradient flow there is a Palais-Smale sequence in Λ associated to each of these values. Using the precise description of the

PS-sequences of J we conclude that, for each $c_k < 2c_\infty$, there is a solution u_k of $(\wp)$ with either $J(u_k) = c_k$ or $J(u_k) = c_k - c_\infty$. Moreover, we may choose the PS-sequence at c_k in a way that will allow us also to conclude that, if two minimax values below $2c_\infty$ coincide, then J has infinitely many critical points. The final crucial step is, thus, to estimate the genus of the sublevel set $\{u \in \Lambda : J(u) < 2c_\infty\}$. We develop criteria (Lemmas 15 and 11) for estimating the genus of the sublevel sets of J in Λ in terms of those in another set $\mathcal{E}$ which is defined explicitely. These criteria might be of further independent interest. The desired estimate then follows by carefully constructing some odd continuous maps into $\mathcal{E}$. The asymptotic condition (A_3) guarantees that these maps fall into the appropriate sublevel set.

The paper is organized as follows: In section 2 we recall the variational structure of $(\wp)$ and we deduce some estimates from condition (A_3). Section 3 is devoted to the proof of Theorem 1. Finally, in the appendix, we prove an invariance property for the negative gradient flow associated with the restriction of J to the Nehari-manifold which is needed for the proof of Theorem 1.

2 Some estimates

For simplicity we assume from now on that $a_\infty = 1$ and $q_\infty = 1$. Let H be the Hilbert space $H := H^1(\mathbb{R}^N)$ with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + a(x)uv)dx.$$

We write $\|\cdot\|$ for the induced norm, which is equivalent to the usual norm in $H^1(\mathbb{R}^N)$. The solutions of $(\wp)$ are the critical points of the C^2-functional

$$J : H \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} q(x)|u|^p dx.$$

Nontrivial critical points of J lie on the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \{u \in H : u \neq 0,\ J'(u)u = 0\}\\ &= \{u \in H : u \neq 0,\ \|u\|^2 = \int_{\mathbb{R}^N} q(x)|u|^p dx\}.\end{aligned}$$

This is a $C^{1,1}$-submanifold of H which is diffeomorphic to the unit sphere of H via the radial projection. It is characterized by the property that, for each

$u \in H \setminus \{0\}$, the ray $\{tu : t \geq 0\}$ intersects $\mathcal{N}$ precisely the point $t_0 u$, where t_0 is the maximum of the function $t \mapsto J(tu)$ on $[0, \infty)$. We set

$$c_0 := \inf_{\mathcal{N}} J = \inf_{u \in H \setminus \{0\}} \sup_{t \geq 0} J(tu).$$

We consider also the limiting problem

$$(\wp_\infty) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}\, u \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

and write

$$J_\infty : H \to \mathbb{R}, \qquad J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx$$

for its associated functional and

$$\mathcal{N}_\infty = \{u \in H : u \neq 0,\ J'_\infty(u)u = 0\}$$

for its Nehari manifold. We set

$$c_\infty := \inf_{\mathcal{N}_\infty} J_\infty = \inf_{u \in H \setminus \{0\}} \sup_{t \geq 0} J_\infty(tu).$$

It is well known that c_∞ is achieved by J_∞ at a radially symmetric positive function $u_\infty \in \mathcal{N}_\infty$ whose asymptotic behavior is given by

$$\left.\begin{aligned} |u_\infty(x)| &= O(|x|^{-\frac{N-1}{2}}\, e^{-|x|}) \\ |\nabla u_\infty(x)| &= O(|x|^{-\frac{N-1}{2}}\, e^{-|x|}) \end{aligned}\right\} \qquad \text{as } |x| \to \infty \tag{3}$$

[19, 15]. Note that $0 < c_0 \leq c_\infty$. We shall show that assumption (A_3) yields $c_0 < c_\infty$. We fix

$$\varepsilon \in (0, \frac{2-\alpha}{\alpha+2})$$

and a cut-off function $\chi \in C^\infty(\mathbb{R}^N)$ with $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1 - \varepsilon$, and $\chi(x) = 0$ if $|x| \geq 1$. For $u \in H$ and $R > 0$ we define $u^R \in H$ by

$$u^R(x) = \chi(\frac{x}{R})u(x), \qquad x \in \mathbb{R}^N.$$

Note that

$$u^R \to u \quad \text{in} \quad H \qquad \text{as } R \to \infty. \tag{4}$$

In the rest of the section, the letter C always stands for some positive constant, but the value of C may change from line to line. We start with some estimates.

Lemma 2 *As $R \to \infty$,*

(i) $\int_{\mathbb{R}^N} ||\nabla u_\infty|^2 - |\nabla u_\infty^R|^2| = O(e^{-2(1-\varepsilon)R})$

(ii) $\int_{|x|\geq R} |u_\infty|^s dx = O(e^{-sR})$ *for $s > 0$.*

Proof. The proof of (ii) is straightforward. We prove (i).

$$\begin{aligned}\int_{\mathbb{R}^N} ||\nabla u_\infty|^2 - |\nabla u_\infty^R|^2| &= \int_{|x|\geq(1-\varepsilon)R} ||\nabla u_\infty|^2 - |\nabla u_\infty^R|^2| \\ &\leq \int_{|x|\geq(1-\varepsilon)R} |\nabla u_\infty|^2 + \int_{|x|\geq(1-\varepsilon)R} |\nabla u_\infty^R|^2.\end{aligned}$$

Since

$$\begin{aligned}\int_{|x|\geq(1-\varepsilon)R} |\nabla u_\infty^R|^2 &= \int_{|x|\geq(1-\varepsilon)R} |\nabla \left(u_\infty(x)\chi(\frac{x}{R})\right)|^2 dx \\ &\leq \int_{|x|\geq(1-\varepsilon)R} \left(|\nabla u_\infty(x)||\chi(\frac{x}{R})| + |u_\infty(x)||\nabla\left(\chi(\frac{x}{R})\right)|\right)^2 dx \\ &\leq C\int_{|x|\geq(1-\varepsilon)R} \left(|\nabla u_\infty(x)| + |u_\infty(x)|\right)^2 dx,\end{aligned}$$

the asymtotic behavior (3) of u_∞ yields

$$\begin{aligned}&\int_{\mathbb{R}^N} ||\nabla u_\infty|^2 - |\nabla u_\infty^R|^2| \\ &\quad\leq \int_{|x|\geq(1-\varepsilon)R} |\nabla u_\infty|^2 + C\int_{|x|\geq(1-\varepsilon)R} \left(|\nabla u_\infty(x)| + |u_\infty(x)|\right)^2 dx \\ &\quad\leq C\int_{|x|\geq(1-\varepsilon)R} |x|^{-(N-1)}e^{-2|x|}dx \\ &\quad\leq C\int_{(1-\varepsilon)R}^{\infty} e^{-2r}dr \\ &\quad= Ce^{-2(1-\varepsilon)R}\end{aligned}$$

for R large enough. This proves (i). □

Next, for $y \in \mathbb{R}^N$ we define $R_y = \frac{\alpha+2}{4}|y|$. Since $\alpha \in (0,2)$,

$$|y| - R_y \to \infty \quad \text{as} \quad |y| \to \infty. \tag{5}$$

For $u \in H$ and $y \in \mathbb{R}^N$ we write

$$y * u := u(\cdot - y) \in H.$$

Proposition 3 *There exists $S_1 > S_0$ such that*

$$J(t(y * u_\infty^{R_y})) \le c_\infty - Ce^{-\alpha|y|} \qquad \textit{for} \;\; |y| \ge S_1, \;\; t \ge 0.$$

Proof. Using (A_2), (4) and (5) it is easy to see that

$$\lim_{|y|\to\infty} J(t(y * u_\infty^{R_y})) = J_\infty(tu_\infty)$$

for every $t \ge 0$. Since $J_\infty(0) = 0$ and $J_\infty(tu_\infty) \to -\infty$ as $t \to \infty$, we may choose $S > 0$ and $t_1 > t_0 > 0$ such that

$$J(t(y * u_\infty^{R_y})) \le \frac{c_\infty}{2} \qquad \text{for } |y| \ge S, \; t \in [0, t_0] \cup [t_1, \infty).$$

Let now $t \in [t_0, t_1]$ and set $u := tu_\infty$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} a(x) \left| t(y * u_\infty^{R_y}) \right|^2 dx &\le \int_{\mathbb{R}^N} a(x) \left| t(y * u_\infty) \right|^2 dx \\
&= \int_{\mathbb{R}^N} a(x+y) \left| tu_\infty \right|^2 dx \\
&= \int_{|x| \le R_y} a(x+y) \left| tu_\infty \right|^2 dx + \int_{|x| > R_y} a(x+y) \left| tu_\infty \right|^2 dx.
\end{aligned}$$

For $|y|$ large enough (5) and (A_3) yield

$$\begin{aligned}
\int_{|x| \le R_y} a(x+y) \left| tu_\infty \right|^2 dx &\le \int_{|x| \le R_y} (1 - C_1 e^{-\alpha|x+y|}) \left| tu_\infty \right|^2 dx \\
&\le \int_{\mathbb{R}^N} \left| tu_\infty \right|^2 dx - C_1 e^{-\alpha|y|} \int_{|x| \le R_y} t_0^2 e^{-\alpha|x|} \left| u_\infty \right|^2 dx \\
&\le \int_{\mathbb{R}^N} \left| tu_\infty \right|^2 dx - C_1 C e^{-\alpha|y|},
\end{aligned}$$

and Lemma 2 (ii) yields

$$\int_{|x| > R_y} a(x+y) \left| tu_\infty \right|^2 dx \le |a|_\infty \, t_1^2 \int_{|x| > R_y} \left| u_\infty \right|^2 dx \le O(e^{-2R_y}).$$

Hence

$$\int_{\mathbb{R}^N} a(x) \left| t(y * u_\infty^{R_y}) \right|^2 dx \le \int_{\mathbb{R}^N} \left| tu_\infty \right|^2 dx - C_1 C e^{-\alpha|y|} + O(e^{-2R_y}) \tag{6}$$

as $|y| \to \infty$. From (A_3) and Lemma 2 (ii) we also obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} q(x) \left|t(y * u_\infty^{R_y})\right|^p dx &\geq \int_{|x|\leq R_y} q(x+y) \left|tu_\infty^{R_y}\right|^p dx \\
&\geq \int_{|x|\leq R_y} (1 + C_2 e^{-\alpha|x+y|}) \left|tu_\infty^{R_y}\right|^p dx \\
&\geq \int_{|x|\leq(1-\varepsilon)R_y} |tu_\infty|^p\, dx + C_2 e^{-\alpha|y|} \int_{|x|\leq(1-\varepsilon)R_y} t_0^p e^{-\alpha|x|} |u_\infty|^p\, dx \\
&= \int_{\mathbb{R}^N} |tu_\infty|^p\, dx + O(e^{-p(1-\varepsilon)R_y}) + C_2 C e^{-\alpha|y|} \qquad (7)
\end{aligned}
$$

as $|y| \to \infty$. Combining inequalities (6) and (7) and Lemma 2 (i), and using the fact that $C_1 + C_2 > 0$, we conclude that

$$
\begin{aligned}
J(t(y * u_\infty^{R_y})) &\leq c_\infty - C(C_1 + C_2)e^{-\alpha|y|} + O(e^{-2(1-\varepsilon)R_y}) \\
&\leq c_\infty - Ce^{-\alpha|y|} \qquad (8)
\end{aligned}
$$

for $|y|$ sufficiently large, since

$$
2(1-\varepsilon)R_y > \frac{4\alpha}{\alpha+2} R_y = \alpha|y|.
$$

The constant C in (8) is independent of $t \in [t_0, t_1]$. This concludes the proof. $\square$

Since the ray $\{tu : t \geq 0\}$ intersects $\mathcal{N}$ for every $u \in H$, Proposition 3 implies, in particular, that $c_0 < c_\infty$. P.-L. Lions' concentration compactness principle immediately yields the following well known result [19, 13].

Corollary 4 *c_0 is achieved by J at a positive function $u_0 \in \mathcal{N}$.*

3 Proof of the main theorem

We start by constructing a set of sign changing functions which is invariant under the negative gradient flow $\varphi : \mathcal{G} \to H$ of J, defined by

$$
\begin{cases} \frac{\partial}{\partial t}\varphi(t,u) = -\nabla J(\varphi(t,u)) \\ \varphi(0,u) = u \end{cases}
$$

where $\mathcal{G} = \{(t,u) : u \in H,\ 0 \le t < T(u)\}$ and $T(u) \in (0,\infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t,u)$. A subset D of H is called *positively invariant under the flow* φ if

$$\varphi(t,u) \in D \quad \text{for every } u \in D \text{ and every } t \in [0,T(u)).$$

If D is positively invariant under φ we set

$$\mathcal{A}(D) := \{u \in H : \varphi(t,u) \in D \ \text{ for some } t \in (0,T(u))\}.$$

We also consider

$$\mathcal{A}_0 := \{u \in H : \varphi(t,u) \to 0 \ \text{ as } t \to T(u)\}.$$

Since 0 is a stable solution of $(\wp)$, $\mathcal{A}_0$ is an open neighborhood of 0. We write $\mathcal{P} := \{u \in H : u \ge 0\}$ for the convex cone of positive functions in H. This is not an open set, in fact, it has empty interior in H. So, for $\alpha > 0$, we consider the open neighborhoods

$$D_+ := \{u \in H : dist(u,\mathcal{P}) < \alpha\} \quad \text{and} \quad D_- := \{u \in H : dist(u,-\mathcal{P}) < \alpha\}$$

of $\mathcal{P}$ and $-\mathcal{P}$, where $dist(u,X)$ denotes the distance from the point u to the set X in H.

Lemma 5 *For $\alpha > 0$ small enough, the following holds:*
(i) The sets D_+ and D_- are positively invariant under the flow φ.
(ii) $\mathcal{A}(D_+) \cap \mathcal{A}(D_-) = \mathcal{A}_0$.

Proof. The gradient $\nabla J : H \to H$ is given by $\nabla J = Id_H - K$ where $K(u) := (-\Delta + a)^{-1}(q|u|^{p-2}u)$ for $u \in H$. In other words, $K(u)$ is uniquely determined by the relation

$$\langle K(u), \psi\rangle = \int_{\mathbb{R}^N} q(x)|u(x)|^{p-2}u(x)\psi(x)dx \quad \text{for all } \psi \in H. \tag{9}$$

In the following, κ_i, $i = 0,1$, denote positive constants, $|\cdot|_p$ denotes the norm in $L^p(\mathbb{R}^N)$, and $u^+ := \max\{u,0\}$, $u^- := \min\{u,0\}$. Note that

$$\left|u^+\right|_p = \min_{w\in-\mathcal{P}} |u-w|_p \le \kappa_0 \min_{w\in-\mathcal{P}} \|u-w\| = \kappa_0\ dist(u,-\mathcal{P}) \tag{10}$$

for every $u \in H$. Writing $v := K(u)$ and using (9) and (10) we obtain

$$\begin{aligned}
dist(v, -\mathcal{P}) \left\|v^+\right\| &\leq \left\|v^+\right\|^2 \\
&= \left\langle v, v^+ \right\rangle \\
&= \int_{\mathbb{R}^N} q(x)|u(x)|^{p-2}u(x)v^+(x)dx \\
&\leq |q|_\infty \int_{\mathbb{R}^N} |u^+(x)|^{p-2}u^+(x)v^+(x)dx \\
&\leq |q|_\infty \left|u^+\right|_p^{p-1} \left|v^+\right|_p \\
&\leq \kappa_1 \; dist(u, -\mathcal{P})^{p-1} \left\|v^+\right\|.
\end{aligned}$$

Hence,

$$dist(K(u), -\mathcal{P}) \leq \kappa_1 \; dist(u, -\mathcal{P})^{p-1} \quad \text{for all } u \in H.$$

In particular, for $\alpha > 0$ small enough,

$$dist(K(u), -\mathcal{P}) \leq \frac{3}{4} \; dist(u, -\mathcal{P}) \quad \text{for all } u \in \overline{D_-}.$$

We now show that, for such an α, the set D_- is positively invariant under φ. Assume, by contradiction, there is a $u_0 \in D_-$ such that $\varphi(t_0, u_0) \in \partial D_-$ for some $t_0 \in (0, T(u_0))$, and that t_0 is the smallest time at which $\varphi(t, u_0)$ crosses ∂D_-. Our choice of α implies that $K(\varphi(t_0, u_0)) \in D_-$. Since D_- is open and convex, there exists a continuous linear functional $\ell \in H^*$ and a $\beta > 0$ such that $\ell(\varphi(t_0, u_0)) = \beta$ and $\ell(u) > \beta$ for $u \in D_-$. It follows that

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} \ell(\varphi(t, u_0)) = \ell(-\nabla J(\varphi(t_0, u_0))) = \ell(K(\varphi(t_0, u_0))) - \beta > 0.$$

Hence there exists $\varepsilon > 0$ such that $\ell(\varphi(t, u_0)) < \beta$ for $t \in (t_0 - \varepsilon, t_0)$. Thus, $\varphi(t, u_0) \notin D_-$ for $t \in (t_0 - \varepsilon, t_0)$. This contradicts our choice of t_0, and proves that D_- is positively invariant under φ. The proof for D_+ is completely analogous. Hence assertion *(i)* is true.

Clearly $\mathcal{A}_0 \subset \mathcal{A}(D_+) \cap \mathcal{A}(D_-)$. So, in order to prove *(ii)*, it suffices to show that

$$D_+ \cap D_- \subset \mathcal{A}_0.$$

Let $r > 0$ be such that

$$\{u \in H : \|u\| < r\} \subset \mathcal{A}_0.$$

Recall (see (10)) that $|u^+|_p \le \kappa_0\ dist(u, -\mathcal{P})$. Similarly, $|u^-|_p \le \kappa_0\ dist(u, \mathcal{P})$. Therefore,

$$|u|_p \le 2\kappa_0\alpha \qquad \text{for all } u \in D_+ \cap D_-.$$

In particular

$$J(u) \ge -\int_{\mathbb{R}^N} q(x)|u(x)|^p dx \ge -|q|_\infty (2\kappa_0\alpha)^p > -\infty \qquad \text{for all } u \in D_+ \cap D_-.$$

Let $u \in D_+ \cap D_-$. Since $D_+ \cap D_-$ is positively invariant under φ, it follows that $T(u) = \infty$ and that J is bounded below on the half-orbit $\{\varphi(t, u) : t \ge 0\}$. Hence there exists a sequence $t_n \ge 0$, $t_n \to \infty$, such that, for $u_n := \varphi(t_n, u)$, $\nabla J(u_n) \to 0$. In particular,

$$\|u_n\|^2 = \langle \nabla J(u_n), u_n\rangle + \int_{\mathbb{R}^N} q(x)|u_n(x)|^p dx \le o(1)\|u_n\| + |q|_\infty (2\kappa_0\alpha)^p.$$

Therefore, if $\alpha < r^{\frac{2}{p}} |q|_\infty^{-\frac{1}{p}} (2\kappa_0)^{-1}$, then $u_n \in \mathcal{A}_0$ for n sufficiently large and hence $u \in \mathcal{A}_0$ as well. This proves *(ii)*. $\square$

In a related setting similar properties have been proved in [7].

We now fix $\alpha > 0$ small enough so that both assertions in Lemma 5 hold and we set

$$\Lambda := \partial\mathcal{A}_0 \setminus (\mathcal{A}(D_+) \cup \mathcal{A}(D_-)).$$

This is a closed subset of H consisting of sign changing functions. It is positively invariant under the negative gradient flow φ, and it is *symmetric*, that is, $-u \in \Lambda$ iff $u \in \Lambda$. Moreover, since $\Lambda \subset \partial\mathcal{A}_0$, $J(u) > 0$ for every $u \in \Lambda$. We consider the minimax values

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} J(u)$$

where Γ_k is the family of all closed symmetric subsets A of Λ with Krasnoselski genus $\gamma(A) \ge k$. Recall that the Krasnoselski genus $\gamma(A)$ of a nonempty symmetric subset A of H is the smallest integer $k \ge 1$ such that there exists an odd continuous map $A \to \mathbb{S}^{k-1}$, where $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$. If no such map exists then $\gamma(A) := \infty$. The flow invariance of Λ allows us to associate a Palais-Smale sequence for J in Λ to each of the minimax values c_k. Moreover, the following holds.

Lemma 6 *Let (A_n) be a sequence of sets in Γ_k such that* $\sup J(A_n) \to c_k$ *as* $n \to \infty$*. Then there exists a sequence (u_n) in Λ such that*

$$J(u_n) \to c_k, \qquad \nabla J(u_n) \to 0, \qquad dist(u_n, A_n) \to 0,$$

as $n \to \infty$.

Proof. Let $\delta_n := \max\{\frac{1}{n}, \sup J(A_n) - c_k\}$ and

$$\tilde{A}_n := \{u \in \Lambda : J(u) \in [c_k - \delta_n, c_k + \delta_n],\ \ dist(u, A_n) \le \sqrt{\delta_n}\}.$$

It suffices to show that

$$\|\nabla J(u_n)\| < 2\sqrt{\delta_n} \qquad \text{for some } u_n \in \tilde{A}_n. \tag{11}$$

Assume, by contradiction, that

$$\|\nabla J(u)\| \ge 2\sqrt{\delta_n} \qquad \text{for all } u \in \tilde{A}_n.$$

We shall show that

$$J(\varphi(1, u)) \le c_k - \delta_n \qquad \text{for every } u \in A_n. \tag{12}$$

Let $u \in A_n$. If $\varphi(t, u) \in \tilde{A}_n$ for all $t \in [0, 1]$ then

$$J(u) - J(\varphi(1, u)) = \int_0^1 \|\nabla J(\varphi(t, u))\|^2 dt \ge 4\delta_n.$$

Hence $J(\varphi(1, u)) \le J(u) - 4\delta_n \le c_k - \delta_n$. If $\varphi(t, u) \notin \tilde{A}_n$ for some $t \in [0, 1]$ then, either $J(\varphi(1, u)) < c_k - \delta_n$, or $t_0 := \inf\{t : dist(\varphi(t, u), A_n) \ge \sqrt{\delta_n}\} < 1$. In the latter case we find that

$$\begin{aligned}
\sqrt{\delta_n} &\le \int_0^{t_0} \|\nabla J(\varphi(t, u))\| dt \\
&\le \frac{1}{2\sqrt{\delta_n}} \int_0^{t_0} \|\nabla J(\varphi(t, u)\|^2 dt \\
&= \frac{1}{2\sqrt{\delta_n}} (J(u) - J(\varphi(t_0, u)),
\end{aligned}$$

and hence that

$$J(\varphi(1, u)) \le J(\varphi(t_0, u)) \le J(u) - 2\delta_n \le c_k - \delta_n.$$

We conclude that (12) holds. This, however, contradicts the definition of c_k, since $\varphi(1, A_n) \in \Gamma_k$. Thus, we have shown (11), as required. $\square$

The precise description of all Palais-Smale sequences for J yields, in particular, the following.

Lemma 7 *Let (u_n) be a sequence in Λ with*

$$J(u_n) \to c \in \mathbb{R}, \qquad \nabla J(u_n) \to 0, \qquad \textit{as } n \to \infty. \tag{13}$$

If $c < 2c_\infty$ then there is a subsequence, still denoted (u_n), and a nontrivial critical point u of J such that $u_n \rightharpoonup u$ weakly in H. If (u_n) does not converge strongly then $c = J(u) + c_\infty \geq c_0 + c_\infty$.

Proof. We apply Benci and Cerami's compactness lemma [9, Lemma 3.1] which asserts that, for every sequence (u_n) in H satisfying (13), there exists a solution u of $(\wp)$, an integer $k \geq 0$, nontrivial solutions $v_1, ..., v_k$ of the limiting problem $(\wp_\infty)$, and k sequences $(y_n^j) \subset \mathbb{R}^N$ such that for some subsequence, still denoted (u_n),

$$|y_n^j| \to \infty \quad \text{and} \quad |y_n^j - y_n^i| \to \infty \;\text{ as } n \to \infty \;\text{ for } 1 \leq i, j \leq k,\; i \neq j,$$

$$\| u_n - u - \sum_{j=1}^{k} y_n^j * v_j \| \to 0 \quad \text{in } H^1(\mathbb{R}^N),$$

$$J(u) + \sum_{j=1}^{k} J_\infty(v_j) = c,$$

where $(y_n^j * v_j)(x) := v_j(x - y_n^j)$. This implies, in particular, that $u_n \rightharpoonup u$ weakly in H. Now assume that $c < 2c_\infty$. If $u_n \to u$ strongly in H then $u \in \Lambda$, hence $u \neq 0$ and we are done. If not, then $u_n = u + (y_n^1 * v_1) + o(1)$ and $J_\infty(v_1) < 2c_\infty$. Since $v_1^+, v_1^- \in \mathcal{N}_\infty$ and $J_\infty(v_1^+) + J_\infty(v_1^-) = J_\infty(v_1) < 2c_\infty$, we may assume without loss of generality that $v_1^- = 0$. Further, the uniqueness [17] of positive solutions of $(\wp_\infty)$ allows us to take $v_1 = u_\infty$. If $u^- = 0$ then $u_n = u + (y_n^1 * v_1) + o(1) \in D_+$ for n large enough, contradicting the fact that $u_n \in \Lambda$. Therefore $u \neq 0$ and, hence, $c_0 + c_\infty \leq J(u) + c_\infty = c$. $\square$

The following lemma is inspired in an analogous result of Devillanova and Solimini [12, Lemma 2.4].

Lemma 8 *If $c_k = c_{k+1} < 2c_\infty$ for some $k \in \mathbb{N}$, then J has infinitely many critical points with value $\leq c_k$.*

Proof. We assume, by contradiction, that J has only finitely many critical points with critical value $\leq c_k$. Then we may choose $v \in H$ such that

$$\langle v, u \rangle \neq 0 \quad \text{for every } u \in H \setminus \{0\} \text{ with } J(u) \leq c_k,\; \nabla J(u) = 0. \tag{14}$$

Let $V := \{u \in H : \langle v, u\rangle = 0\}$. Take a sequence (A_n) of sets in Γ_{k+1} such that $\sup J(A_n) \to c_{k+1}$ as $n \to \infty$, and set

$$\widetilde{A}_n := \{u \in A_n : \langle v, u\rangle = 0\}, \qquad n \in \mathbb{N}.$$

Well known properties of the genus yield

$$\gamma(\widetilde{A}_n) \geq \gamma(A_n) - \gamma(H \setminus V) \geq (k+1) - 1 = k.$$

Thus, $\widetilde{A}_n \in \Gamma_k$ and, therefore,

$$c_k \leq \sup J(\widetilde{A}_n) \leq \sup J(A_n) \to c_{k+1} = c_k \quad \text{ as } \; n \to \infty.$$

By Lemma 6 there is a sequence (u_n) in Λ such that

$$J(u_n) \to c_k, \qquad \nabla J(u_n) \to 0, \qquad dist(u_n, \widetilde{A}_n) \to 0,$$

as $n \to \infty$ and, by Lemma 7, a subsequence (u_n) converges weakly in H to a nontrivial critical point u of J with $J(u) \leq c_k$. Since $dist(u_n, \widetilde{A}_n) \to 0$, it follows that

$$\langle v, u\rangle = \lim_{n\to\infty} \langle v, u_n\rangle = 0,$$

contradicting (14). Hence J has infinitely many critical points with value $\leq c_k$, as claimed. $\square$

Now, we wish to see how many of the minimax values c_k lie below $2c_\infty$. We consider the set

$$\mathcal{E} := \{u \in H : u^+ \in \mathcal{N} \;\text{ and }\; u^- \in \mathcal{N}\}.$$

$\mathcal{E}$ is a symmetric subset of the Nehari manifold $\mathcal{N}$ consisting of sign changing functions. It is characterized by the property that, for every sign changing function $u \in H$, the set $\{su^+ + tu^- : s, t \in [0,\infty)\}$ intersects $\mathcal{E}$ precisely at the point $s_0u^+ + t_0u^-$, where (s_0, t_0) is the maximum of the function $(s,t) \mapsto J(su^+ + tu^-)$ on $[0,\infty) \times [0,\infty)$ (details may be found, for example, in [10]). All sign changing critical points of J lie in $\mathcal{E}$. We write

$$\mathcal{E}^c := \{u \in \mathcal{E} : J(u) \leq c\}, \qquad c \in \mathbb{R},$$

$\mathbb{S}^{k-1}$ for the unit sphere, and $\mathbb{D}^k$ for the unit ball in $\mathbb{R}^k$.

Lemma 9 *If there exists an odd continuous map $h : \mathbb{S}^{k-1} \to \mathcal{E}^c$, then $c_k \le c$.*

Proof. Let $T := (\mathbb{S}^{k-1} \times [-1,1]) \cup (\mathbb{D}^k \times \{-1,1\}) \subset \mathbb{R}^k \times \mathbb{R} \equiv \mathbb{R}^{k+1}$, and fix $v_0 \in H \setminus \{0\}$, $v_0 \ge 0$. We extend h to a map $\overline{h} : T \to H$ as follows: For $\theta \in \mathbb{S}^{k-1}$, $r \in [0,1]$, $s \in [-1,1]$ we set

$$\overline{h}(r\theta, s) = \begin{cases} (1-s)h(\theta)^- + (1+s)h(\theta)^+ & \text{if } r = 1 \\ r2h(\theta)^+ + (1-r)v_0 & \text{if } s = 1 \\ r2h(\theta)^- - (1-r)v_0 & \text{if } s = -1 \end{cases}$$

Note that $\overline{h}(\mathbb{D}^k \times \{\pm 1\}) \subset \pm\mathcal{P} \subset \mathcal{A}(D_\pm)$. Now we extend $\overline{h}$ radially to a map $\overline{h} : \mathbb{R}^{k+1} \to H$ such that

$$\overline{h}(t\xi) = t\overline{h}(\xi) \quad \text{for } \xi \in T,\ t \in [0,\infty).$$

The map $\overline{h}$ is odd and continuous. Therefore the set

$$\mathcal{O} := \{x \in \mathbb{R}^{k+1} : \overline{h}(x) \in \mathcal{A}_0\}$$

is a symmetric neighborhood of the origin. Moreover, since $J(\overline{h}(t\xi)) = J(t\overline{h}(\xi)) \to -\infty$ as $t \to \infty$ for every $\xi \in T$, $\mathcal{O}$ is bounded. The Borsuk-Ulam theorem yields $\gamma(\partial\mathcal{O}) = k+1$. Let

$$B := \overline{h}(\partial\mathcal{O}) \setminus [\mathcal{A}(D_+) \cup \mathcal{A}(D_-)] \subset \Lambda.$$

Since $\partial\mathcal{A}_0 \cap \mathcal{A}(D_+) \cap \mathcal{A}(D_-) = \partial\mathcal{A}_0 \cap \mathcal{A}_0 = \emptyset$ (by Lemma 5) and $\mathcal{A}(D_-) = -\mathcal{A}(D_+)$, we have that

$$\gamma(\partial\mathcal{A}_0 \cap [\mathcal{A}(D_+) \cup \mathcal{A}(D_-)]) = 1. \tag{15}$$

Therefore, since $\mathcal{A}(D_\pm)$ are open, well known properties of the genus (see for example [4, Proposition 2.15]) yield

$$\gamma(B) \ge \gamma(\overline{h}(\partial\mathcal{O})) - \gamma(\partial\mathcal{A}_0 \cap [\mathcal{A}(D_+) \cup \mathcal{A}(D_-)]) \ge \gamma(\partial\mathcal{O}) - 1 \ge k.$$

Since

$$\begin{aligned} B &\subset \{\overline{h}(t\xi) : \xi \in \mathbb{S}^{k-1} \times [-1,1], t \ge 0\} \\ &= \{t_1 h(\theta)^+ + t_2 h(\theta)^- : \theta \in \mathbb{S}^{k-1}, t_1, t_2 \ge 0\}, \end{aligned}$$

and since $h(\theta) \in \mathcal{E}$ for $\theta \in \mathbb{S}^{k-1}$, we obtain

$$c_k \le \sup_{u \in B} J(u) \le \sup_{\substack{\theta \in \mathbb{S}^{k-1} \\ t_1, t_2 \ge 0}} J(t_1 h(\theta)^+ + t_2 h(\theta)^-) = J(h(\theta)) \le c$$

as claimed. $\square$

Using the estimates from Section 2 we obtain the following.

Corollary 10 *$c_N < 2c_\infty$ and $2c_0 \le c_1$.*

Proof. To obtain the first inequality we define an odd continuous map $h : \mathbb{S}^{N-1} \to \mathcal{E}$ by

$$h(\theta) = \rho(y * u_\infty^{R_y}) - \rho((-y) * u_\infty^{R_y}) \tag{16}$$

where $\theta \in \mathbb{S}^{N-1}$, $y = S_1\theta$, S_1 and R_y are as in Proposition 3, and $\rho : H \setminus \{0\} \to \mathcal{N}$ is the radial projection onto the Nehari manifold. Proposition 3 yields

$$J(h(\theta)) = J(\rho(y * u_\infty^{R_y})) + J(\rho((-y) * u_\infty^{R_y})) \le 2(c_\infty - Ce^{-\alpha S_1}) < 2c_\infty$$

for every $\theta \in \mathbb{S}^{N-1}$. By Lemma 9, $c_N < 2c_\infty$ as claimed. On the other hand, Lemma 6 gives us a sequence (u_n) in Λ with $J(u_n) \to c_1$ and $\nabla J(u_n) \to 0$. By Lemma 7, a subsequence converges weakly in H to a nontrivial critical point u of J. If (u_n) does not converge strongly, then $c_1 = J(u) + c_\infty \ge 2c_0$. If $u_n \to u$ strongly in H then u is a sign changing critical point of J. Therefore $u^+, u^- \in \mathcal{N}$ and $c_1 = J(u) = J(u^+) + J(u^-) \ge 2c_0$. So $c_1 \ge 2c_0$ as claimed. □

In addition, the following holds. Let u_0 be a positive minimizer of J on $\mathcal{N}$ (see Corollary 4).

Lemma 11 *If $c_1 < c_\infty$ and if u_0 is the only positive critical point of J with value $< c_\infty$, then $c_{N+1} < 2c_\infty$.*

Proof. By Lemma 9 it suffices to show that there exists an odd and continuous map $\mathbb{S}^N \to \mathcal{E}^c$ for some $c < 2c_\infty$. The existence of such a map follows easily once we have shown that the map $h : \mathbb{S}^{N-1} \to \mathcal{E}$ defined in (16) is homotopic to a constant in $\{u \in \mathcal{E} : J(u) < 2c_\infty\}$. To prove this last assertion we first note that, by Lemmas 6 and 7, the value $c_1 < c_\infty$ is attained by J at a sign changing critical point $u_1 \in \Lambda$.
Since $\mathcal{N}$ is a $C^{1,1}$-manifold and J is bounded below on $\mathcal{N}$, there exists a global semiflow $\varphi_\mathcal{N} : [0, \infty) \times \mathcal{N} \to \mathcal{N}$ defined by

$$\begin{cases} \frac{\partial}{\partial t}\varphi_\mathcal{N}(t, u) = -\nabla_T J(\varphi_\mathcal{N}(t, u)) \\ \varphi_\mathcal{N}(0, u) = u \end{cases}$$

where $\nabla_T J(u)$ denotes the orthogonal projection of $\nabla J(u)$ onto the tangent space $T_u\mathcal{N}$ of $\mathcal{N}$ at the point $u \in \mathcal{N}$. The set $\mathcal{N} \cap \mathcal{P}$ is positively invariant under the flow $\varphi_{\mathcal{N}}$, that is,

$$u \in \mathcal{N}, \quad u \geq 0 \qquad \text{implies} \qquad \varphi_{\mathcal{N}}(t,u) \geq 0 \;\; \text{for every } t \geq 0. \tag{17}$$

A proof of this fact is given in the appendix. Since J satisfies the Palais-Smale condition at every $c < c_\infty$ [19] and since u_0 is the only positive critical point of J with value $< c_\infty$, it follows that, for every $u \in \mathcal{N}$ with $u \geq 0$ and $J(u) < c_\infty$,

$$\varphi_{\mathcal{N}}(t,u) \to u_0 \qquad \text{as } t \to \infty.$$

This is true, in particular, for u_1^+ and for each $\rho(y * u_\infty^{R_y})$ with $y = S_1\theta$, $\theta \in \mathbb{S}^{N-1}$. We fix a homeomorphism $\tau : [0,1) \to [0,\infty)$ and define $\sigma : [0,1] \to \mathcal{N}$ by

$$\sigma(t) = \begin{cases} \varphi_{\mathcal{N}}(\tau(t), u_1^+) & \text{if } t \in [0,1) \\ u_0 & \text{if } t = 1 \end{cases}$$

and $g : \mathbb{S}^{N-1} \times [0,1] \to \mathcal{N}$ by

$$g(\theta,t) = \begin{cases} \varphi_{\mathcal{N}}(\tau(t), \rho(y * u_\infty^{R_y})) & \text{if } t \in [0,1) \\ u_0 & \text{if } t = 1 \end{cases}$$

Both σ and g are continuous and satisfy

$$J(\sigma(t)) < c_\infty \quad \text{and} \quad J(g(\theta,t)) < c_\infty \qquad \text{for all } t \in [0,1],\; \theta \in \mathbb{S}^{N-1}.$$

Since $[0,1]$ and $\mathbb{S}^{N-1} \times [0,1]$ are compact, by (4) we may choose $R > 2S_1$ such that $\sigma(t)^R$, $g(\theta,t)^R$, $(u_1^-)^R \in H \setminus \{0\}$ and

$$J(\rho(\sigma(t)^R)) < c_\infty, \quad J(\rho(g(\theta,t)^R)) < c_\infty, \quad J(\rho((u_1^+)^R)) + J(\rho((u_1^-)^R)) < c_\infty, \tag{18}$$

for all $t \in [0,1]$, $\theta \in \mathbb{S}^{N-1}$. Now we define four homotopies $H_i : \mathbb{S}^{N-1} \times [0,1] \to \mathcal{E}$, $i = 1, ..., 4$, as follows. First we fix $S_2 > \frac{4R}{2-\alpha}$ and define

$$H_1(\theta,t) = \rho(y * u^{R_y}) - \rho((-z) * u^{R_z})$$

with $y = S_1\theta$ and $z = tS_2\theta + (1-t)S_1\theta$ for each $\theta \in \mathbb{S}^{N-1}$, $t \in [0,1]$. Note that the support of $H_1(\theta,1)^-$ is contained in the complement of the ball of radius R centered at 0. Next we define

$$\begin{aligned} H_2(\theta,t) &= \rho(g(\theta,t)^R) + H_1(\theta,1)^- \\ H_3(\theta,t) &= \rho(\sigma(1-t)^R) + H_1(\theta,1)^- \\ H_4(\theta,t) &= \rho((u_1^+)^R) + \rho\left[t(u_1^-)^R + (1-t)H_1(\theta,1)^-\right] \end{aligned}$$

for $\theta \in \mathbb{S}^{N-1}$, $t \in [0,1]$. Observe that the two summands which appear in the definition of each $H_i(\theta,t)$ have disjoint supports, the first summand is positive and the second one is negative, and both of them belong to $\mathcal{N}$. Therefore $H_i(\theta,t) \in \mathcal{E}$ and

$$J(H_i(\theta,t)) = J(H_i(\theta,t)^+) + J(H_i(\theta,t)^-) < 2c_\infty.$$

This inequality follows from (18) together with the fact that

$$\begin{aligned} J(\rho[t(u_1^-)^R + (1-t)H_1(\theta,1)^-]) &\leq J(\rho(t(u_1^-)^R)) + J(\rho((1-t)H_1(\theta,1)^-)) \\ &= J(\rho(u_1^-)^R) + J(H_1(\theta,1)^-), \end{aligned}$$

because $(u_1^-)^R$ and $H_1(\theta,1)^-$ have disjoint supports. Since $H_i(\theta,1) = H_{i+1}(\theta,0)$ we may apply $H_1,...,H_4$ successively to obtain a homotopy $H : \mathbb{S}^{N-1} \times [0,1] \to \mathcal{E}$ such that $H(\theta,0) = h(\theta)$, $H(\theta,1) = \rho((u_1^+)^R) + \rho((u_1^-)^R) \in \mathcal{E}$, and $J(H(\theta,t)) < 2c_\infty$, as claimed.
Finally, we extend H to an odd map $(\mathbb{S}^{N-1} \times [-1,1]) \cup (\mathbb{D}^N \times \{-1,1\}) \to \mathcal{E}$ which is constant on $\mathbb{D}^N \times \{1\}$ and on $\mathbb{D}^N \times \{-1\}$. Composing this map with the radial projection $\mathbb{S}^N \cong (\mathbb{S}^{N-1} \times [-1,1]) \cup (\mathbb{D}^N \times \{-1,1\})$ yields an odd continuous map $\mathbb{S}^N \to \mathcal{E}^c$ with $c < 2c_\infty$. The proof is finished. $\square$

We are ready to prove Theorem 1.

Proof of Theorem 1. Let $K := \{u \in H \setminus \{0\} : J(u) < 2c_\infty, \nabla J(u) = 0\}$. If K is infinite, the proof is finished. If not, then Lemma 8 and Corollary 10 yield

$$c_0 < c_1 < \cdots < c_N < 2c_\infty.$$

Let k_0 be such that $c_{k_0-1} < c_\infty \leq c_{k_0}$. By Lemmas 6 and 7, J has at least $k_1 := \max\{k_0, N-k_0+1\}$ nontrivial critical points with value $< 2c_\infty$. Note that $k_1 \geq \frac{N+2}{2}$ except when N is odd and $k_0 = \frac{N+1}{2}$.
So assume that N is odd and that $k_0 = \frac{N+1}{2}$. Corollary 4 and Lemmas 6 and 7 yield k_0 nontrivial solutions $u_0, u_1, ..., u_{k_0-1}$ of $(\wp)$ with $J(u_k) = c_k$, such

that u_0 is positive and $u_1, ..., u_{k_0-1}$ are sign changing. If there is another positive solution $\widetilde{u} \neq u_0$ with $J(\widetilde{u}) < 2c_\infty$, then $(\wp)$ has at least $k_0 + 1 = \frac{N+3}{2}$ solutions in K. If u_0 is the only positive solution in K, then Lemma 11 asserts that $c_{N+1} < 2c_\infty$. Hence $(\wp)$ has at least $N + 2 - k_0 = \frac{N+3}{2}$ solutions in K. The proof is finished. □

Remark 12 The proof given above actually yields the following additional information:

$$\begin{aligned} &\sharp\{\text{pairs of nontrivial solutions of } (P) \text{ satisfying } (2)\} \\ &+ \sharp\{\text{pairs of sign changing solutions of } (P) \text{ satisfying } (2)\} \ \geq \ N, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\sharp\{\text{positive solutions of } (P) \text{ satisfying } (2)\} \\ &+ 2\sharp\{\text{pairs of sign changing solutions of } (P) \text{ satisfying } (2)\} \ \geq \ N. \end{aligned}$$

This in particular implies that, if (P) has no sign changing solutions satisfying (2), then there are at least N positive solutions.

4 Appendix

This appendix is devoted to the proof of assertion (17) which we now restate.

Proposition 13 *Let $\varphi_{\mathcal{N}} : [0, \infty) \times \mathcal{N} \to \mathcal{N}$ be the negative gradient flow of the restriction of J to the Nehari manifold $\mathcal{N}$. Then*

$$\varphi_{\mathcal{N}}(t, u) \geq 0 \quad \textit{for every } u \in \mathcal{N},\ u \geq 0, \textit{ and every } t \geq 0.$$

Let $\Psi : H \to \mathbb{R}$ be the C^2-function

$$\Psi(u) := J'(u)u = \|u\|^2 - \int_{\mathbb{R}^N} q(x)\, |u|^p \, dx.$$

For each $u \in H$, consider the closed linear subspace of H

$$T_u := \{v \in H : \langle \nabla\Psi(u), v\rangle = 0\} = \ker \Psi'(u),$$

and let $\nabla_T J(u)$ be the orthogonal projection of $\nabla J(u)$ onto T_u. Note that 0 is a regular value of Ψ on $H \setminus \{0\}$, and that $\Psi^{-1}(0) = \mathcal{N}$ is the Nehari manifold. If $u \in \mathcal{N}$, then T_u is the tangent space of $\mathcal{N}$ at u and $\nabla_T J(u)$ is the gradient at u of the restriction of J to $\mathcal{N}$.

For every $u \in H$,

$$\nabla J(u) = u - K(u) \quad \text{and} \quad \nabla\Psi(u) = 2u - pK(u)$$

where $K(u)$ is defined by the relation (9). Thus, by the maximum principle, $K(u) \geq 0$ if $u \geq 0$. As before we write $\mathcal{P} = \{u \in H : u \geq 0\}$ for the convex cone of positive functions in H. Note that both $\nabla J(u)$ and $\nabla\Psi(u)$ lie in the linear subspace generated by u and $K(u)$ and, hence, also does $\nabla_T J(u)$.

Lemma 14 *There is an open subset $\mathcal{O}$ of H which contains $\{u \in \mathcal{N} : \nabla J(u) \neq 0\}$ and, for every $u \in \mathcal{O} \cap \mathcal{P}$, there is an $s_u > 0$ with the property that*

$$u - s\nabla_T J(u) \in \mathcal{P}$$

for every $0 \leq s \leq s_u$.

Proof. For $u \in H \setminus \{0\}$ let

$$K_\perp(u) := K(u) - \frac{\langle u, K(u)\rangle}{\|u\|^2} u$$

be the orthogonal projection of $K(u)$ onto the orthogonal complement of $\{tu : t \in \mathbb{R}\}$ in H, and set

$$\mathcal{O} := \{u \in H \setminus \{0\} : \langle K_\perp(u), -\nabla_T J(u)\rangle > 0\}.$$

For every $u \in \mathcal{N}$ we have $\langle \nabla J(u), u\rangle = 0$ and $\langle \nabla\Psi(u), u\rangle \neq 0$. Hence $K_\perp(u) = -\nabla J(u)$, and

$$\langle K_\perp(u), -\nabla_T J(u)\rangle = \langle \nabla J(u), \nabla_T J(u)\rangle > 0$$

for every $u \in \mathcal{N}$ with $\nabla J(u) \neq 0$.
Now, if $u \in \mathcal{O}$, then u and $K(u)$ are linearly independent. The line generated by u splits the plane generated by u and $K(u)$ into two half planes, and $u - \nabla_T J(u)$ is in the same half plane as $K(u)$ for every $u \in \mathcal{O}$. It follows that

$$u - s\nabla_T J(u) \in \{\alpha u + \beta K(u) : \alpha, \beta > 0\} \subset \mathcal{P}$$

for every $0 < s < s_u := \|\nabla_T J(u)\|^{-1}\, dist(u, \{tK(u) : t \in \mathbb{R}\})$. □

Proof of Proposition 13. Let $C > 0$ be such that

$$\|\nabla\Psi(u)\| \geq 2C \quad \text{for every } u \in \mathcal{N}.$$

We apply Volkmann's theorem [24] to the data $E = H$, $F = \mathcal{P}$, $G = \{u \in \mathcal{O} : \|\nabla\Psi(u)\| > C\}$, and $f = -\nabla_T J : G \to H$. Lemma 14 above implies condition (3) in Volkmann's theorem. Conditions (4) and (5) are immediate for the uniqueness function $\omega(t,s) = s$. Therefore, $\varphi_{\mathcal{N}}(t,u) \in \mathcal{P}$ for every $u \in \mathcal{N} \cap \mathcal{P}$, $t \in [0,\infty)$. $\square$

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