TWO SOLUTIONS OF THE BAHRI-CORON PROBLEM IN PUNCTURED DOMAINS VIA THE FIXED POINT TRANSFER

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Abstract. We consider the problem

$$-\Delta u = |u|^{2^*-2} u \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. We assume that Ω is annular shaped, i.e. there are constants $R_2 > R_1 > 0$ such that $\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega$ and $\{x \in \mathbb{R}^N : |x| < R_1\} \smallsetminus \Omega \neq \varnothing$. Coron (1984) showed that there is one positive solution to this problem if R_2/R_1 is large enough. We establish the existence of at least two pairs of nontrivial solutions in this case. The proof combines a deformation argument on the Nehari manifold with cohomological information derived from Dold's fixed point transfer. To deal with the lack of compactness, an energy estimate recently proved by one of the authors is used.

1. Introduction

We consider the problem

$$(\wp) \quad \begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

This problem has been subject of extensive research in the last few decades. It arises in a geometric context in the Yamabe problem and the prescribed scalar curvature problem. Its invariance under dilations produces a lack of compactness of the associated variational functional and turns it into a quite interesting and challenging problem. The first result is due to Pohožaev [17] who showed that $(\wp)$ has no nontrivial solution if Ω is strictly starshaped.

Much progress has been made concerning the existence of one solution. The first existence result is due to Coron [7]. He showed that, if Ω is annular-shaped, that is, if $\Omega \supset \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\}$ and $\{x \in \mathbb{R}^N : |x| < R_1\} \smallsetminus \Omega \neq \varnothing$, and if R_2/R_1 is large enough, then problem $(\wp)$ has at least one positive solution. The most remarkable existence result is due to Bahri and Coron [1] who showed that the same is true for every domain Ω having nontrivial homology. On the other hand, Dancer [9], Ding [10], and Passaseo [15, 16] gave examples of contractible domains Ω for which problem $(\wp)$ has at least one nontrivial solution.

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Concerning the existence of more than one pair $\pm u$ of solutions less progress has been made. If Ω is a annulus or, more generally, if Ω is invariant under a group of symmetries whose orbits in Ω are all infinite, then compactness is restored and problem $(\wp)$ has infinitely many symmetric solutions [12, 4]. For some specific domains, multiplicity results have been obtained by Passaseo [15], Marchi and Pacella [13], and the authors [5]. The common feature of these domains is that they possess certain symmetry properties, and these properties play an important role in the proofs. Without symmetry assumptions, Rey [18] proved existence of multiple solutions for domains with multiple small holes, and Passaseo [16] considered domains with holes and tunnels. In this paper we go back to the annular-shaped domains considered by Coron, and we prove the following.

Theorem 1. *Assume there are constants $R_2 > R_1 > 0$ such that*

$$\Omega \supset \{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \quad \text{and} \quad \{x \in \mathbb{R}^N : |x| < R_1\} \smallsetminus \Omega \neq \varnothing$$

Then, if R_2/R_1 is large enough, problem $(\wp)$ has at least two pairs $\pm u$ of nontrivial solutions.

This result provides two pairs of solutions for domains with just one single (sufficiently small) hole. At least one pair of solutions does not change sign by Coron's result [7]. We conjecture that $(\wp)$ admits a sign-changing solution under the assumptions of Theorem 1. The existence of a sign-changing solution was proved in [5] under the additional symmetry assumption $x \in \Omega \iff -x \in \Omega$. For a domain Ω having this symmetry and containing the origin, Musso and Pistoia [14] recently established the existence of solutions with any number of nodal domains in $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0)$ for ε sufficiently small.

One may ask whether $(\wp)$ has multiple pairs of solutions on every topologically nontrivial domain. This general question, however, is completely open.

Our proof of Theorem 1 combines variational and topological methods. More precisely, we first modify the variational principle introduced by Coron [7] in his proof of the existence of one pair of solutions, and then we use Dold's fixed point transfer [11] and a recent result from [21] to get another pair of solutions. In contrast to its well documented importance in algebraic topology, transfer maps have not been used much, so far, in the variational context. Bahri and Coron applied a different kind of transfer to prove their result in [1]. Very recently, in a joint paper with T. Bartsch [3], we applied the fixed point transfer to semiclassical nonlinear Schrödinger equations.

We recall that the main difficulty in obtaining multiplicity results for problem $(\wp)$ is the lack of compactness of the variational funcional associated to it. This lack of compactness was fully described by Struwe [19] in terms of the solutions of the limit problem $-\Delta u = |u|^{2^*-2}u$ in $\mathbb{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$. Whereas positive solutions to this last problem are well known, little is known about sign changing solutions. Recently in [21] the second named author obtained a lower bound for the energy of the sign changing solutions of the limit problem. This lower bound will play a crucial role in our approach.

The paper is organized as follows. In Section 2 we recall the variational framework for $(\wp)$ and the corresponding limit problem, and we prove a deformation lemma on the Nehari manifold. This lemma is not standard since it includes information with respect to the C^1-topology obtained via a bootstrap argument. Here extra care is needed to deal with the critical nonlinearity. Section 3 contains somewhat technical lemmas concerning punctured domains and the barycenter map. These lemmas are clearly inspired by the approach in [7], but we need additional properties which are more difficult to obtain. In Section 4 we prove Theorem 1 by a contradiction argument. Here we use Dold's fixed point transfer. The definition and some properties of the transfer are recalled in the Appendix.

2. Preliminaries

First we fix some notation. If $u \in L^p(\mathbb{R}^N)$ for some $1 \le p \le \infty$, we write $|u|_p$ for the usual L^p-norm of u. As usual, we let $D^{1,2}(\mathbb{R}^N)$ denote the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2$. We consider the limit problem

$$(\wp_\infty) \qquad -\Delta u = |u|^{2^*-2} u, \quad u \in D^{1,2}(\mathbb{R}^N),$$

and we denote by

$$J_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}, \qquad J_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*},$$

its associated energy functional. The nontrivial critical points of J lie on the Nehari manifold

$$\begin{aligned}\mathcal{N}_\infty &= \{u \in D^{1,2}(\mathbb{R}^N) : u \ne 0,\ J'_\infty(u)u = 0\} \\ &= \{u \in D^{1,2}(\mathbb{R}^N) : u \ne 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\}.\end{aligned}$$

$\mathcal{N}_\infty$ is a $C^{1,1}$-manifold which is radially diffeomorphic to the unit sphere in $D^{1,2}(\mathbb{R}^N)$. More precisely, the radial projection $\rho : D^{1,2}(\mathbb{R}^N) \smallsetminus \{0\} \to \mathcal{N}_\infty$ onto the Nehari manifold is given by

$$\rho(u) = \left(\frac{\|u\|^2}{|u|_{2^*}^{2^*}}\right)^{\frac{N-2}{4}} u. \tag{2.1}$$

We recall that J_∞ and $\mathcal{N}_\infty$ are invariant under translations and dilations, i.e., under transformations of the form

$$D^{1,2}(\mathbb{R}^N) \to D^{1,2}(\mathbb{R}^N), \quad u \mapsto (\lambda, y) * u \quad \text{for } y \in \mathbb{R}^N,\ \lambda > 0,$$

where $[(\lambda, y) * u](x) := \lambda^{\frac{2-N}{2}} u(\frac{x-y}{\lambda})$ for $x \in \mathbb{R}^N$. For brevity, we will write $\lambda * u$ instead of $(\lambda, 0) * u$ and $y * u$ instead of $(1, y) * u$ for $\lambda > 0, y \in \mathbb{R}^N$ and $u \in D^{1,2}(\mathbb{R}^N)$. The positive solutions of $(\wp_\infty)$ are precisely the functions $(\lambda, y) * U$, $y \in \mathbb{R}^N$, $\lambda > 0$, where $U \in \mathcal{N}_\infty \cap C^\infty(\mathbb{R}^N)$ is the instanton defined by

$$U(z) = a_N \left(\frac{1}{1 + |y|^2}\right)^{\frac{N-2}{2}}, \qquad a_N := [N(N-2)]^{\frac{N-2}{4}}. \tag{2.2}$$

These solutions are also precisely the least energy solutions of $(\wp_\infty)$. Indeed, we recall that

$$J_\infty(U) = \inf_{\mathcal{N}_\infty} J_\infty = c_\infty,$$

where $c_\infty := \frac{1}{N} S^{N/2}$ and S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \smallsetminus \{0\}} \frac{\|u\|^2}{|u|_{2^*}^2}. \tag{2.3}$$

The following energy bound for sign changing solutions of ($\wp_\infty$), obtained recently in [21], plays a crucial role in the proof of our main result.

Theorem 2. *There exists an $\varepsilon_0 > 0$ such that every sign changing solution u of problem ($\wp_\infty$) satisfies $J_\infty(u) > 2c_\infty + \varepsilon_0$.*

Next we consider a bounded domain Ω, and via trivial extension we regard $H_0^1(\Omega)$ as a closed subspace of $D^{1,2}(\mathbb{R}^N)$. The solutions of problem ($\wp$) are the critical points of the functional

$$J : H_0^1(\Omega) \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*},$$

which is the restriction of J_∞ to $H_0^1(\Omega)$. The nontrivial critical points of J belong to

$$\mathcal{N} := \mathcal{N}_\infty \cap H_0^1(\Omega) = \{u \in H_0^1(\Omega) : u \neq 0, \ \|u\|^2 = |u|_{2^*}^{2^*}\}.$$

It is known that

$$\inf_{\mathcal{N}} J = \inf_{\mathcal{N}_\infty} J_\infty = c_\infty$$

independently of Ω and that c_∞ is not attained by J in $\mathcal{N}$.
The gradient of $J : H_0^1(\Omega) \to \mathbb{R}$ with respect to the scalar product

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v$$

has the form $\nabla J(u) = u - K(u)$ where $v = K(u)$ is the unique solution of the Poisson problem

$$-\Delta v = |u|^{2^*-2}u, \qquad v \in H_0^1(\Omega).$$

Consider the function

$$F : H_0^1(\Omega) \smallsetminus \{0\} \to \mathbb{R}, \qquad F(u) := \|u\|^2 - |u|_{2^*}^{2^*},$$

Note that $\mathcal{N} = F^{-1}(0)$. By definition of S in (2.3),

$$F(u) \geq \|u\|^2 - S^{-\frac{2*}{2}}\|u\|^{2^*} = \|u\|^2(1 - S^{-\frac{2*}{2}}\|u\|^{2^*-2}),$$

so that for $u \in \mathcal{N}$ we have $\|u\| \geq S^{\frac{2^*}{2(2^*-2)}} = S^{\frac{N}{4}}$ and thus $\langle \nabla F(u), u\rangle = -\frac{4}{N-2}\|u\|^2 \leq -\frac{4}{N-2}S^{\frac{N}{4}}\|u\|$. Hence $\|\nabla F(u)\| \geq \frac{4}{N-2}S^{\frac{N}{4}}$ for every $u \in \mathcal{N}$. The tangent space of $\mathcal{N}$ at u is the subspace

$$T_u\mathcal{N} := \{v \in H_0^1(\Omega) : \langle \nabla F(u), v\rangle = 0\},$$

and the projection of $\nabla J(u)$ onto the tangent space is given by

$$\nabla_{\mathcal{N}} J(u) = \nabla J(u) - \frac{\langle \nabla J(u), \nabla F(u)\rangle}{\|\nabla F(u)\|^2}\nabla F(u).$$

Since $\nabla F(u) = 2u - 2^* K(u)$, we have that

$$\nabla_{\mathcal{N}} J(u) = a(u)u - b(u)K(u),$$

where

$$a(u) := 1 - \frac{2\langle \nabla J(u), \nabla F(u)\rangle}{\|\nabla F(u)\|^2}, \qquad b(u) := \left(1 - \frac{2^*\langle \nabla J(u), \nabla F(u)\rangle}{\|\nabla F(u)\|^2}\right). \tag{2.4}$$

The negative gradient flow $\eta : [0,\infty) \times \mathcal{N} \to \mathcal{N}$ of J on $\mathcal{N}$ is defined by

$$\begin{cases} \frac{\partial}{\partial t}\eta(t,u) = -\nabla_{\mathcal{N}} J(\eta(t,u)) \\ \quad \eta(0,u) = u \end{cases}$$

Recall that J is said to satisfy the Palais-Smale condition $(\mathrm{PS})_c$ at c if every sequence (u_k) in $H_0^1(\Omega)$ such that $J(u_k) \to c$ and $\nabla J(u_k) \to 0$ contains a convergent subsequence. Let $C_0^1(\overline{\Omega})$ denote the Banach space of real C^1-functions on $\overline{\Omega}$ which vanish on $\partial\Omega$, endowed with the norm $\|u\|_{C^1} := \sup_\Omega |u| + \sup_\Omega |\nabla u|$ for $u \in C_0^1(\overline{\Omega})$. As usual, we also write $\mathcal{N}^c := \{u \in \mathcal{N} : J(u) \le c\}$ for $c \in \mathbb{R}$.

Lemma 1. *Let $[b,d] \subset \mathbb{R}$ be a closed interval containing no critical values of J and such that J satisfies $(PS)_c$ for every $b \le c \le d$. Let*

$$e(u) := \inf\{t \ge 0 : \eta(t,u) \in \mathcal{N}^b\}$$

be the entrance time of $u \in \mathcal{N}^d$ in $\mathcal{N}^b$. Then the following holds.
(a) $e : \mathcal{N}^d \to [0,\infty)$ is continuous, and $e_* := \sup_{u\in\mathcal{N}^d} e(u) < \infty$
(b) If $u \in \mathcal{N}^d \cap C_0^1(\overline{\Omega})$, then $\eta(t,u) \in C_0^1(\overline{\Omega})$ for every $t \in [0, e(u)]$.
(c) If $B \subset \mathcal{N}^d \cap C_0^1(\overline{\Omega})$ is bounded in $C_0^1(\overline{\Omega})$, then $\widehat{B} := \{\eta(t,u) : u \in B,\ t \in [0, e(u)]\}$ is bounded in $C_0^1(\overline{\Omega})$.
(d) If $u \in \mathcal{N}^d \cap C_0^1(\overline{\Omega})$ and $u \ge 0$, then $\eta(t,u) \ge 0$ for every $t \ge 0$.

Proof. (a) Since $\mathcal{N}^b$ is closed, it immediately follows that e is a lower semi-continuous function. To show upper semi-continuity, let $u \in \mathcal{N}^d$. Then $J(\eta(e(u),u)) = b$, hence $\eta(e(u),u)$ is not a critical point of J by assumption. Consequently, if $\varepsilon > 0$ is given, then $J(\eta(e(u)+\varepsilon,u)) < b$, and thus $J(\eta(e(u)+\varepsilon,v)) < b$ for v sufficiently close to u. We conclude that $e(v) \le e(u) + \varepsilon$ for v sufficiently close to u, and thus e is upper semi-continuous in u.
To show that $\sup_{u\in\mathcal{N}^d} e(u) < \infty$, we assume by contradiction that there is a sequence (u_n) in $\mathcal{N}^d$ with $e(u_n) \to \infty$. Let $t_n \in [0, e(u_n)]$ be such that

$$\begin{aligned} e(u_n)\,\|\nabla_{\mathcal{N}} J(\eta(t_n,u_n))\|^2 &= \int_0^{e(u_n)} \|\nabla_{\mathcal{N}} J(\eta(s,u_n))\|^2\, ds \\ &= -\int_0^{e(u_n)} \frac{d}{ds} J(\eta(s,u_n))ds = J(u_n) - b \le d - b. \end{aligned}$$

Then $\nabla_{\mathcal{N}} J(\eta(t_n,u_n)) \to 0$ and, since $J(\eta(t_n,u_n)) \in [b,d]$ and J satisfies $(\mathrm{PS})_c$ at every $c \in [b,d]$, a subsequence of $(\eta(t_n,u_n))$ converges to a critical point u of J with $J(u) \in [b,d]$, contradicting our assumption that J has no critical values in this interval.
(c) The negative gradient flow of J on $\mathcal{N}$ is given by

$$\eta(t,u) = e^{-A(t,u)}u + e^{-A(t,u)} \int_0^t e^{A(s,u)} b(\eta(s,u)) K(\eta(s,u)) ds, \tag{2.5}$$

where $A(t,u) := \int_0^t a(\eta(s,u))ds$ and a and b are the functions defined in (2.4). Since $J(u) = \frac{1}{N}\|u\|^2$ for every $u \in \mathcal{N}$, we have that $\mathcal{N}^d$ is bounded in $H_0^1(\Omega)$. Thus, it is easy to see that

$$a_* := \sup_{u\in\mathcal{N}^d} a(u) < \infty \qquad \text{and} \qquad b_* := \sup_{u\in\mathcal{N}^d} b(u) < \infty.$$

Let $\|\cdot\|_{k,r}$ denote the usual norm of the Sobolev space $W^{k,r}(\Omega)$. It follows from (2.5) that

$$\|\eta(t,u)\|_{1,r} \le e^{a_* e_*} \|u\|_{1,r} + e^{2a_* e_*} b_* \int_0^t \|K(\eta(s,u))\|_{1,r}\, ds \tag{2.6}$$
$$\forall\ u \in W^{1,r}(\Omega) \cap \mathcal{N}^d,\ t \in [0, t_b(u)].$$

We prove *(c)* by a bootstrap argument. In the following $c_1, c_2, c_3, \ldots$ denote positive constants which depend only on d, b, N and Ω. Let $p := \frac{3N}{7}$. The Sobolev embedding theorem yields

$$W^{2,\frac{3N}{7}}(\Omega) \hookrightarrow W^{1,\frac{3N}{4}}(\Omega) \hookrightarrow L^{3N}(\Omega).$$

Note that $(2^* - 1)p = (2^* - 2)p + p = \frac{6}{7}2^* + \frac{1}{7}3N$. Hence,

$$\begin{aligned}
\|K(u)\|_{1,\frac{3N}{4}}^p &\le c_1 \|K(u)\|_{2,p}^p \le c_2 \left| |u|^{2^*-1} \right|_p^p \\
&= c_2 \int_\Omega |u|^{(2^*-1)p} \le c_2 \left(\int_\Omega |u|^{2^*} \right)^{6/7} \left(\int_\Omega |u|^{3N} \right)^{1/7} \\
&\le c_3 \|u\|^{\frac{6}{7}2^*} \|u\|_{1,\frac{3N}{4}}^p \le c_4 \|u\|_{1,\frac{3N}{4}}^p \qquad \forall u \in W^{1,\frac{3N}{4}}(\Omega) \cap \mathcal{N}^d.
\end{aligned}$$

So if $B \subset W^{1,\frac{3N}{4}}(\Omega) \cap \mathcal{N}^d$ is bounded in $W^{1,\frac{3N}{4}}(\Omega)$ then, combining the above inequality with (2.6), we obtain

$$\begin{aligned}
\|\eta(t,u)\|_{1,\frac{3N}{4}} &\le e^{a_* e_*} \|u\|_{1,\frac{3N}{4}} + e^{2a_* e_*} b_* \int_0^t \|K(\eta(s,u))\|_{1,\frac{3N}{4}} \, ds \\
&\le c_5 + c_6 \int_0^t \|\eta(s,u)\|_{1,\frac{3N}{4}} \, ds \qquad \forall u \in B, \ t \in [0, t_b(u)],
\end{aligned}$$

and Gronwall's inequality yields

$$\|\eta(t,u)\|_{1,\frac{3N}{4}} \le c_5 e^{c_6 t} \le c_5 e^{c_6 e_*} \qquad \forall u \in B, \ t \in [0, t_b(u)].$$

Hence $\widehat{B} := \{\widehat{\eta}(s,u) : u \in B, \, s \in [0,1]\}$ is bounded in $W^{1,\frac{3N}{4}}(\Omega)$.
Next, let $q := \frac{3N(N-2)}{4} > \frac{N}{2}$ and $q^* := \frac{Nq}{N-q} > N$. We have the following embeddings

$$W^{2,q}(\Omega) \hookrightarrow W^{1,q^*}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Since $(2^* - 1)q = (2^* - 2)q + q = 3N + q$ we obtain

$$\begin{aligned}
\|K(u)\|_{1,q^*}^q &\le c_7 \|K(u)\|_{2,q}^q \le c_8 \left| |u|^{2^*-1} \right|_q^q \\
&= c_8 \int_\Omega |u|^{(2^*-1)q} \le c_8 |u|_\infty^q \int_\Omega |u|^{3N} \qquad (2.7) \\
&\le c_9 \|u\|_{1,q^*}^q \|u\|_{1,\frac{3N}{4}}^{3N} \qquad \forall u \in W_0^{1,q^*}(\Omega).
\end{aligned}$$

If $B \subset W^{1,q^*}(\Omega) \cap \mathcal{N}^d$ is bounded in $W^{1,q^*}(\Omega)$ then it is also bounded in $W^{1,\frac{3N}{4}}(\Omega)$ and, by the previous step, $\widehat{B} := \{\widehat{\eta}(s,u) : u \in B, \, s \in [0,1]\}$ is bounded in $W^{1,\frac{3N}{4}}(\Omega)$. So inequality (2.7) yields

$$\|K(\eta(t,u))\|_{1,q^*} \le c_9^{1/q} \left(\sup_{v \in \widehat{B}} \|v\|_{1,\frac{3N}{4}}^{3N} \right)^{1/q} \|\eta(t,u)\|_{1,q^*} \qquad \forall u \in B, \ t \in [0, t_b(u)].$$

Combining this last inequality with (2.6) and using Gronwall's inequality as we did before, we obtain

$$\|\eta(t,u)\|_{1,q^*} \le c_{10} < \infty \qquad \forall u \in B, \ t \in [0, t_b(u)].$$

Hence $\widehat{B}$ is bounded in $W^{1,q^*}(\Omega)$.
Finally, consider the embeddings

$$W^{2,q^*}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \hookrightarrow W^{1,q^*}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Hence we have

$$\|K(u)\|_{C^1}^{q^*} \le c_{11}\,\|K(u)\|_{2,q^*}^{q^*} \le c_{12}\left||u|^{2^*-1}\right|_{q^*}^{q^*} \le c_{13}\,|u|_\infty^{(2^*-1)q^*} \le c_{14}\,\|u\|_{1,q^*}^{(2^*-2)q^*}\,\|u\|_{C^1}^{q^*} \qquad \forall u \in C_0^1(\overline{\Omega}). \tag{2.8}$$

Let $B \subset \mathcal{N}^d \cap C_0^1(\overline{\Omega})$ be bounded in $C_0^1(\overline{\Omega})$. Then B is also bounded in $W^{1,q^*}(\Omega)$ and, by the preceeding step, so is $\widehat{B}$. So (2.8) yields

$$\|K(\eta(t,u))\|_{C^1} \le c_{14}^{1/q^*}\left(\sup_{v\in\widehat{B}}\|v\|_{1,q^*}^{2^*-2}\right)\|\eta(t,u)\|_{C^1} \qquad \forall u \in B,\ t\in[0,t_b(u)].$$

Using (2.6) and Gronwall's inequality as before, we obtain

$$\|\eta(t,u)\|_{C^1} \le c_{15} < \infty \qquad \forall u \in B,\ t \in [0,t_b(u)].$$

Therefore $\widehat{B}$ is bounded in $C_0^1(\overline{\Omega})$, and the proof of (c) is finished.
(b) is a special case of *(c)*.
Property *(d)* has already been proved by the authors in [6, Appendix] for a different superlinear problem. Since the proof is the same in the present situation, we refer the reader to [6] for details. □

3. Punctured domains

For $x \in \mathbb{R}^N$ and $\delta > 0$, set $B_\delta(x) := \{y \in \mathbb{R}^N : |y-x| \le \delta\}$. Fix a radial function $\phi \in C^\infty(\mathbb{R}^N)$ such that $\phi(z) = 1$ if $|z| \ge 2$ and $\phi(z) = 0$ if $|z| \le 1$. For $\delta > 0$ set

$$\phi_\delta(z) := \phi\left(\frac{z}{\delta}\right).$$

Let

$$\mathcal{U}_\delta := \{(x,u) \in \mathbb{R}^N \times \mathcal{N} : \phi_\delta(\cdot - x)u \neq 0\}$$

and consider the *puncturing map*

$$\pi_\delta : \mathcal{U}_\delta \to \mathcal{N}, \qquad \pi_\delta(x,u) := \rho\left(\phi_\delta(\cdot - x)u\right), \tag{3.1}$$

where ρ is the radial projection onto the Nehari manifold defined in (2.1). Note that $\pi_\delta(x,u) \in C_0^1(\overline{\Omega})$ if $u \in C_0^1(\overline{\Omega})$, $\pi_\delta(x,u)\,|_{B_\delta(x)} \equiv 0$ and

$$\pi_\delta(x,u) = u \quad \text{if} \quad \operatorname{dist}(x, \operatorname{supp}(u)) \ge 2\delta. \tag{3.2}$$

The following holds.

Lemma 2. *Let $\mathcal{K}$ be a compact subset of $H_0^1(\Omega)$ such that $\mathcal{K} \subset \mathcal{N} \cap C_0^1(\overline{\Omega})$ and $\mathcal{K}$ is bounded in $C_0^1(\overline{\Omega})$. Then, given $\gamma > 0$, there exists $\delta > 0$ such that $\mathbb{R}^N \times \mathcal{K} \subset \mathcal{U}_\delta$ and the puncturing map π_δ satisfies*

$$|J(\pi_\delta(x,u)) - J(u)| < \gamma \qquad \forall (x,u) \in \mathbb{R}^N \times \mathcal{K}.$$

Proof. Put $\psi_\delta := 1-\phi_\delta \in C_c^\infty(\mathbb{R}^N)$ for $\delta>0$. Since $\mathcal{K}$ is bounded in $C_0^1(\overline{\Omega})$, there exists $C>0$ such that, for all $(x,u)\in\mathbb{R}^N\times\mathcal{K}$, the following holds:

$$\begin{aligned}\|\phi_\delta(\cdot-x)u-u\|^2 &= \|\psi_\delta(\cdot-x)u\|^2\\ &= \int_\Omega |\psi_\delta(\cdot-x)\nabla u + u\nabla\psi_\delta(\cdot-x)|^2\\ &\le 4\int_\Omega \left(|\psi_\delta(\cdot-x)|^2|\nabla u|^2 + |\nabla\psi_\delta(\cdot-x)|^2|u|^2\right)\\ &\le C\left(|\psi_\delta|_2^2+\|\psi_\delta\|^2\right).\end{aligned}$$

Moreover, an easy computation shows that $\|\psi_\delta\|^2\to 0$ and $|\psi_\delta|_2^2\to 0$ as $\delta\to 0$. Hence there is $\delta_0>0$ with $\phi_\delta(\cdot-x)u\neq 0$ for $0<\delta\le\delta_0$, $(x,u)\in\mathbb{R}^N\times\mathcal{K}$, and $\|\phi_\delta(\cdot-x)u-u\|\to 0$ uniformly in $(x,u)\in\mathbb{R}^N\times\mathcal{K}$ as $\delta\to 0$. Note that $\phi_\delta(\cdot-x)u=u$ if $\operatorname{dist}(x,\Omega)\ge 2\delta$. Therefore the set

$$\mathcal{K}_0 := \{u,\phi_\delta(\cdot-x)u \,:\, (x,u)\in\mathbb{R}^N\times\mathcal{K}, 0<\delta\le\delta_0\}$$

is compact, and the function $J\circ\rho$ is uniformly continuous on $\mathcal{K}_0$. We conclude that, for $\delta>0$ small enough,

$$|J(\pi_\delta(x,u))-J(u)| = |(J\circ\rho)(\phi_\delta(\cdot-x)u))-(J\circ\rho)(u)|<\gamma \quad \forall (x,u)\in\mathbb{R}^N\times\mathcal{K},$$

as claimed. □

Next, let $\beta: H_0^1(\Omega)\setminus\{0\}\to\mathbb{R}^N$ be the barycenter map defined by

$$\beta(u)=\frac{\int_{\mathbb{R}^N} x\,|u(x)|^{2^*}\,dx}{\int_{\mathbb{R}^N}|u(x)|^{2^*}\,dx}. \tag{3.3}$$

We shall consider subdomains of Ω with the following property.

Definition 1. *Let $\varepsilon>0$. We shall say that a domain $\Omega_0\subset\Omega$ is an ε-punctured subdomain of Ω if there exists a closed ball $B\subset\mathbb{R}^N$ with $\partial B\subset\Omega_0$, $B\not\subset\Omega$, such that, for every $\delta>0$, there is a continuous map $h_\delta: B\to C_0^1(\overline{\Omega}_0)$ with the following properties:*
(i) $h_\delta(x)\in\mathcal{N}$, and it is a nonnegative function for every $x\in B$.
(ii) $J(h_\delta(x))\le c_\infty+\varepsilon$ for each $x\in B$.
(iii) $J(h_\delta(x))\le c_\infty+\delta$ for each $x\in\partial B$.
(iv) $\operatorname{supp}(h_\delta(x))\subset B_\delta(x)$ for each $x\in\partial B$.
(v) $\beta(h_\delta(x))=x$ for each $x\in\partial B$.
We then say that B is a closed ball associated to Ω_0.

We show that annular-shaped domains contain ε-punctured subdomains. More precisely, the following holds.

Lemma 3. *Let $\varepsilon>0$. If R_2/R_1 is large enough, then the annulus $A_{R_1,R_2}:=\{x\in\mathbb{R}^N : R_1<|x|<R_2\}$ is an ε-punctured subdomain of any domain Ω with $A_{R_1,R_2}\subset\Omega$ and $\{x\in\mathbb{R}^N:|x|<R_1\}\setminus\Omega\neq\varnothing$.*

The result is not surprising in view of the original construction in [7], but extra work is needed to verify all properties (i)–(v) listed in Definition 1.

Proof of Lemma 3. By dilation invariance, it suffices to show that, for $R>0$ sufficiently large and $r>0$ sufficiently small, $A_{r,R}$ is an ε-punctured subdomain of

any domain Ω with $A_{r,R} \subset \Omega$ and $\{x \in \mathbb{R}^N : |x| < r\} \setminus \Omega \neq \varnothing$.
To show this, we fix a radial cut-off function $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \chi \le 1$ in $\mathbb{R}^N$, $\chi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\chi(x) = 0$ for $|x| \ge 1$. Moreover, for $R > 0$ we consider the functions $U_R := \rho\big(\chi(\cdot/R)U(\cdot)\big) \in C_c^\infty(\mathbb{R}^N) \cap \mathcal{N}$, where U is the instanton defined in (2.2). A straightforward calculation shows that $U_R \to U$ in $D^{1,2}(\mathbb{R}^N)$ as $R \to \infty$ and therefore

$$J_\infty(U_R) \to J_\infty(U) = c_\infty \qquad \text{as } R \to \infty. \tag{3.4}$$

Next we let B be the unit ball centered at zero, and for fixed $R > 0$ we define $h_R : \operatorname{int}(B) \to \mathcal{N} \cap C_c^\infty(\mathbb{R}^N)$ by

$$h_R(y) = \rho\big[((1-|y|)^2, y) * U_{R/(1-|y|)}\big]$$

Then

$$\beta(h_R(y)) = y \qquad \text{for every } R > 0,\ y \in \operatorname{int}(B).$$

Moreover, (3.4) and the dilation invariance of J_∞ imply that, for every $R > 0$,

$$J_\infty(h_R(y)) \to c_\infty \qquad \text{as } |y| \to 1.$$

Furthermore, the support of $h_R(y)$ is contained in the ball $B_{R(1-|y|)}(y) \subset B_R(0)$. Note also that

$$\sup_{y \in \operatorname{int}(B)} J_\infty(h_R(y)) \to c_\infty \qquad \text{as } R \to \infty.$$

Now fix $R > 0$ large enough such that

$$\sup_{y \in \operatorname{int}(B)} J_\infty(h_R(y)) < c_\infty + \varepsilon,$$

and fix $s_R \in (0,1)$ such that

$$\operatorname{supp}(h_R(y)) \cap B_{\frac{1}{2}}(0) = \varnothing \qquad \text{if } |y| \ge s_R. \tag{3.5}$$

By Lemma 2 there is $0 < r < \frac{1}{2}$ sufficiently small such that

$$J\big(\pi_r(0, h_R(y))\big) < c_\infty + \varepsilon \qquad \text{for } |y| \le s_R. \tag{3.6}$$

Since $\pi_r(0, h_R(y)) = h_R(y)$ for $|y| \ge s_R$ by (3.5), we actually get (3.6) for all $y \in \operatorname{int}(B)$. In particular, if we define

$$\hat{h} : \operatorname{int}(B) \to \mathcal{N} \cap C_c^\infty(\mathbb{R}^N), \qquad \hat{h}(y) = \pi_r(0, h_R(y)),$$

then we have

$$\begin{aligned}
&\operatorname{supp}(\hat{h}(y)) \subset B_{R(1-|y|)}(y) \cap A_{r,R} \qquad \text{for every } y \in \operatorname{int}(B),\\
&\beta(\hat{h}(y)) = y \qquad \text{if } |y| \ge s_R,\\
&\sup_{y \in \operatorname{int}(B)} J_\infty(\hat{h}(y)) < c_\infty + \varepsilon,\\
&J_\infty(\hat{h}(y)) \to c_\infty \qquad \text{as } |y| \to 1.
\end{aligned}$$

Now let $0 < \delta < R - 1$ be arbitrary, and choose $s_\delta \in (s_R, 1)$ such that

$$\sup_{|y| = s_\delta} J_\infty(\hat{h}(y)) < c_\infty + \delta \qquad \text{and} \qquad R(1 - s_\delta) < \delta.$$

We then define $h_\delta : B \to \mathcal{N} \cap C_0^1(A_{r,R})$ by

$$h_\delta(y) = \begin{cases} \hat{h}(y), & |y| \le s_\delta, \\ \dfrac{(|y| - s_\delta)y}{|y|} * \hat{h}(s_\delta \dfrac{y}{|y|}), & s_\delta < |y| \le 1. \end{cases}$$

It is now easy to check that h_δ satisfies properties (i)–(iv) from Definition 1. Since $\delta > 0$ was arbitrarily small, we conclude that $A_{r,R}$ is an ε-punctured subdomain of any domain Ω with $A_{r,R} \subset \Omega$ and $\{x \in \mathbb{R}^N : |x| < r\} \smallsetminus \Omega \neq \varnothing$. The proof is finished. □

Next, let

$$\mathcal{E} := \{u \in \mathcal{N} : u^+, u^- \in \mathcal{N}\} \tag{3.7}$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. We need the following technical lemma.

Lemma 4. *Assume that, for some $\varepsilon < c_\infty$, Ω contains two disjoint ε-punctured subdomains Ω_1 and Ω_2 with associated balls B_1 and B_2. Let $c_\infty < c_0 < c_1 < c_\infty + \varepsilon$, and assume that J has no critical values in $(c_1, 2c_\infty)$. Then there exists a map $g : B_1 \times B_2 \to \mathcal{E}$ with the following properties:*
(i) $J(g(x, y)) \le 2c_\infty + 2\varepsilon$ for each $(x, y) \in B_1 \times B_2$.
(ii) $J(g(x, y)) \le c_0 + c_1$ for each $(x, y) \in (\partial B_1 \times B_2) \cup (B_1 \times \partial B_2)$.
(iii) $J(g(x, y)^+) \le c_0$ and $\beta(g(x, y)^+) = x$ for each $(x, y) \in \partial B_1 \times B_2$.
(iv) $J(g(x, y)^-) \le c_0$ and $\beta(g(x, y)^-) = y$ for each $(x, y) \in B_1 \times \partial B_2$.

Remark 1. *Since $J(g(x, y)) = J(g(x, y)^+) + J(g(x, y)^-)$ and $g(x, y)^\pm \in \mathcal{N}$, property (i) implies that*

$$c_\infty < J(g(x, y)^\pm) < c_\infty + 2\varepsilon \qquad \text{for all } (x, y) \in B_1 \times B_2. \tag{3.8}$$

Proof of Lemma 4. Set $d_i :=\text{dist}(\partial B_i, \mathbb{R}^N \smallsetminus \Omega_i), i = 1, 2$, and

$$\delta_0 := \frac{1}{2} \min\{\varepsilon, d_1, d_2, c_0 - c_\infty, c_\infty + \varepsilon - c_1\}.$$

Let

$$h^i_{\delta_0} : B_i \to C^1_0(\overline{\Omega}_i), \quad i = 1, 2.$$

be maps having properties *(i)*-*(v)* of Definition 1. It is well known that J satisfies $(\text{PS})_c$ at every $c \in (c_\infty, 2c_\infty)$, and J has no critical values in $[c_1 + \delta_0, c_\infty + \varepsilon]$ by assumption. Hence Lemma 1 applies to the interval $[c_1 + \delta_0, c_\infty + \varepsilon]$, and we may define the continuous deformation

$$[0, 1] \times \mathcal{N}^{c_\infty+\varepsilon} \to \mathcal{N}^{c_\infty+\varepsilon}, \qquad (s, u) \mapsto \eta^s(u) := \eta(se(u), u),$$

where $e(u) := \inf\{t \ge 0 : \eta(t, u) \in \mathcal{N}^{c_1+\delta_0}\} < \infty$ is the entrance time of u in $\mathcal{N}^{c_1+\delta_0}$. In particular,

$$\eta^0(u) = u,\ \eta^t(v) = v,\ \eta^1(u) \in \mathcal{N}^{c_1+\delta_0} \qquad \forall\, u \in \mathcal{N}^{c_\infty+\varepsilon},\ v \in \mathcal{N}^{c_1+\delta_0},\ t \in [0, 1].$$

Set

$$\mathcal{K}_i := \{\eta^t(h^i_{\delta_0}(x)) : x \in B'_i,\ t \in [0, 1]\}.$$

It follows from Lemma 1 that $\mathcal{K}_i$ is a bounded subset of $C^1_0(\overline{\Omega})$ and that $\mathcal{K}_i \subset \mathcal{N}^+ := \{u \in \mathcal{N} : u \ge 0\}$. We apply Lemma 2 to obtain a $0 < \delta < \delta_0$ such that

$\mathcal{K}_1 \cup \mathcal{K}_2 \subset \mathcal{U}_\delta$ and the puncturing map $\pi_\delta : \mathcal{U}_\delta \to \mathcal{N}$, defined in (3.1), satisfies $\pi(x,u)\,|_{B_\delta(x)} \equiv 0$ and

$$|J(\pi_\delta(x,u)) - J(u)| < \delta_0 \quad \forall (x,u) \in \mathbb{R}^N \times (\mathcal{K}_1 \cup \mathcal{K}_2)\,.$$

Let z_i be the center of the ball B_i, and set $r := \frac{\delta_0}{\delta}$. Consider the ball

$$B_i' := \{z_i + \lambda(x - z_i) : x \in \partial B_i,\ \lambda \in [0,r]\}.$$

To simplify notation we write $[x,\lambda]_i := z_i + \lambda(x - z_i)$. We extend $h^i_{\delta_0}$ to a map $g_i : B_i' \to C^1_0(\overline{\Omega}_i)$ as follows:

$$g_i([x,\lambda]_i) := \begin{cases} h^i_{\delta_0}([x,\lambda]_i) & \text{if } \lambda \in [0,1],\ x \in \partial B_i \\ \lambda^{-1} * h^i_{\delta_0}(x) & \text{if } \lambda \in [1,r],\ x \in \partial B_i. \end{cases}$$

Then,

$$\begin{aligned} & g_i([x,\lambda]_i) \in \mathcal{N}^+ && \forall \lambda \in [0,r],\ x \in \partial B_i, \\ & \operatorname{supp}(g_i([x,r]_i)) \subset B_\delta(x) && \forall x \in \partial B_i, \\ & J(g_i([x,\lambda]_i)) = J(h^i_{\delta_0}(x)) \le c_\infty + \delta_0 && \forall \lambda \in [1,r],\ x \in \partial B_i, \\ & \beta(g_i([x,\lambda]_i)) = \beta(h^i_{\delta_0}(x)) = x && \forall \lambda \in [1,r],\ x \in \partial B_i. \end{aligned} \tag{3.9}$$

Let

$$B_i'' := \{z_i + \lambda(x - z_i) : x \in \partial B_i,\ \lambda \in [0,1+r]\}$$

and define $\widehat{g} : B_1'' \times B_2'' \to \mathcal{E}$ as follows: For $x \in \partial B_1$, $y \in \partial B_2$, $\lambda, \mu \in [0,1+r]$, let

$$\widehat{g}([x,\lambda]_1, [y,\mu]_2) = \begin{cases} g_1([x,\lambda]_1) - g_2([y,\mu]_2) & \text{if } \lambda,\mu \le r \\ \pi_\delta\big(y, \eta^{\mu-r}(g_1([x,\lambda]_1))\big) - g_2([y,r]_2) & \text{if } \lambda \le r \le \mu \\ g_1([x,r]_1) - \pi_\delta\big(x, \eta^{\lambda-r}(g_2([y,\mu]_2))\big) & \text{if } \mu \le r \le \lambda \\ g_1([x,r]_1) - g_2([y,r]_2) & \text{if } \lambda,\mu \ge r \end{cases}$$

In all four cases, the first summand occuring in the definition of $\widehat{g}(z)$ is $\widehat{g}(z)^+$ and the second one is $\widehat{g}(z)^-$, and $\widehat{g}(z)^+, \widehat{g}(z)^- \in \mathcal{N}$ for each $z \in B_1'' \times B_2''$. It follows from (3.2) and the third relation in (3.9) that

$$\pi_\delta\big(y, \eta^{\mu-r}(g_1([x,\lambda]_1))\big) = \pi_\delta(y, g_1([x,\lambda]_1)) = g_1([x,\lambda]_1) \quad \text{if } \lambda \le r = \mu,$$
$$\pi_\delta\big(y, \eta^{\mu-r}(g_1([x,\lambda]_1))\big) = \pi_\delta(y, g_1([x,r]_1)) = g_1([x,r]_1) \quad \text{if } \lambda = r \le \mu.$$

The other cases needed to show that $\widehat{g}$ is well defined and continuous are similar. Since $J(u) = J(u^+) + J(u^-)$, it is straightforward to verify that $\widehat{g}$ satisfies

(i') $J(\widehat{g}(x,y)) \le 2c_\infty + 2\varepsilon$ for each $(x,y) \in B_1'' \times B_2''$.

(ii') $J(\widehat{g}(x,y)) \le c_0 + c_1$ for each $(x,y) \in (\partial B_1'' \times B_2'') \cup (B_1'' \times \partial B_2'')$.

(iii') $J(\widehat{g}(x,y)^+) \le c_0$ and $\beta(\widehat{g}(x,y)^+) = (1+r)^{-1}x$ for each $(x,y) \in \partial B_1'' \times B_2''$.

(iv') $J(\widehat{g}(x,y)^-) \le c_0$ and $\beta(\widehat{g}(x,y)^-) = (1+r)^{-1}y$ for each $(x,y) \in B_1'' \times \partial B_2''$.

Composing $\widehat{g}$ with the reparametrization

$$\varrho : B_1 \times B_2 \to B_1'' \times B_2'', \qquad \varrho(x,y) = ((1+r)x, (1+r)y),$$

we obtain a map $g := \widehat{g} \circ \varrho : B_1 \times B_2 \to \mathcal{E}$ having all the desired properties. □

Annular-shaped domains satisfy the hypotheses of the previous lemma.

Lemma 5. *Let $\varepsilon > 0$ and assume that there are constants $R_2 > R_1 > 0$ such that*

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega \quad \text{and} \quad \{x \in \mathbb{R}^N : |x| < R_1\} \smallsetminus \Omega \neq \varnothing.$$

If R_2/R_1 is large enough, then Ω contains two disjoint ε-punctured subdomains.

Proof. Let $R_0 := \sqrt{R_1 R_2}$. Then $R_2/R_0 = R_0/R_1 = \sqrt{R_2/R_1}$. So, by Lemma 3, A_{R_1,R_0} and A_{R_0,R_2} are disjoint ε-punctured subdomains of Ω if R_2/R_1 is large enough. □

4. The main theorem

We fix a constant $0 < \varepsilon_0 < c_\infty$ such that the statement of Theorem 2 holds. Our main result is the following.

Theorem 3. *Assume that Ω contains two disjoint $\frac{\varepsilon_0}{3}$-punctured subdomains. Then problem $(\wp)$ has at least two solutions.*

Theorem 1 follows immediately from Theorem 3 and Lemma 5. The rest of this section is devoted to the proof of Theorem 3.

Let Ω_1 and Ω_2 be disjoint $\frac{\varepsilon_0}{3}$-punctured subdomains in Ω. Let $B_1, B_2 \subset \mathbb{R}^N$ be closed balls associated to Ω_1 and Ω_2, as in Definition 1. Fix $x_i \in \operatorname{int}(B_i) \smallsetminus \overline{\Omega}$ for $i = 1, 2$. Then one has that

$$c_\infty < \inf\{J(u) : u \in \mathcal{N},\ \beta(u) \in \{x_1, x_2\}\},$$

[22, Lemma 5.23]. We fix $c_0 < c_\infty + \frac{\varepsilon_0}{3}$ such that

$$c_\infty < c_0 < \inf\{J(u) : u \in \mathcal{N},\ \beta(u) \in \{x_1, x_2\}\}, \tag{4.1}$$

and define

$$c_i := \inf\{c \geq c_0 : \quad \beta^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_i\}) \to H^N(\mathcal{N}^c, \mathcal{N}^{c_0}) \ \text{ is a monomorphism}\}$$

where $\mathcal{N}^c := \{u \in \mathcal{N} : J(u) \leq c\}$, H^* is Alexander-Spanier (or Čech) cohomology with integer coefficients, and β^* is the homomorphism induced in cohomology by the barycenter map. Note that $c_0 < c_i$ for $i = 1, 2$ by (4.1). The following holds.

Proposition 1. $c_i \leq c_\infty + \frac{\varepsilon_0}{3}$, *and there exists a positive solution u_i to problem $(\wp)$ such that $J(u_i) = c_i$, $i = 1, 2$.*

Proof. Set $\varepsilon := \frac{\varepsilon_0}{3}$ and $\delta := c_0 - c_\infty > 0$. By Definition 1, there exists a map

$$h_\delta : (B_i, \partial B_i) \to (\mathcal{N}^{c_\infty + \varepsilon}, \mathcal{N}^{c_0})$$

such that $\beta(h_\delta(x)) = x$ for every $x \in \partial B_i$. Since $x_i \in \operatorname{int}(B_i)$, the inclusion $\partial B_i \hookrightarrow \mathbb{R}^N \smallsetminus \{x_i\}$ induces an isomorphism in cohomology and the exactness property of cohomology yields that $(\beta \circ h_\delta)^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_i\}) \to H^N(B_i, \partial B_i)$ is an isomorphism. Therefore,

$$\beta^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_i\}) \to H^N(\mathcal{N}^{c_\infty + \varepsilon}, \mathcal{N}^{c_0})$$

is a monomorphism and, hence, $c_i \leq c_\infty + \varepsilon < 2c_\infty$.
Assume, by contradiction, that c_i is not a critical value of J. Then, since J satisfies $(\mathrm{PS})_c$ for $c \in (c_\infty, 2c_\infty)$, we infer that J contains no critical values in $[c_i - \gamma, c_i + \gamma]$ for $0 < \gamma < c_i - c_0$ small enough. Thus we may use Lemma 1 to define the continuous map

$$[0,1] \times \mathcal{N}^{c_i + \gamma} \to \mathcal{N}^{c_i + \gamma}, \qquad (t, u) \mapsto \eta(te(u), u),$$

where $e(u) := \inf\{t \geq 0 : \eta(t,u) \in \mathcal{N}^{c_i-\gamma}\} < \infty$ is the entrance time of u in $\mathcal{N}^{c_i-\gamma}$. This defines a deformation of $\mathcal{N}^{c_i+\gamma}$ into $\mathcal{N}^{c_i-\gamma}$ which keeps $\mathcal{N}^{c_i-\gamma}$ fixed. Hence, the inclusion $\iota : (\mathcal{N}^{c_i-\gamma}, \mathcal{N}^{c_0}) \hookrightarrow (\mathcal{N}^{c_i+\gamma}, \mathcal{N}^{c_0})$ induces an isomorphism in cohomology. Since $\beta^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_i\}) \to H^N(\mathcal{N}^{c_i+\gamma}, \mathcal{N}^{c_0})$ is a monomorphism and $\beta \circ \iota = \beta$, it follows that $\beta^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_i\}) \to H^N(\mathcal{N}^{c_i-\gamma}, \mathcal{N}^{c_0})$ is a monomorphism too, contradicting the definition of c_i. Hence, c_i is a critical value of J.

Finally, a well known argument [2, Proof of Theorem A] shows that no solution u of problem $(\wp)$ with $J(u) \in (c_\infty, 2c_\infty)$ changes sign. Since solutions occur in pairs $\pm u$, problem $(\wp)$ has a positive solution u_i with $J(u_i) = c_i$. □

Proposition 1 does not guarantee that $u_1 \neq u_2$. So, in order to prove Theorem 3, we still need to show the existence of a second solution.

Without loss of generality we assume from now on that

$$c_1 \geq c_2 \qquad \text{and} \qquad x_2 = 0.$$

Lemma 6. *If J has no critical values in $(c_1, 2c_\infty)$ then J satisfies $(PS)_c$ for every $c \in (c_\infty + c_1, 2c_\infty + \varepsilon_0]$.*

Proof. Let (u_k) be a sequence in $H_0^1(\Omega)$ such that $J(u_k) \to c \in (c_\infty + c_1, 2c_\infty + \varepsilon_0]$ and $\nabla J(u_k) \to 0$. Theorem 2 implies that, up to translations and dilations, the instanton U defined in (2.2) is the only solution of the limit problem $(\wp_\infty)$ in $J_\infty^{-1}(0, 2c_\infty + \varepsilon_0]$. Thus, by Struwe's compactness lemma [19, 20], one of the following assertions holds:

(i) (u_k) contains a convergent subsequence, or

(ii) $c = c' + mc_\infty$, where c' is a critical value of J and $m \geq 1$.

Note that $2c_\infty < c < 3c_\infty$, since $\varepsilon_0 < c_\infty$. So, if *(ii)* holds, then $c' \neq 0$, which implies $c' > c_\infty$ and therefore $m = 1$. But then $c_1 < c - c_\infty = c' < 2c_\infty$, contradicting our assumption that J has no critical values in $(c_1, 2c_\infty)$. Therefore, *(i)* must hold. □

From now on we make the following

Assumption: J has no critical values in $(c_1, 2c_\infty + \varepsilon_0]$.

We will show that this assumption leads to a contradiction. We define

$$\gamma_0 : H_0^1(\Omega) \to \mathbb{R}, \qquad \gamma_0(u) = \begin{cases} \frac{|u|_{2^*}^{2^*}}{\|u\|^2} - 1 & \text{if } u \neq 0 \\ -1 & \text{if } u = 0 \end{cases}$$

Since $S\,|u|_{2^*}^2 \leq \|u\|^2$ for all $u \in H_0^1(\Omega)$, this function is continuous. Note that $\gamma_0(u) = 0$ if and only if $u \in \mathcal{N}$. We also define

$$\gamma : \mathcal{N} \to \mathbb{R}, \qquad \gamma(u) = \gamma_0(u^+) - \gamma_0(u^-).$$

Then

$$\text{(4.2)} \qquad \gamma(u) = -1 \text{ iff } u \leq 0, \qquad \gamma(u) = 1 \text{ iff } u \geq 0, \qquad \text{and} \quad \gamma(u) = 0 \text{ iff } u \in \mathcal{E},$$

where $\mathcal{E}$ is defined in (3.7). Set $\varepsilon := \frac{\varepsilon_0}{2}$. By Proposition 1, we have that

$$\text{(4.3)} \qquad c_\infty < c_0 < c_1 \leq c_\infty + \frac{\varepsilon_0}{3} < c_\infty + \varepsilon < c_\infty + 2\varepsilon < 2c_\infty.$$

Since we are assuming that J has no critical values in $(c_1, 2c_\infty)$, we may fix a map $g : B_1 \times B_2 \to \mathcal{E}$ satisfying properties *(i)*-*(iv)* of Lemma 4. Moreover, by Lemma 6 and Lemma 1, applied to the interval $[c_0+c_1, 2c_\infty+\varepsilon_0] \subset (c_\infty+c_1, 2c_\infty+\varepsilon_0]$, we may define a continuous map

$$[0,1]\times \mathcal{N}^{2c_\infty+\varepsilon_0} \to \mathcal{N}^{2c_\infty+\varepsilon_0}, \qquad (t,u)\mapsto \eta^t(u) := \eta(te(u),u),$$

where $e(u) := \inf\{t\geq 0 \;:\; \eta(t,u)\in \mathcal{N}^{c_0+c_1}\} < \infty$ is the entrance time of u in $\mathcal{N}^{c_0+c_1}$. In particular,

$$\eta^0(u)=u,\; \eta^t(v)=v,\; \eta^1(u)\in \mathcal{N}^{c_0+c_1} \qquad \forall\, u\in \mathcal{N}^{2c_\infty+\varepsilon_0},\; v\in \mathcal{N}^{c_0+c_1},\; t\in[0,1].$$

We now consider, for $t\in[0,1]$, the map $\widetilde{g}_t : B_1\times B_2\times[-1,1]\to \mathcal{N}^{2c_\infty+\varepsilon_0}$ defined by

$$\widetilde{g}_t(x,y,\lambda) := \eta^t\big(\rho[(1-\lambda)g(x,y)^+ + (1+\lambda)g(x,y)^-]\big).$$

Here we recall that ρ is the projection onto the Nehari manifold, see (2.1). Note that, since $g(x,y)\in \mathcal{E}\cap \mathcal{N}^{2c_\infty+\varepsilon_0}$, we have that

$$J(\rho[(1-\lambda)g(x,y)^+ + (1+\lambda)g(x,y)^-]) \leq J(g(x,y)) \leq 2c_\infty+\varepsilon_0 \qquad \text{for } -1\leq\lambda\leq 1,$$

so $\widetilde{g}_t$ is well defined. Property *(ii)* of g, stated in Lemma 4, together with (3.8) and (4.3), yields that

$$\begin{aligned} &\widetilde{g}_t(x,y,\lambda) = \rho[(1-\lambda)g(x,y)^+ + (1+\lambda)g(x,y)^-] \\ &\text{if } (x,y,\lambda)\in \partial\left(B_1\times B_2\times[-1,1]\right). \end{aligned} \tag{4.4}$$

Next we define, for $t\in[0,1]$, the map $\widetilde{\beta}_t : B_1\times B_2\times[-1,1]\to\mathbb{R}^N$ by

$$\widetilde{\beta}_t(\mathbf{z}) = \begin{cases} [1-\gamma(\widetilde{g}_t(\mathbf{z}))]\beta(\widetilde{g}_t(\mathbf{z})^-) + \gamma(\widetilde{g}_t(\mathbf{z}))y, & \text{if } \widetilde{g}_t(\mathbf{z})^-\neq 0,\\ y, & \text{if } \widetilde{g}_t(\mathbf{z})^- = 0. \end{cases}$$

Here and in the following, $\mathbf{z}=(x,y,\lambda)\in B_1\times B_2\times[-1,1]$. Then (4.2) implies that $\widetilde{\beta}_t$ is continuous and depends continuously on t. Now we define $\theta_t : B_1\times B_2\times [-1,1]\to\mathbb{R}^N\times\mathbb{R}$ for $t\in[0,1]$ by

$$\theta_t(\mathbf{z}) = \big(\theta_t^1(\mathbf{z}),\theta_t^2(\mathbf{z})\big) := \big(\widetilde{\beta}_t(\mathbf{z}), \gamma(\widetilde{g}_t(\mathbf{z}))\big).$$

Set

$$\mathcal{E}^* := \{u\in\mathcal{E} : \beta(u^-)=0\}.$$

Lemma 7. *Let* $\mathbf{z}=(x,y,\lambda)\in B_1\times B_2\times[-1,1]$. *Then:*
(a) If $\theta_t(\mathbf{z})=0$, *then* $\widetilde{g}_t(\mathbf{z})\in\mathcal{E}^*$.
(b) If $\lambda\in\{-1,1\}$, *then* $\theta_t^2(\mathbf{z})=\lambda$.
(c) If $y\in\partial B_2$, *then* $\theta_t^1(\mathbf{z})=y$.
(d) If $(y,\lambda)\in\partial\left(B_2\times[-1,1]\right)$, *then* $\theta_t(\mathbf{z})\neq 0$.

Proof. (a) If $\theta_t(\mathbf{z})=0$, then $\gamma(\widetilde{g}_t(\mathbf{z}))=0$ and thus $\widetilde{g}_t(\mathbf{z})\in\mathcal{E}$ by (4.2) and $\beta(\widetilde{g}_t(\mathbf{z})^-)=\widetilde{\beta}_t(\mathbf{z})=0$.
(b). Suppose first that $\lambda=1$. Then $\rho[(1-\lambda)g(x,y)^+ + (1+\lambda)g(x,y)^-] = g(x,y)^+$, and (4.4) yields $\widetilde{g}_t(\mathbf{z})=g(x,y)^+$, which implies that

$$\theta_t^2(\mathbf{z}) = \gamma(g(x,y)^+) = 1.$$

If $\lambda=-1$, a similar argument yields $\theta_t^2(\mathbf{z})=\gamma(g(x,y)^-)=-1$.
(c). Property *(iv)* of g yields that $\beta(g(x,y)^-)=y$ if $y\in\partial B_2$. Thus, by property *(ii)* of g, (4.4) and the definition of $\widetilde{\beta}_t$, we obtain $\theta_t^1(\mathbf{z})=\widetilde{\beta}_t(\mathbf{z})=y$ for $y\in\partial B_2$.
Assertion *(d)* follows immediately from *(b)* and *(c)*. □

We now extend $\theta_t : B_1 \times B_2 \times [-1,1] \to \mathbb{R}^N \times \mathbb{R}$ to the parameter interval $[-1,1]$ by defining

$$\theta_t(x,y,\lambda) := -t(y,\lambda) + (1+t)\theta_0(x,y,\lambda) \qquad \text{for } -1 < t < 0.$$

Note that this extension still satisfies

$$\theta_t(x,y,\lambda) \neq 0 \quad \text{if} \quad (y,\lambda) \in \partial\left(B_2 \times [-1,1]\right),\ t \in [-1,1], \tag{4.5}$$

while

$$\theta_{-1}(x,y,\lambda) = (y,\lambda) \qquad \text{for every } (x,y,\lambda). \tag{4.6}$$

We now complete the

Proof of Theorem 3. Consider the map

$$f_t : B_1 \times B_2 \times [-1,1] \to B_1 \times \mathbb{R}^N \times \mathbb{R}, \qquad t \in [-1,1].$$

given by

$$f_t(x,y,\lambda) := (x, (y,\lambda) - \theta_t(x,y,\lambda)).$$

Its fixed point set

$$\mathrm{Fix}(f_t) := \{(x,y,\lambda) \in B_1 \times B_2 \times [-1,1] : f_t(x,y,\lambda) = (x,y,\lambda)\}$$

is the set of zeroes of θ_t. Note that $\mathrm{Fix}(f_t)$ is compact. Because of (4.5) the map f_t is compactly fixed over B_1, and there are transfer homomorphisms

$$\tau_{f_t} : \begin{array}{lcl} H^*(\mathrm{Fix}(f_t), \mathrm{Fix}(f_t) \cap p^{-1}(\partial B_1)) & \to & H^*(B_1, \partial B_1), \\ H^*(\mathrm{Fix}(f_t) \cap p^{-1}(\partial B_1)) & \to & H^*(\partial B_1), \end{array} \qquad t \in [-1,1],$$

where $p : B_1 \times \mathbb{R}^N \times \mathbb{R} \to B_1$ is the projection. The definition and properties of the transfer are given in the Appendix. Since $f_{-1}(x,y,\lambda) = (x,0,0)$ by (4.6), the homotopy and unit properties of the transfer yield that

$$\tau_{f_1} \circ p^* = \tau_{f_{-1}} \circ p^* = id : H^{N-1}(\partial B_1) \to H^{N-1}(\partial B_1). \tag{4.7}$$

Moreover, since $\widetilde{g}_1(\mathbf{z}) \in \mathcal{E}^* \cap \mathcal{N}^{c_0+c_1}$ for $\mathbf{z} = (x,y,\lambda) \in \mathrm{Fix}(f_1)$, one has that

$$\begin{aligned} c_0 + c_1 &\geq J(\widetilde{g}_1(\mathbf{z})) = J(\widetilde{g}_1(\mathbf{z})^+) + J(\widetilde{g}_1(\mathbf{z})^-) \\ &> J(\widetilde{g}_1(\mathbf{z})^+) + c_0. \end{aligned}$$

Therefore, there exists $d < c_1$ such that $\widetilde{g}_1(\mathbf{z})^+ \in \mathcal{N}^d$ for all $\mathbf{z} \in \mathrm{Fix}(f_1)$. Properties *(ii)* and *(iii)* of g, stated in Lemma 4, give that

$$\left\{ \begin{array}{l} \widetilde{g}_1(\mathbf{z})^+ \in \mathcal{N}^{c_0} \quad \text{and} \quad \beta(\widetilde{g}_1(\mathbf{z})^+) = \beta(g(x,y)^+) = x = p(\mathbf{z}) \\ \text{for } \mathbf{z} = (x,y,\lambda) \in \mathrm{Fix}(f_1) \cap p^{-1}(\partial B_1). \end{array} \right. \tag{4.8}$$

Since $x_1 \in \mathrm{int}(B_1)$, the inclusion $\partial B_1 \to \mathbb{R}^N \smallsetminus \{x_1\}$ induces an isomorphism $H^{N-1}(\mathbb{R}^N \smallsetminus \{x_1\}) \cong H^{N-1}(\partial B_1)$. So (4.7) and (4.8) imply that the left column in the following commutative diagram is an isomorphism.

$$\begin{array}{ccc}
H^{N-1}(\mathbb{R}^N \smallsetminus \{x_1\}) & \overset{\delta^*}{\cong} & H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_1\}) \\
\downarrow \beta^* & & \downarrow \beta^* \\
H^{N-1}(\mathcal{N}^{c_0}) & \overset{\delta^*}{\longrightarrow} & H^N(\mathcal{N}^d, \mathcal{N}^{c_0}) \\
\downarrow (\widetilde{g}_1^+)^* & & \downarrow (\widetilde{g}_1^+)^* \\
H^{N-1}(\mathrm{Fix}(f_1) \cap p^{-1}(\partial B_1)) & \overset{\delta^*}{\longrightarrow} & H^N(\mathrm{Fix}(f_1), \mathrm{Fix}(f_1) \cap p^{-1}(\partial B_1)) \\
\downarrow \tau_{f_1} & & \downarrow \tau_{f_1} \\
H^{N-1}(\partial B_1) & \overset{\delta^*}{\cong} & H^N(B_1, \partial B_1)
\end{array}$$

It follows that the right column must be an isomorphism too and, hence, that

$$\beta^* : H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{x_1\}) \to H^N(\mathcal{N}^d, \mathcal{N}^{c_0})$$

is a monomorphism. Since $d < c_1$, this contradicts the definition of c_1. Therefore J must have a critical value in $(c_1, 2c_\infty + \varepsilon_0]$. Together with Proposition 1, this shows that J has at least two critical values. ■

Appendix A. The fixed point transfer

The fixed point transfer was introduced by Dold in [11]. For the reader's convenience we give here the definition and the properties which we use in this paper.

Let B be a metric space, let U be an open subset of $B \times \mathbb{R}^n$, and let $p : B \times \mathbb{R}^n \to B$ and $\pi : B \times \mathbb{R}^n \to \mathbb{R}^n$ be the projections. A map $f : \overline{U} \to B \times \mathbb{R}^n$ is said to be *compactly fixed over* B if $p(f(z)) = p(z)$ for every $z \in \overline{U}$, $\mathrm{Fix}(f) := \{z \in \overline{U} : f(z) = z\} \subset U$, and there exists a continuous function $\varrho : B \to (0, \infty)$ such that

$$\mathrm{Fix}(f) \subset T_\varrho := \{(b, x) \in B \times \mathbb{R}^n : |x| \le \varrho(b)\}.$$

Let A be a closed subset of B, and let $Y \subset X$ be open subsets of U such that

$$X \supset \mathrm{Fix}(f), \quad Y \supset \mathrm{Fix}(f) \cap (A \times \mathbb{R}^n), \quad Y \supset \mathrm{Fix}(f) \cap (p(Y) \times \mathbb{R}^n). \tag{A.1}$$

Set $B' := B \smallsetminus p(\mathrm{Fix}(f))$, and consider the following sequence of maps

$$\begin{array}{lcl}
(X, Y) \times (\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) & \overset{(id, i-f)}{\longleftarrow} & (X, (X \smallsetminus \mathrm{Fix}(f)) \cup Y) \\
& \overset{i_1}{\hookrightarrow} & (B \times \mathbb{R}^n, (B \times \mathbb{R}^n \smallsetminus \mathrm{Fix}(f)) \cup (p(Y) \times \mathbb{R}^n)) \\
& \overset{i_2}{\hookleftarrow} & (B \times \mathbb{R}^n, (B \times \mathbb{R}^n \smallsetminus T_\varrho) \cup ((p(Y) \cup B') \times \mathbb{R}^n)) \\
& \overset{i_3}{\hookrightarrow} & (B, p(Y) \cup B') \times (\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) \\
& \overset{i_4}{\hookleftarrow} & (B, A) \times (\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0)
\end{array}$$

where $(id, i-f)(b, x) := (b, x, x - \pi(f(b, x)))$ and all other maps are inclusions. Here we write, as usual, $(B, A) \times (D, C) := (B \times D, B \times C \cup A \times D)$. Let H^* be Alexander-Spanier cohomology (or any continuous cohomology theory). The inclusion i_1 is an excision, so it induces an isomorphism in cohomology. The excision, homotopy and exactness properties of cohomology ensure that the inclusion i_3 induces an isomorphism too. Hence, applying cohomology to this sequence of maps, and composing both ends with the suspension isomorphism $H^{i+n}((B, A) \times (\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0)) \cong H^i(B, A)$ we obtain a homomorphism

$$t_f^{X,Y} : H^*(X, Y) \to H^*(B, A).$$

Note that the set of all open subsets $Y \subset X$ of U which satisfy (A.1) is a cofinal subset of $\mathcal{U} := \{(X,Y) : X, Y \text{ open in } B \times \mathbb{R}^n,\ X \supset \mathrm{Fix}(f),\ Y \supset \mathrm{Fix}(f) \cap (A \times \mathbb{R}^n)\}$. So passing to the direct limit

$$H^*(\mathrm{Fix}(f), \mathrm{Fix}(f) \cap (A \times \mathbb{R}^n)) \cong \lim_{\rightarrow}\{H^*(X,Y) : (X,Y) \in \mathcal{U}\}$$

we obtain a homomorphism

$$\tau_f : H^*(\mathrm{Fix}(f), \mathrm{Fix}(f) \cap (A \times \mathbb{R}^n)) \to H^*(B, A)$$

called the *(relative) fixed point transfer of* f [11]. It has many useful properties. We state only those which we need for our purposes.

Proposition 2. *The transfer has the following properties:*
Naturality: *It commutes with connecting homomorphisms, that is, the diagram*

$$\begin{array}{ccc} H^i(Fix(f) \cap (A \times \mathbb{R}^n)) & \xrightarrow{\delta^*} & H^{i+1}(Fix(f), Fix(f) \cap (A \times \mathbb{R}^n)) \\ \tau_f \downarrow & & \downarrow \tau_f \\ H^i(A) & \xrightarrow{\delta^*} & H^{i+1}(B, A) \end{array}$$

commutes.
Units: *If $s : B \to B \times \mathbb{R}^n$ is a section of p (i.e. $p \circ s = id$) and $f = s \circ p : B \times \mathbb{R}^n \to B \times \mathbb{R}^n$, then $Fix(f) = s(B)$ and $\tau_f = s^* : H^*(s(B)) \to H^*(B)$.*
Homotopy: *Let $h : \overline{W} \to B \times [0,1] \times \mathbb{R}^n$ be a compactly fixed map over $B \times [0,1]$. For each $t \in [0,1]$ set $W_t := \{(b,x) \in B \times \mathbb{R}^n : (b,t,x) \in W\}$ and let $h_t : W_t \to B \times \mathbb{R}^n$ be given by $h_t(b,x) := q(h(b,t,x))$ where $q : B \times [0,1] \times \mathbb{R}^n \to B \times \mathbb{R}^n$ is the projection. Then*

$$\tau_{h_0} \circ p^* = \tau_{h_1} \circ p^*.$$

The proof is straightforward. We omit the details and refer the reader to [3, 11].

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