SYMMETRIC QUASI-SCHURIAN ALGEBRAS.

Octavio Mendoza Hernández
Departamento de matemática
Universidad Nacional del Sur
8000 Bahía Blanca
Argentina.*

Abstract

Let $k$ denote an algebraically closed field. We say that a finite dimensional $k$-algebra $\Lambda$ is quasi-schurian, if it satisfies the following two conditions:

QS1) $\dim_k\text{Hom}_\Lambda(P, Q) \leq 1$ if $P, Q$ are not isomorphic indecomposable projective $\Lambda$-modules.

QS2) $\dim_k\text{End}_\Lambda(P) = 2$ for each indecomposable projective $\Lambda$-module $P$.

An important class of quasi-schurian algebras is the trivial extensions of finite representation type.

In this paper, we give necessary and sufficient conditions for a given quasi-schurian algebra $\Lambda$ to be weakly-symmetric or symmetric. These conditions are given in a combinatorial approach using a graph $GS(\Lambda)$ associated to $\Lambda$, and a function $\phi_{\Lambda} : \text{Ch}(GS(\Lambda)) \to k$ where $\text{Ch}(GS(\Lambda))$ is the set of chains of the graph $GS(\Lambda)$. Finally we give some connections between symmetric quasi-schurian algebras and trivial extensions of algebras.

1 Introduction

Throughout this paper, we let $k$ denote a fixed algebraically closed field. By algebra is always meant a finite dimensional associative $k$-algebra with an identity, which we assume moreover to be basic and connected, and by module is meant a finitely generated left $A$-module.

Let $A$ be a schurian triangular algebra. It is well known that the trivial extension $T(A)$ of $A$ satisfies $\dim_k\text{Hom}_{T(A)}(P, Q) \leq 1$ and $\dim_k\text{End}_{T(A)}(P) = 2$ where $P, Q$ are non isomorphic indecomposable projective $T(A)$-modules. In this way, we are interested

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in the class of algebras Λ satisfying the above property. Thus, we say that an algebra Λ is quasi-schurian if it satisfies the following two conditions:

QS1) \( \dim_k \text{Hom}_\Lambda(P, Q) \leq 1 \) if \( P, Q \) are not isomorphic indecomposable projective \( \Lambda \)-modules.

QS2) \( \dim_k \text{End}_\Lambda(P) = 2 \) for each indecomposable projective \( \Lambda \)-module \( P \).

The aim of this paper is both to give necessary and sufficient conditions for a given quasi-schurian algebra to be weakly-symmetric or symmetric, and to say when a symmetric quasi-schurian algebra arises from a trivial extension of a schurian triangular algebra.

Let \( \Lambda = \frac{kQ}{I} \) where \( Q \) is the ordinary quiver associated with \( \Lambda \) and \( I \) is an admissible ideal. If \( \delta \) is a path in the quiver \( Q \), we will denote by \( \delta \) the sub quiver of \( Q \) having as vertices and arrows those which belong to \( \delta \), this \( \delta \) is called the support of \( \delta \).

Let \( C \) be an oriented cycle. Each vertex \( j \) in the support \( C \) of \( C \) determines a cycle with origin \( j \) which we call \( C(j) \).

Finally we denote by \( \gamma \) the congruence class \( \gamma + I \) in \( \Lambda = \frac{kQ}{I} \).

In section 3 we prove the following theorems

**Theorem.** Let \( \Lambda = \frac{kQ}{I} \) be a quasi-schurian algebra. Then the following conditions are equivalent:

I) \( \Lambda \) is weakly-symmetric.

II) For every non zero path \( \gamma \) there exists a path \( \delta \) such that \( \delta \gamma \) is a non zero minimal oriented cycle.

III) For each non zero \( f \) in \( \text{Hom}_\Lambda(P, Q) \) the induced morphism

\[ \text{Hom}_\Lambda(Q, f) : \text{Hom}_\Lambda(Q, P) \rightarrow \text{End}_\Lambda(Q) \]

is non zero, if \( P \) and \( Q \) are indecomposable non isomorphic projective \( \Lambda \)-modules.

IV) \( \Lambda \) satisfies the following conditions

a) If a minimal oriented cycle \( C \) is non zero, then \( \overline{C(t)} \neq 0 \) for each vertex \( t \) in the support \( C \) of \( C \).

b) Let \( \{C_1, C_2, \ldots, C_m\} \) be the set of supports corresponding to the non zero oriented cycles. Then \( Q = \cup_{i=1}^{m} C_i \).

**Theorem.** Let \( \Lambda = \frac{kQ}{I} \) be a quasi-schurian weakly-symmetric algebra. Let \( \{C_1, C_2, \ldots, C_m\} \) be the set of supports of the non zero minimal oriented cycles. The following statements are equivalent:

I) \( \Lambda \) is a symmetric algebra.
II) There are non zero elements $a_1, \ldots, a_m$ in the field $k$ such that, for each $i$ and $j$ with $(C_i)_0 \cap (C_j)_0 \neq \emptyset$ the following condition holds

$$\overline{C}_i(t) = a_i a_j^{-1} \overline{C}_j(t) \quad \forall t \in (C_i)_0 \cap (C_j)_0.$$ 

In section 4 we give a combinatorial approach to the above last theorem using a graph $GS(\Lambda)$ associated to $\Lambda$, and a function $\phi_\Lambda : Ch(GS(\Lambda)) \to k$ where $Ch(GS(\Lambda))$ is the set of chains of the graph $GS(\Lambda)$. In this way, the existence of the non zero constants $a_1, \ldots, a_m$ which are required in the last theorem, is very closely related with the structure of the graph $GS(\Lambda)$ and with the function $\phi_\Lambda : Ch(GS(\Lambda)) \to k$. In fact, we prove that the quasi-schurian Weakly-Symmetric $k$-algebra $\Lambda$ is symmetric if either the graph $GS(\Lambda)$ is a tree or $\phi_\Lambda$ satisfies $\phi_\Lambda(C) = 1$ for each minimal cycle $C$ in $GS(\Lambda)$ with at least three vertices.

In section 5 we give a connexion between symmetric quasi-schurian algebras and trivial extensions of algebras, which we state next.

**Theorem.** Let $\Lambda$ be basic connected finite dimensional $k$-algebra. The following statements are equivalent

1) There exists a schurian basic triangular algebra $\Lambda'$ such that $\Lambda \simeq T(\Lambda')$.

2) $\Lambda$ is symmetric quasi-schurian, and there exists a set $\mathcal{C}(\Lambda)$ consisting of exactly one arrow in each non zero minimal oriented cycle, such that $Q_{\mathcal{C}}$ has non oriented cycles, where $Q_{\mathcal{C}}$ is the quiver obtained from $Q_\Lambda$ by deleting the arrows in $\mathcal{C}(\Lambda)$.

If these conditions hold, then $\Lambda' \simeq \Lambda/I_C$ where $I_C$ is the ideal generated by $\mathcal{C}(\Lambda)$ in $\Lambda$. □

In the case that $Q$ is an oriented tree and $\Lambda = T(kQ)$ we can always choose a set $\mathcal{C}(\Lambda)$ as in 2) in the theorem. Moreover, we prove that for any such choice the factor algebra $\Lambda/I_C$ is iterated tilted of type $Q$. This is a useful approach to obtain iterated tilted algebras of a given tree class.

## 2 Preliminaries

It is well known that each basic finite dimensional algebra $\Lambda$ over an algebraically closed field $k$ is isomorphic to $k$-algebra $kQ/I$ where $Q$ is the finite quiver associated with $\Lambda$ and $I$ is an admissible ideal of the path algebra $kQ$.

Let $Q$ be a quiver. We will denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows of $Q$. Given an arrow $\alpha \in Q_1$, we say it starts at the vertex $o(\alpha)$ and ends at $e(\alpha)$. A path in the quiver $Q$ is either an oriented sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $e(\alpha_i) = o(\alpha_{i+1})$ for $1 \leq t < n$, or the symbol $e_i$ for $i \in Q_0$. We call the paths $e_i$ trivial paths and we define $o(e_i) = e(e_i)$. For a nontrivial path $p = \alpha_n \cdots \alpha_1$ we define $o(p) = o(\alpha_1)$ and $e(p) = e(\alpha_n)$. If $\delta$ is a path in $Q$, we will denote by $\delta$ the support of $\delta$ in $Q$. Thus, $\delta$ is a sub quiver of $Q$ having as vertices and arrows those which belong to $\delta$. A nontrivial path $p$ is said to be an oriented cycle if $o(p) = e(p)$. 

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Let $C = \alpha_n\alpha_{n-1} \cdots \alpha_2\alpha_1$ be an oriented cycle in $Q$. We will call $C$ minimal oriented cycle if $n = 1$ or all the vertices $o(\alpha_1), o(\alpha_2), \ldots, o(\alpha_n)$ are different in case $n > 1$. Let $j$ be a vertex in the support $C$ of $C$, then the arrows of $C$ determine a cycle with origin $j$, which we call $C(j)$. That is, $C(j) = \alpha_{r-1} \cdots \alpha_2\alpha_1\alpha_n \cdots \alpha_{r+1}\alpha_r$ where $j = o(\alpha_r)$ is the origin of $\alpha_r$.

Let $\Lambda$ be a finite dimensional $k$-algebra, we denote by $\text{mod}(\Lambda)$ the category of finitely generated left-$\Lambda$ modules, by $Q_\Lambda$ the ordinary quiver associated with $\Lambda$, by $S(a)$ the simple $\Lambda$-module corresponding to the vertex $a$ in $Q_\Lambda$, by $P(a)$ the projective cover, and by $I(a)$ the injective envelope of $S(a)$. Let $\gamma$ be a path in $Q_\Lambda$. By $\overline{\gamma}$ we denote the congruence class $\gamma + I$ in $\Lambda = kQ_\Lambda/I$. We will say that the path $\gamma$ is zero if $\gamma = 0$.

**Definition:** An algebra $\Lambda$ is called quasi-schurian, if it satisfies the following two conditions:

QS1) $\dim_k \text{Hom}_\Lambda(P,Q) \leq 1$ if $P,Q$ are non isomorphic indecomposable projective $\Lambda$-modules.

QS2) $\dim_k \text{End}_\Lambda(P) = 2$ for each indecomposable projective $\Lambda$-module.

An important class of quasi-schurian algebras consists of the trivial extensions of Cartan type $D$, with $D$ a Dynkin quiver. These algebras are closely related with the iterated tilted algebras of Dynkin type $D$, see [1],[2]. More generally, consider a schurian algebra $\Lambda$ such that $Q_\Lambda$ has no oriented cycles. Then the trivial extension $T(\Lambda)$ of $\Lambda$ will be quasi-schurian.

**2.1 Symmetric algebras.** Let $\Lambda$ be a $k$-algebra. We denote by $D_\Lambda$ the usual duality

$$
\text{Hom}_k(-,k) : \text{mod}(\Lambda) \to \text{mod}(\Lambda^{op}).
$$

The algebra $\Lambda$ is called symmetric if there exists an isomorphism $\varphi : \Lambda \cong D_\Lambda(\Lambda)$ as $\Lambda-\Lambda$ bimodules. It is well known that $\Lambda$ is symmetric if and only if there is a non-degenerate $\Lambda$-balanced symmetric $k$-bilinear mapping $\theta : \Lambda \times \Lambda \to k$, see [4]. We will point out the following equivalent version of the above property.

**Proposition 1** Let $\Gamma$ be a finite dimensional $k$-algebra and $f \in D_\Gamma(\Gamma)$. Then there exists a $\Gamma-\Gamma$ bimodule isomorphism $\varphi : \Gamma \cong D_\Gamma(\Gamma)$ such that $\varphi(1) = f$ if and only if $f$ satisfies:

$\alpha$) For each $\gamma_1 , \gamma_2 \in \Gamma$ we have that $\gamma_2 \gamma_1 = 0$ is equivalent to $\gamma_1 \Gamma \gamma_2 \subseteq \text{Ker}f$.

$\beta$) $\gamma_1 \gamma_2 - \gamma_2 \gamma_1 \in \text{Ker}f$ for every $\gamma_1 , \gamma_2 \in \Gamma$.

**Proof:** straightforward calculations. $\square$

**Remarks:**

1) The condition $\alpha$) may be changed by one of the following conditions

$\alpha'$) If $\gamma \Gamma \subseteq \text{Ker}f$, then $\gamma = 0$.  

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α’’ If Γγ ⊆ Ker f, then γ = 0.

2) Let {e1, · · · , en} be a complete family of orthogonal idempotents in Γ. Then the condition α) implies that
i) f(ejΓei) = 0 for i ̸= j.
ii) f(eiΓei) ̸= 0 for each i.

2.2 The Supplement Property for quasi-schurian algebras.

Definition: Let Λ = kQΛ/I be a quasi-schurian algebra. We will say that Λ satisfies the Supplement Property if for every non zero path γ there exists a non zero minimal oriented cycle C such that
1) o(γ) = o(C).
2) All the arrows in γ lie in the support C of the cycle C.
The path δ such that δγ = C is called the supplement of γ in the cycle C.
E. Fernández and M.I. Platzeck proved that this property holds for the trivial extension T(Λ) of a schurian algebra Λ (see [3]).

Lemma 2 Let Λ be a quasi-schurian algebra and δ a nontrivial path in kQΛ. If C is an oriented cycle then Cδ = δC = 0.

Proof: Suppose that Cδ ̸= 0. Then we will prove that the set {δ, Cδ} is linearly independent over k. This gives a contradiction since Λ is quasi-schurian.
Let aδ + bCδ = 0 where a and b lie in k. If a ̸= 0 then (1 + ba−1C)δ = 0. But ba−1C lies in the radical of Λ and so 1 + ba−1C is invertible in Λ. Thus δ = 0, a contradiction. So, a must be zero. This means that bCδ = 0 which also gives that b = 0. Then, the set {δ, Cδ} is linearly independent. □

3 Main results.

Let Λ be a finite dimensional k-algebra. Recall that Λ is called weakly-symmetric if for any indecomposable projective Λ-module P we have that soc(P) ≃ top(P). It can be proven (see [4]) that a weakly-symmetric algebra is self-injective. Moreover, symmetric implies weakly-symmetric. In case Λ is a quasi-schurian algebra, we give in this section an answer to the following questions.
1) When is Λ weakly-symmetric?.
2) When is Λ symmetric?.

The Supplement Property which was defined above for quasi-schurian algebras is very closely related with these questions, as we will see in this section.

Theorem 3 Let Λ = kQΛ/I be a quasi-schurian algebra. Then the following conditions are equivalent
I) $\Lambda$ is weakly-symmetric.

II) $\Lambda$ satisfies the Supplement Property.

III) For each non zero $f$ in $\text{Hom}_\Lambda(P,Q)$ the induced morphism

$$\text{Hom}_\Lambda(Q,f) : \text{Hom}_\Lambda(Q,P) \to \text{End}_\Lambda(Q)$$

is non zero, if $P$ and $Q$ are indecomposable non isomorphic projective $\Lambda$-modules.

IV) $\Lambda$ satisfies the following conditions

a) If a minimal oriented cycle $C$ is non zero, then $C(t) \neq 0$ for each vertex $t$ in the support $C$ of $C$.

b) Let $\{C_1, C_2, \ldots, C_m\}$ be the set of supports corresponding to the non zero oriented cycles. Then $Q_\Lambda = \bigcup_{i=1}^{m} C_i$.

Before proving the theorem, we will need the following result.

**Lemma 4** Let $\Lambda = kQ_\Lambda/I$ be a finite dimensional $k$-algebra, let $i$ be a vertex in $Q_\Lambda$ and $\gamma$ a non trivial path in $Q_\Lambda$, non zero in $\Lambda$.

If $\text{soc}(P(i)) \simeq S(i)$ and $\overline{\gamma} \in \text{soc}(P(i))$, then $\gamma$ is a cycle with origin at the vertex $i$.

**Proof:** Assume that $\text{soc}(P(i)) \simeq S(i)$ and $\overline{\gamma}$ lies in $\text{soc}(P(i))$. Let $j = e(\gamma)$. Then $\overline{\gamma} \in I(j)$. But $k\overline{\gamma} = \text{soc}(P(i)) \simeq S(i)$, hence $k\overline{\gamma} \simeq S(i)$. But $\overline{\gamma}$ is in $I(j)$, then $k\overline{\gamma} = \text{soc}(I(j)) \simeq S(j)$. This means that $S(i) \simeq S(j)$ and hence $i = j$. 

**Remark:** We recall that, if $M$ is a $\Lambda$ module then the socle of $M$ is equal to the right annihilator of $\text{rad}(\Lambda)$ in $M$ (see [4]). This property will be used in the next proof.

**Proof of Theorem 3:**

I) $\Rightarrow$ II) Assume that $\Lambda$ is weakly-symmetric. Let $\gamma = \alpha_r \alpha_{r-1} \cdots \alpha_1$ be a non zero path such that $o(\gamma) \neq e(\gamma)$. Therefore $\overline{\gamma} \notin \text{soc}(P(o(\gamma)))$ : indeed, if this is not the case, then Lemma 4 would imply that $o(\gamma) = e(\gamma)$, a contradiction. Then there exists an arrow $\beta$ such that $\beta \gamma$ is non zero. So, multiplying $\gamma$ by the necessary number of arrows $\beta_1, \ldots, \beta_m$, we may assume that the non zero path $\delta = \beta_m \beta_{m-1} \cdots \beta_1 \gamma$ is an oriented cycle or $\delta$ lies in the socle of $P(o(\delta))$. Hence the assertion is now a consequence of Lemma 2 and Lemma 4.

II) $\Rightarrow$ I) Assume that $\Lambda$ satisfies the Supplement Property. Let $i$ be a vertex in $Q_\Lambda$ and $\gamma$ a non zero path in $\Lambda$ such that $\overline{\gamma} \in \text{soc}(P(i))$. By the Supplement Property, there exists a non zero minimal oriented cycle $C$ containing the path $\gamma$ and such that $o(C) = i$. If $\gamma \neq C$, then there is an arrow $\beta$ in $C$ such that $\beta \overline{\gamma} \neq 0$. Hence $\overline{\gamma}$ does not lie in $\text{soc}(P(i))$, giving a contradiction. Thus, $\gamma = C$ and hence $\text{soc}(P(i)) = kC$. So, the socle of $P(i)$ is isomorphic to the simple $S(i)$.

II) $\Leftrightarrow$ III) III) is just a restatement of II).

II) $\Rightarrow$ IV)

a) Let $C = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$ be a non zero oriented cycle. Assume that $t = o(\alpha_i)$. Since $\overline{C} \neq \overline{0}$ we have that the path $\gamma = \alpha_i \cdots \alpha_{i+1} \alpha_i$ is non zero. Then by the supplement
property there is a path \( \delta \) such that \( \delta \gamma \) is a non zero minimal oriented cycle. Since the paths \( \delta \) and \( \alpha_{i-1} \cdots \alpha_1 \) have the same starting and ending vertices we obtain that \( \overline{\delta} = a\overline{\alpha_{i-1}} \cdots \overline{\alpha_1} \) where \( a \in k - \{0\} \). Then \( \overline{0} \neq \overline{\delta \gamma} = a\overline{C}(t) \) and hence \( \overline{0} \neq \overline{C}(t) \).

b) Each arrow of \( Q \Lambda \) is non zero in \( \Lambda \). Hence by the Supplement Property we get that \( Q \Lambda = \bigcup_{i=1}^{m} \overline{C}_i \).

IV) \( \Rightarrow \) II) Let \( \gamma \) be a non zero path. By b) and Lemma 2 we get that \( \gamma \) belongs to some non zero minimal oriented cycle \( C \). Thus the Supplement Property holds since by a) we have that \( \overline{C}(o(\gamma)) \neq 0 \).

**Corollary 5** Let \( \Lambda = kQ \Lambda / I \) be a quasi-schurian weakly-symmetric algebra. Then the ordinary quiver \( Q \Lambda \) is the union of all non zero minimal oriented cycles.

The other main result in this section is the following theorem.

**Theorem 6** Let \( \Lambda = kQ \Lambda / I \) be a quasi-schurian weakly-symmetric algebra. Let \( \{C_1, C_2, \ldots, C_m\} \) be the set of supports of the non zero minimal oriented cycles. The following statements are equivalent:

I) \( \Lambda \) is a symmetric algebra.

II) There are non zero elements \( a_1, \ldots, a_m \) in the field \( k \) such that, for each \( i \) and \( j \) with \( (C_i)_0 \cap (C_j)_0 \neq \emptyset \) the following condition holds

\[
\overline{C}_i(t) = a_ia_j^{-1}\overline{C}_j(t) \quad \forall t \in (C_i)_0 \cap (C_j)_0.
\]

We will need the next lemma to give a proof of this theorem.

**Lemma 7** Let \( \Lambda = kQ \Lambda / I \) be a symmetric k-algebra. Let \( \varphi : \Lambda \to D(\Lambda) \) be an isomorphism of \( \Lambda - \Lambda \) bimodules and \( f = \varphi(1) \). Then the following conditions hold for every non zero minimal oriented cycle \( C \).

a) If \( \dim_k End_{\Lambda}(P(i)) = 2 \) where \( o(C) = i \), then \( f(C) \neq 0 \).

b) \( f(\overline{C}(j)) = f(C) \) for every \( j \in (C)_0 \).

**Proof:**

b): Follows from \( \beta \) in Proposition 1 since \( \gamma_1 \gamma_2 - \gamma_2 \gamma_1 \in Ker f \) for every \( \gamma_1, \gamma_2 \in \Lambda \).

a): By b) above it is sufficient to prove that \( f(C) \neq 0 \). Since \( \dim_k End_{\Lambda}(P(i)) = 2 \) we get that \( \{\overline{e_i}C, \overline{C}C\} \) is a k-basis of \( End_{\Lambda}(P(i)) \) and \( \overline{C}^2 = 0 \).

We know that \( \overline{e_i}C \neq 0 \). Then by Proposition 1 it follows that there exists \( \lambda \in \Lambda \) such that \( f(\overline{C} \overline{e_i}) \neq 0 \). In particular \( 0 \neq \lambda \overline{e_i} \in End_{\Lambda}(P_i) \), and we get that \( \lambda \overline{e_i} = r\overline{e_i} + s\overline{C} \) where \( r, s \in k \). Then \( \overline{C} \lambda \overline{e_i} = r\overline{C} \overline{e_i} + s\overline{C}^2 = r\overline{C} \) and this means that \( f(\overline{C}) \neq 0 \) since \( 0 \neq f(\overline{C} \lambda \overline{e_i}) = f(r\overline{C}) \). \( \square \)
Remark: Let $f : \Lambda \to k$ be as in Lemma 7, and $C$ be a non zero minimal oriented cycle. It is clear by Lemma 7 that $f(C(i)) = f(C(j))$ for all vertices $i, j$ in $C$. Hence $f$ can be defined on the support $C$ as follows, fix a vertex $j$ in $C$ and let $f(C) = f(C(j))$. In this way, we say that $f$ is constant and non zero on $C$.

Proof of Theorem 6:

$I) \Rightarrow II$: Assume that $\Lambda$ is a symmetric algebra. Let $\varphi : \Lambda \to D(\Lambda)$ be an isomorphism of $\Lambda - \Lambda$ bimodules and $f = \varphi(1)$. To obtain the nonzero constants $a_1, \cdots, a_m$ we can use the above remark and define $a_i = f(C_i)$.

$II) \Rightarrow I$: The idea of the proof is to construct a linear functional $f : \Lambda \to k$ such that the properties $\alpha', \beta$ in Proposition 1 hold. Let us start with the linear functional $F : kQ\Lambda \to k$ defined on the basis of the paths in $Q\Lambda$ as follows: $F(\gamma) = a_i$ if there are $i$ and $t$ such that $\gamma = C_i(t)$, and zero otherwise. Then $II$ implies that $\overline{\gamma} = F(\gamma)(F(\gamma'))^{-1}\overline{\gamma'}$, for nonzero cycles $\gamma$ and $\gamma'$ with the same origin. The next step is to check that $F : kQ\Lambda \to k$ factors through the canonical epimorphism $\pi : kQ\Lambda \to \Lambda$, that is, that $I \subseteq KerF$. Let $\gamma = \sum_{i=1}^{n} c_{\gamma_i} \in I$ be a linear combination of paths $\gamma_i$ starting at the vertex $a$ and ending at the vertex $b$ for $1 \leq i \leq n$. We may assume that $a = b$ and $\gamma_i$ is a non zero oriented cycle for $i = 1, \cdots, n$. Since $\overline{\gamma} = (F(\gamma_i)/F(\gamma_1))\overline{\gamma_i}$ $i = 2, 3, \cdots, n$, we get that $0 = \overline{\gamma} = \sum_{i=1}^{n} c_{\gamma_i} \overline{\gamma_i} = (\sum_{i=1}^{n} c_i F(\gamma_i)/F(\gamma_1))\overline{\gamma_1}$. But $\overline{\gamma_1} \neq 0$. So $\sum_{i=1}^{n} c_i F(\gamma_i)/F(\gamma_1) = 0$ and then $\gamma = \sum_{i=1}^{n} c_i (\gamma_i - (F(\gamma_i)/F(\gamma_1)) \gamma_1)$, therefore $F(\gamma) = 0$. Hence there exists $f : \Lambda \to k$ such that $f = F_{\pi}$.

We will prove that $\alpha'$ holds, that is $\lambda_1 \lambda \subseteq Kerf$ implies $\lambda_1 = 0$. Assume that $\lambda_1 = \sum_{j=1}^{n} c_j \overline{\gamma_j}$ be such that $\lambda_1 \Lambda \subseteq Kerf$ where $\gamma_j$ is a path in $Q\Lambda$ for $j = 1, 2, \cdots, n$. Observe that $\lambda_1 = \sum_{j=1}^{n} c_j \overline{\gamma_j}, \lambda_1 \overline{\gamma} \subseteq \lambda \Lambda \subseteq Kerf$. Hence it is enough to prove $\alpha'$ only for each $\lambda_i e_i$; that is, for all linear combination of paths starting at the vertex $i$. Then we may assume without loose of generality that $i = 1$ and $o(\gamma_j) = 1$ for $j = 1, 2, \cdots, n$.

Let $\{b_1, \cdots, b_r\}$ be the set of end points of the paths $\gamma_j$, for $j = 1, 2, \cdots, n$. Let $A_j = \{i \mid e(\gamma_i) = b_j\}$. Then we can write $\lambda_1 = \sum_{j=1}^{r} \sum_{i \in A_j} c_i \overline{\gamma_i}$. Let us prove that $\sum_{i \in A_j} c_i \overline{\gamma_i} = 0$. Assume that $\nu$ is a supplement to the paths $\{\gamma_i : i \in A_j\}$. This path exists since $\Lambda$ is quasi-schurian and the Supplement Property holds. Fix an index $i_1$ in $A_1$, then $\overline{\gamma_j} = d_j \overline{\gamma_i}$ for some $d_j \in k$ and each $j \in A_1 - \{i_1\}$. Multiplying both sides of the above equality by $\overline{\nu}$ and applying $f$ we get $d_j = f(\overline{\gamma_j})/f(\overline{\gamma_i})$. Now, by Lemma 2 and the fact that $f = F_{\pi}$ we obtain $f(\overline{\gamma_j}) = 0$ for all $i \in A_j$ with $j > 1$. Hence $0 = f(\lambda_1) = \sum_{i \in A_j} c_i f(\overline{\gamma_i})$, and this implies that $\sum_{i \in A_j} c_i \overline{\gamma_i} = (\sum_{i \in A_j} c_i d_i) \overline{\gamma_i} = (\sum_{i \in A_j} c_i f(\overline{\gamma_i}))\overline{\gamma_i}/f(\overline{\gamma_i}) = 0$. We point out that the equality $\sum_{i \in A_j} c_i \overline{\gamma_i} = 0$ for $j = 2, 3, \cdots, r$ can be obtained in an analogous way. Hence $\lambda_1 = 0$, as we wanted.

We will prove that $\beta$ holds, that is $\lambda_1 \lambda_2 - \lambda_2 \lambda_1 \in Kerf$ for all $\lambda_1, \lambda_2 \in \Lambda$.

Let $\lambda_1 = \sum_{i} c_i \overline{\gamma_i}$ and $\lambda_2 = \sum_{j} d_j \overline{\gamma_j}$, where $\gamma_i, \gamma_j$ are paths for each $i$ and $j$. Assume that $\overline{\gamma_i \gamma_j} \neq 0$, hence $\overline{\gamma_i \gamma_j}$ lies in a non zero minimal oriented cycle $C$ such that $o(\gamma_i \gamma_j) = o(C)$. If $\gamma_i \gamma_j = C$ we obtain that the supports of $\gamma_i \gamma_j$ and $\gamma_j \gamma_i$ coincide. Hence $F(\gamma_i \gamma_j) = F(\gamma_j \gamma_i)$ and this implies that $f(\overline{\gamma_i \gamma_j}) = f(\overline{\gamma_j \gamma_i})$. In case $\overline{\gamma_i \gamma_j} \neq e(C)$ we have $\overline{\gamma_j \gamma_i} = 0$ and also $F(\gamma_i \gamma_j) = 0$.

Therefore, $\overline{\gamma_i \gamma_j} \neq 0$ implies that $f(\overline{\gamma_i \gamma_j}) = f(\overline{\gamma_j \gamma_i})$. 

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In the same way it can be proved that $\gamma_i \gamma_j = 0$ implies that $f(\gamma_i \gamma_j) = f(\gamma_j \gamma_i)$. Hence the assertion follows.

4 A combinatorial approach to Theorem 6. Let $\Lambda = kQ_{\Lambda}/I$ be a weakly-symmetric and quasi-schurian $k$-algebra. We associate to $\Lambda$ a graph $GS(\Lambda)$. The construction is as follows. Let $\{C_1, C_2, \ldots, C_m\}$ be the set of supports of the non-zero minimal oriented cycles. Then the set of vertices of $GS(\Lambda)$ is $\{1, 2, \ldots, m\}$ and the edges are determined as follows.

a) If $m = 1$, the set of edges is empty.

b) If $m > 1$, there is only one edge with vertices $\{i, j\}$ in case $(C_i)_0 \cap (C_j)_0 \neq \emptyset$ and $i \neq j$.

It is not difficult to see that $GS(\Lambda)$ is a connected graph without loops and non parallel edges.

Notation: a chain $C$ in $GS(\Lambda)$ joining the vertices $v_1$ and $v_k$ is a sequence of vertices and edges $v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k$ where for each $i$ the edge $A_i$ has vertices $v_i, v_{i+1}$. We say that the length of $C$ is $k - 1$. Let $B = w_1 B_1 w_2 B_2 \cdots w_n B_{n-1} w_n$ be another chain in $GS(\Lambda)$. We will say that the composition $A \circ B$ is defined if $v_k = w_1$ and we let $A \circ B$ be the chain $v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k B_1 w_2 B_2 \cdots w_n B_{n-1} w_n$.

A chain $v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k$ is called reduced if $v_{i-1} \neq v_{i+1}$ for each $i = 2, 3, \ldots, k-1$. The set of all chains in $GS(\Lambda)$ is denoted by $Ch(GS(\Lambda))$. Usually we shall only be interested in reduced chains, and unless the contrary is explicitly stated, we shall assume that all chains under discussion are reduced.

A cycle $C$ in $GS(\Lambda)$ is a chain of the form $v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_1$. If the vertices $v_1, v_2, \ldots, v_{k-1}$ are all different, then the chain $C$ is called a minimal cycle. We observe that a minimal cycle has at least three vertices.

Let $C$ be the chain $v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k$ in $GS(\Lambda)$. We denote by $C$ the support of $C$ which is defined as the subgraph of $GS(\Lambda)$ with vertices $v_1, \ldots, v_k$ and edges $A_1, \ldots, A_{k-1}$. Given a minimal cycle $C = v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k$ in $GS(\Lambda)$, we denote by $C(v_i)$ the cycle $v_i A_i v_{i+1} \cdots v_{k-1} A_{k-1} v_1 v_2 A_2 \cdots v_{i-1} A_{i-1} v_i$, for $1 \leq i \leq k-1$.

The set $\{C_1, C_2, \ldots, C_m\}$ of supports of the non-zero minimal oriented cycles induces a family of nonzero constants $\lambda_{ij}(t) \in k$, such that $C_i(t) = \lambda_{ij}(t) C_j(t)$ for $t \in (C_i)_0 \cap (C_j)_0$. If the algebra $\Lambda$ is symmetric we obtain by Theorem 6 that $\lambda_{ij} = \lambda_{ij}(t) \forall t \in (C_i)_0 \cap (C_j)_0$. Thus, the non-zero constants $\lambda_{ij}(t)$ do not depend on the common vertices $t \in (C_i)_0 \cap (C_j)_0$.

Having this property as a motivation we will assume that the family of constants $\lambda_{ij}(t) \in k$, satisfies the following condition

$$\lambda_{ij} = \lambda_{ij}(t) \forall t \in (C_i)_0 \cap (C_j)_0.$$  

Now, we can define a map $\phi_{\Lambda} : Ch(GS(\Lambda)) \rightarrow k$ in the following way

$$\phi_{\Lambda}(v_1 A_1 v_2 A_2 \cdots v_{k-1} A_{k-1} v_k) = \lambda_{v_1 v_2} \lambda_{v_2 v_3} \cdots \lambda_{v_{k-1} v_k}.$$  

We point out that, if $C_1$ and $C_2$ are chains such that their composition is defined, then we have that $\phi_{\Lambda}(C_1 \circ C_2) = \phi_{\Lambda}(C_1) \phi_{\Lambda}(C_2)$.
Let $D$ be the chain $v_1A_1v_2A_2\cdots v_{k-1}A_{k-1}v_k$. We denote by $D^{-1}$ the chain $v_kA_{k-1}v_{k-1}\cdots v_2A_1v_1$. In this way $\phi_\Lambda(D^{-1}) = \phi_\Lambda(D^{-1} \circ D) = 1$ and hence $\phi_\Lambda(D^{-1}) = \phi_\Lambda(D)^{-1}$.

Let $C$ be a minimal cycle in $GS(\Lambda)$ with support $C$. It is clear that $\phi_\Lambda(C(v_i)) = \phi_\Lambda(C(v_j))$ for each $v_i, v_j$ in $C$. Hence $\phi_\Lambda$ can be defined on $C$ as follows, fix a vertex $v$ in $C$ and let $\phi_\Lambda(C) = \phi_\Lambda(C(v))$.

The existence of the non zero constants $a_1, a_2, \cdots, a_m$ which are required in Theorem 6 for $\Lambda$ to be symmetric is very closely related with the structure of the graph $GS(\Lambda)$ and with the function $\phi_\Lambda : Ch(GS(\Lambda)) \rightarrow k$ as can be seen in the next theorem.

**Theorem 8** Let $\Lambda$ be a quasi-schurian weakly-symmetric $k$-algebra, and $\{C_1, C_2, \cdots, C_m\}$ be the set of supports of the non zero minimal oriented cycles. Suppose that the non zero constants $\lambda_{ij}(t)$ above defined do not depend on the common vertices $t \in (C_0) \cap (C_j)_0$.

Then

a) If the graph $GS(\Lambda)$ is a tree, then $\Lambda$ is symmetric.

b) Suppose that $GS(\Lambda)$ is not a tree. Then $\Lambda$ is symmetric if and only if the function $\phi_\Lambda : Ch(GS(\Lambda)) \rightarrow k$ satisfies $\phi_\Lambda(C) = 1$ for each minimal cycle $C$ in $GS(\Lambda)$.

**Proof:**

a): Assume that $GS(\Lambda)$ is a tree. Let us prove that in this case the required non zero constants always exist. Since $GS(\Lambda)$ is a tree we have that for each vertex $j \neq 1$ in the graph $GS(\Lambda)$ there exists only one nontrivial chain $D_j$ in $GS(\Lambda)$ joining the vertex $j$ with the vertex 1. Hence we can define $a_1, a_2, \cdots, a_m$ in the following way, let $a_1 = 1$ and $a_j = \phi_\Lambda(D_j)$ if $j \neq 1$. The next step is to prove that $\lambda_{ij} = a_ia_j^{-1}$ in case $(C_0) \cap (C_j)_0 \neq \emptyset$ and $i \neq j$. Let $A$ be the edge with vertices $i$ and $j$. Since $GS(\Lambda)$ is a tree we have that $D_i = (iA_j) \circ D_j$ and hence $a_i = \phi_\Lambda(D_i) = \phi_\Lambda((iA_j) \circ D_j) = \phi_\Lambda(iA_j)\phi_\Lambda(D_j) = \lambda_{ij}a_j$. Thus, $\lambda_{ij} = a_ia_j^{-1}$ as we wanted.

b):,$(\Rightarrow)$ assume that $\Lambda$ is symmetric, that is the non zero constants $a_1, a_2, \cdots, a_m$ of Theorem 6 exist.

Let $C = v_1A_1v_2A_2\cdots v_{k-1}A_{k-1}v_1$ be a minimal cycle in $GS(\Lambda)$. Then $\phi_\Lambda(C) = \lambda_{v_1v_2, \cdots, \lambda_{v_{k-1}v_1}} = a_{v_1}a_{v_2}^{-1}a_{v_3}^{-1}\cdots a_{v_{k-1}}^{-1}a_{v_1}^{-1} = 1$.

$(\Leftarrow)$ Assume that $\phi_\Lambda(C) = 1$ for each minimal cycle $C$ in $GS(\Lambda)$. We will need the next Lemma.

**Lemma:** if $B_j$ and $D_j$ are two chains in $GS(\Lambda)$ joining the vertex $j$ with the vertex 1 where $j \neq 1$, then $\phi_\Lambda(B_j) = \phi_\Lambda(D_j)$.

Proof: If $B_j \circ D_j^{-1}$ is a minimal cycle, then by hypothesis we have $1 = \phi_\Lambda(B_j \circ D_j^{-1}) = \phi_\Lambda(D_j)\phi_\Lambda(B_j)^{-1}$. Hence $\phi_\Lambda(B_j) = \phi_\Lambda(D_j)$.

Assume that $B_j \circ D_j^{-1}$ is not a minimal cycle. Then it is not difficult to see that $B_j$ and $D_j$ have a decomposition $B_j = F_1 \circ F_2$, $D_j = G_1 \circ G_2$ in such way that $F_1 \circ G_1^{-1}$ is a minimal cycle in $GS(\Lambda)$. Hence $B_j \circ D_j^{-1} = F_1 \circ F_2 \circ G_2 \circ G_1^{-1}$ implies $\phi_\Lambda(B_j \circ D_j^{-1}) = \phi_\Lambda(F_1 \circ G_1^{-1})\phi_\Lambda(F_2 \circ G_2^{-1})$. But we know by construction that $\phi_\Lambda(F_1 \circ G_1^{-1}) = 1$. Then $\phi_\Lambda(B_j \circ D_j^{-1}) = \phi_\Lambda(F_2 \circ G_2^{-1})$ but now, the length of the chain
$F_2 \circ G_2^{-1}$ is smaller than the length of $B_j \circ D_j^{-1}$. Hence by induction we can obtain that 

$$\phi_{\Lambda}(B_j \circ D_j^{-1}) = 1$$

and conclude that $\phi_{\Lambda}(B_j) = \phi_{\Lambda}(D_j)$. □

Now, using this lemma it is possible to define the required constants. Let $D_j$ be a chain in $GS(\Lambda)$ joining the vertices $j$ and 1 where $j \neq 1$. Then we define $a_1 = 1$ and $a_j = \phi_{\Lambda}(D_j)$ if $j \neq 1$. By the above lemma we have that the constant $a_j$ is well defined. Now we will check that $\lambda_{ij} = a_ia_j^{-1}$ in case $(\mathcal{C}_i)_0 \cap (\mathcal{C}_j)_0 \neq \emptyset$ and $i \neq j$. Let $A$ be the edge with vertices $i$ and $j$, let $D_j$ be a chain joining the vertex $j$ with the vertex 1. Then the chain $B_i = (iAj) \circ D_j$ is a chain in $GS(\Lambda)$ joining the vertices $i$ and 1. Hence $a_i = \phi_{\Lambda}(B_i) = \phi_{\Lambda}((iAj) \circ D_j) = \phi_{\Lambda}(iAj)\phi_{\Lambda}(D_j) = \lambda_{ij}a_j$. This implies that $\lambda_{ij} = a_i a_j^{-1}$. □

From the above theorem we obtain the following corollaries.

Let $\Lambda$ be a quasi-schurian and weakly-symmetric $k$-algebra, $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be the set of supports of the non zero minimal oriented cycles and $\lambda_{ij}(t) \in k$ be the family of non zero constants such that $\overrightarrow{\mathcal{C}_i(t)} = \lambda_{ij}(t)\overrightarrow{\mathcal{C}_j(t)}$ for $t \in (\mathcal{C}_i)_0 \cap (\mathcal{C}_j)_0$.

**Corollary 9** If the graph $GS(\Lambda)$ associated to the quasi-schurian weakly-symmetric $k$-algebra $\Lambda$ is a tree, then the following conditions are equivalent

I) $\Lambda$ is a symmetric algebra.

II) $\lambda_{ij} = \lambda_{ij}(t) \forall t \in (\mathcal{C}_i)_0 \cap (\mathcal{C}_j)_0$, $i \neq j$.

**Corollary 10** Suppose that the graph $GS(\Lambda)$ associated to the quasi-schurian and weakly-symmetric $k$-algebra $\Lambda$ is not a tree. Then the following conditions are equivalent.

I) $\Lambda$ is a symmetric algebra.

II) $\lambda_{ij} = \lambda_{ij}(t) \forall t \in (\mathcal{C}_i)_0 \cap (\mathcal{C}_j)_0$, $i \neq j$, and the function $\phi_{\Lambda}: Ch(GS(\Lambda)) \to k$ defined above satisfies $\phi_{\Lambda}(C) = 1$ for each minimal cycle $C$.

**Example:** Let $\Lambda$ be the factor algebra of the path algebra $kQ$ for $Q$ the quiver

$$\begin{align*}
1 & \xrightarrow{\alpha_1} 0 \xrightarrow{\alpha_2} 2 \\
& \quad \alpha_3 \\
& \quad \alpha_4 \xrightarrow{\alpha_5} 3 \\
& \quad \alpha_6
\end{align*}$$

modulo the ideal $I = \langle \alpha_1 \alpha_0 - a\alpha_4 \alpha_3, \alpha_0 \alpha_2 \alpha_4, \alpha_3 \alpha_2 \alpha_1, \alpha_0 \alpha_0, \alpha_6 \alpha_5 \alpha_6, \alpha_5 \alpha_3, \alpha_1 \alpha_5, \alpha_4 \alpha_6, \alpha_5 \alpha_6 \alpha_5, b\alpha_0 \alpha_2 \alpha_1 - \alpha_5 \alpha_6, \alpha_2 \alpha_1 \alpha_0 \alpha_2, \alpha_3 \alpha_2 \alpha_4 - \alpha_0 \alpha_5 \rangle$ where $a, b, c \in k - \{0\}$.

$\Lambda$ is quasi-schurian and moreover weakly-symmetric since this algebra satisfies the Supplement Property (see Theorem 3).

We will prove that $\Lambda$ is symmetric if and only if $abc = 1$. 

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Let $$C_1 = \alpha_1\alpha_0\alpha_2$$, $$C_2 = \alpha_4\alpha_3\alpha_2$$ and $$C_3 = \alpha_6\alpha_5$$. Then $$\{C_1, C_2, C_3\}$$ is the set of supports of the non zero minimal oriented cycles. Hence the graph $$GS(\Lambda)$$ is

![Diagram](image)

Let us now compute the family of non zero constants $$\lambda_{ij}(t) \in k$$, such that $$\overline{C_i(t)} = \lambda_{ij}(t)\overline{C_j(t)}$$ for $$t \in (C_1)_0 \cap (C_2)_0$$. In this case $$C_1(0) = \alpha_1\alpha_0\alpha_2$$, $$C_2(0) = \alpha_4\alpha_3\alpha_2$$, $$C_1(2) = \alpha_2\alpha_0\alpha_0$$, and $$C_2(2) = \alpha_2\alpha_4\alpha_3$$. Using the relations given in the ideal $$I$$ we have $$\overline{C_1(0)} = a\overline{C_2(0)}$$, $$\overline{C_1(2)} = a\overline{C_2(2)}$$. Hence $$\lambda_{12} = \lambda_{12}(0) = \lambda_{12}(2) = a$$. In an analogous way we obtain that $$\lambda_{13} = \lambda_{13}(1) = b^{-1}$$, $$\lambda_{23} = \lambda_{23}(3) = c$$.

Consider the following minimal cycle $$C$$ in $$GS(\Lambda)$$, $$C = 1A_12A_23A_31$$. Then $$\phi(\Lambda)(C) = \lambda_{12}\lambda_{23}\lambda_{31} = abc$$, and $$\mathcal{C}_i$$ is the set of supports of the minimal cycles in $$GS(\Lambda)$$. So, by Corollary 10 we get $$\Lambda$$ is symmetric if and only if $$abc = 1$$.

5 A connexion with trivial extensions of algebras.

Let $$\Lambda$$ be a quasi-schurian weakly-symmetric algebra, $$\{\mathcal{C}_1, \cdots, \mathcal{C}_m\}$$ be the set of supports of the non zero minimal oriented cycles. In case it is possible to select exactly one arrow in each of the $$\mathcal{C}_i$$'s, we fix such a choice, and denote by $$\mathcal{C}(\Lambda)$$ the set consisting of the chosen arrows. Thus, $$\mathcal{C}(\Lambda)$$ is a set of arrows of $$Q_\Lambda$$ such that $$\mathcal{C}(\Lambda) \cap \mathcal{C}_i$$ has only one arrow for each $$i = 1, 2, \cdots, m$$. The ideal generated in $$\Lambda$$ by $$\mathcal{C}(\Lambda)$$ will be denoted by $$I_C$$. Moreover, $$\mathcal{C}(\Lambda)$$ induces a sub quiver $$Q_C$$ of $$Q_\Lambda$$ as follows $$(Q_C)_0 = (Q_\Lambda)_0$$ and $$(Q_C)_1 = (Q_\Lambda)_1 - \mathcal{C}(\Lambda)$$. Let $$\beta$$ be an arrow in $$Q_\Lambda$$. We will denote by $$Suppl(\beta)$$ the set of supplements of $$\beta$$. Thus, a path $$\delta$$ lies in $$Suppl(\beta)$$ if and only if $$\delta\beta$$ is a non zero minimal oriented cycle.

Our aim in this section is to give a proof and some consequences of the following result.

**Theorem 11** Let $$\Lambda$$ be a quasi-schurian symmetric algebra. If there exists a choice of arrows $$\mathcal{C}(\Lambda)$$ as above then $$\Lambda/I_C$$ is a schurian algebra, and moreover $$\Lambda \simeq T(\Lambda/I_C)$$ where $$T(\Lambda/I_C)$$ is the trivial extension of $$\Lambda/I_C$$.

In the proof of this theorem we will need the following lemmas.

**Lemma 12** Let $$\Lambda$$ be a quasi-schurian weakly-symmetric algebra. If there exists a choice of arrows $$\mathcal{C}(\Lambda)$$ as above, then $$\Lambda/I_C$$ is a schurian algebra and $$Q_C$$ is the ordinary quiver associated with $$\Lambda/I_C$$.

**Proof:** Assume that $$\Lambda = kQ_\Lambda I$$ where $$I$$ is an admissible ideal. Let $$\varphi : kQ_C \to \Lambda/I_C$$ be defined as follows $$\varphi(\delta) = \pi(\delta)$$ where $$\pi : \Lambda \to \Lambda/I_C$$ is the canonical epimorphism. We get that $$rad(\Lambda/I_C) = rad\Lambda/I_C$$ and $$\Lambda/rad\Lambda \simeq (\Lambda/I_C)/rad(\Lambda/I_C)$$ since $$I_C \subseteq rad\Lambda$$ and $$\pi$$ is an epimorphism. Hence $$\Lambda/I_C$$ is basic and $$\{\varphi(i) : i \in (Q_C)_0\}$$ is a complete family of
orthogonal primitive idempotents of $\Lambda/I_C$. So, to obtain that $Q_C = Q_{\Lambda/I_C}$ it is enough to prove that $\{\varphi(\alpha) : \alpha \in (Q_C)_1\}$ is a $k$-basis of $\text{rad}(\Lambda/I_C)/\text{rad}^2(\Lambda/I_C)$. First, observe that $\alpha \not\in C(\Lambda)$ implies that $\varphi(\alpha)$ is non zero and also does not lie in $\text{rad}^2(\Lambda/I_C)$. Therefore $\{\varphi(\alpha) : \alpha \in (Q_C)_1\}$ is a $k$-basis of $\text{rad}(\Lambda/I_C)/\text{rad}^2(\Lambda/I_C)$ since $\text{rad}(\Lambda/I_C) = \text{rad}A/I_C$. Finally let us prove that $\Lambda/I_C$ is schurian. Using the $k$-module isomorphisms 

$$\pi(e_i)(\Lambda/I_C)\pi(e_i) \simeq \overline{e_i}C_i\overline{e_i}/\overline{e_i}C_i\overline{e_i}$$

for all $i, j$. It is enough to prove that $\text{dim}\_k\overline{e_i}C_i\overline{e_i} = 1$ for each $i$, since $\Lambda$ is quasi-schurian. But this follows easily because each non zero oriented cycle contains an arrow from $C(\Lambda)$. □

**Definition.** Let $\Gamma$ be a finite dimensional $k$-algebra. We say that $x \in \Gamma$ is maximal in case $x \neq 0$ and $wx = wx = 0$ for all $w \in \text{rad}\Gamma$.

**Remark.** Let $\Gamma = Q_\Gamma/I$ with $I$ an admissible ideal. Let $\delta$ be a non zero path in $Q_\Gamma$. Then $\delta$ is maximal in $\Gamma$ if an only if $\overline{\delta} = 0$ for all arrows $\alpha \in Q_\Gamma$.

**Lemma 13** Let $\Lambda = Q_{\Lambda}/I$ be a quasi-schurian weakly-symmetric algebra with $I$ an admissible ideal. If there exists a choice of arrows $C(\Lambda)$ as above and $Q_{\Lambda}$ has no loops then the next statements hold, where $\pi : \Lambda \rightarrow \Lambda/I_C$ is the canonical epimorphism.

a) $\overline{e_\beta}C_\beta\overline{e_\beta} = 0$ for all $\beta \in C(\Lambda)$. Therefore $\pi(\overline{\gamma}) \neq 0$ for any supplement $\gamma$ of an arrow $\beta$ in $C(\Lambda)$.

b) Let $\gamma$ be a path in $Q_{\Lambda}$. Then $\pi(\overline{\gamma})$ is maximal if and only if there exists $\beta$ in $C(\Lambda)$ such that $\gamma \in \text{Suppl}(\beta)$.

c) Let $C(\Lambda) = \{\beta_1, \beta_2, \cdots, \beta_r\}$. Then the set $\{\pi(\overline{e_i}) : \delta_i \in \text{Suppl}(\beta_i), 1 \leq i \leq r\}$ is a $k$-basis of the vector space generated by all the maximal paths in $\Lambda/I_C$.

**Proof:**

a): Suppose that $\gamma$ is a non zero path and $\overline{\gamma} \in \overline{e_\beta}C_\beta\overline{e_\beta}$ for some $\beta \in C(\Lambda)$. Let $\delta$ be a supplement of $\gamma$, which exists since $\Lambda$ is weakly-symmetric (see Theorem 3). Now, we may assume that $\gamma$ contains an arrow $\beta'$ of $C(\Lambda)$ because $\overline{\gamma} \in I_C$ and $\Lambda$ is quasi-schurian. But $\beta$ and $\delta$ have the same starting and ending vertices. Hence $\delta = \beta$ and the non zero oriented cycle $\beta\gamma$ has two arrows in $C(\Lambda)$, a contradiction. So, $\overline{\gamma} = 0$ and therefore $\overline{e_\beta}C_\beta\overline{e_\beta} = 0$.

b): Let $\gamma$ be a path in $Q_{\Lambda}$. Assume that $\pi(\overline{\gamma})$ is maximal. So, $\overline{\gamma} \neq 0$. Let $\delta$ be a supplement of $\gamma$. Then $\delta$ contains an arrow $\beta$ in $C(\Lambda)$ since $\overline{\gamma} \not\in I_C$ and $\delta\gamma$ is a non zero minimal oriented cycle. Now we will prove that $\delta = \beta$ using the fact that $\pi(\overline{\gamma})$ is maximal. Considering a decomposition of $\delta$ as $\gamma_1\beta\gamma_2$ we obtain by a) that $0 \neq \pi(\overline{\gamma_2\gamma_1}) = \pi(\overline{\gamma_2})\pi(\overline{\gamma})\pi(\overline{\gamma_1})$ and therefore $\gamma_1, \gamma_2$ are trivial paths by the maximality of $\pi(\overline{\gamma})$. Hence $\delta = \beta$ and this means that $\gamma \in \text{Suppl}(\beta)$.

Assume now that $\gamma \in \text{Suppl}(\beta)$ with $\beta$ in $C(\Lambda)$, and let us prove that $\pi(\overline{\gamma})$ is maximal. From a) we obtain that $\pi(\overline{\gamma}) \neq 0$. Let $\alpha$ be an arrow in $Q_{\Lambda}$ such that $\pi(\overline{\alpha})\pi(\overline{\gamma}) \neq 0$. So $\overline{\alpha\gamma} \neq 0$ and therefore there exists a supplement $\mu$ of $\alpha\gamma$. But $\beta$ and $\mu\gamma$ have the same starting an ending vertices. Then there is $c$ in $k - \{0\}$ such that $\overline{\beta} = c\overline{\alpha\gamma}$ since $\Lambda$ is quasi-schurian, a contradiction because $I$ is an admissible ideal. Hence $\pi(\overline{\alpha})\pi(\overline{\gamma}) = 0$ for all arrow $\alpha \in Q_{\Lambda}$.

In an analogous way we obtain that $\pi(\overline{\alpha})\pi(\overline{\gamma}) = 0$ for all arrow $\alpha \in Q_{\Lambda}$.
c): The set \( \{ \pi(\delta_i) \mid 1 \leq i \leq r \} \) generates all the maximal paths since b) holds and \( \Lambda/I_C \) is schurian by Lemma 12. Let \( \sum_{i=1}^{r} a_i \pi(\delta_i) = 0 \) with \( a_i \in k \). Then \( \sum_{i=1}^{r} a_i \delta_i \in I_C \), and using a) we obtain \( a_j \delta_j = \frac{c_{\alpha(\beta_j)}}{c_{\delta(\beta_j)}}(\sum_{i=1}^{r} a_i \delta_i) = 0 \) since \( \Lambda \) is quasi-schurian and \( \beta_i, \beta_j \) do not have the same starting and ending vertices for \( i \neq j \). So, \( a_j = 0 \) for \( j = 1, 2, \ldots, r \). Therefore \( \{ \pi(\delta_i) \mid 1 \leq i \leq r \} \) is a linearly independent set and hence a \( k \)-basis since we knew that it generates all the maximal paths. \( \square \)

**Lemma 14** Let \( \Lambda = kQ_\Lambda/I \) be a quasi-schurian weakly-symmetric algebra with \( I \) an admissible ideal. If \( Q_\Lambda \) has a loop then \( \Lambda \simeq k[x]/<x^2> \).

**Proof:** Let \( n \) be the number of vertices of \( Q_\Lambda \). If \( n = 1 \) then \( Q_\Lambda \) has only one loop since \( \Lambda \) is quasi-schurian and \( I \) is an admissible ideal. Therefore in this case \( \Lambda \simeq k[x]/<x^2> \) since \( \Lambda \) is quasi-schurian. Assume \( n > 1 \) and let \( \alpha \) be a loop. As before we have only one loop at the vertex \( o(\alpha) \). Let \( \beta \) be another arrow starting at \( o(\alpha) \) and let \( \delta \) be a supplement of \( \beta \) (see Theorem 3). Hence there exists \( c \in k - \{0\} \) such that \( \pi = c\delta\beta \) since \( \Lambda \) is quasi-schurian, a contradiction because \( I \) is admissible. So, \( n \) has to be 1 and in this case we have already proved the lemma. \( \square \)

Before giving a proof of Theorem 11, let us recall from [3] (see also in [7],[8]) the description of the ordinary quiver and relations of the trivial extension \( T(A) \) of a schurian algebra \( A \).

Let \( A = kQ_A/I \) be a schurian algebra with \( I \) an admissible ideal, and \( p_1, p_2, \ldots, p_t \) be paths in \( Q_A \) such that \( \{ \overline{p_1}, \overline{p_2}, \ldots, \overline{p_t} \} \) is a \( k \)-basis of the vector space generated by all the maximal paths in \( A \). Then the vertices of \( Q_{T(A)} \) are the vertices of \( Q_A \) and \( (Q_{T(A)})_1 = (Q_A)_1 \cup \{ \beta_{p_1}, \beta_{p_2}, \ldots, \beta_{p_t} \} \), where \( \beta_{p_i} \) is an arrow starting at \( e(p_i) \), ending at \( o(p_i) \) and not belonging to \( Q_A \) for \( i = 1, 2, \ldots, t \). We observe that all arrows of \( Q_{T(A)} \) are in oriented cycles. An oriented cycle \( C \) in \( Q_{T(A)} \) is called elementary if there exists a vertex \( j \) in \( C \) with \( C(j) = q\beta_{p_i} \) for some \( i = 1, 2, \ldots, t \) and some path \( q \) maximal in \( A \) with the same starting and ending vertices as \( p_i \). We can describe now the relations of \( T(A) \) given in [3].

**Theorem 15** ([3]) Let \( A = kQ_A/I \) be a schurian algebra with \( I \) an admissible ideal. Let \( I_{T(A)} \) be the ideal of \( kQ_{T(A)} \) generated by the following relations:

i) The composition of \( n + 1 \) arrows in an elementary oriented cycle of length \( n \).

ii) The composition of arrows not belonging to a same elementary oriented cycle.

iii) The elements \( q - bq' \) where \( q, q' \) are paths in \( Q_{T(A)} \) having the same ending and starting vertices, and such that one of the following conditions holds.

a) \( \overline{\pi} = b\overline{q'} \) with \( b \in k - \{0\} \) and \( q, q' \) paths in \( Q_A \).

b) There is a path \( \nu \) in \( Q_A \) such that \( \nu q = \alpha_{r-1} \cdots \alpha_2 \alpha_1 \beta_{p_1} \alpha_m \cdots \alpha_{r+1} \alpha_r \) and \( \nu q' = \alpha'_{s-1} \cdots \alpha'_2 \alpha'_1 \beta_{p'_1} \alpha'_m \cdots \alpha'_{s+1} \alpha'_s \) are elementary cycles. Then \( b \) is defined by \( b = a_1/a_2 \) for non zero \( a_1, a_2 \in k \) with \( \overline{\alpha_n} \cdots \overline{\alpha_1} = a_1 \overline{p_1} \) and \( \overline{\alpha'_m} \cdots \overline{\alpha'_1} = a_2 \overline{p'_1} \).

Then the ideal \( I_{T(A)} \) is admissible and \( T(A) \simeq kQ_{T(A)}/I_{T(A)} \).

\( \square \)
Now recall the well known isomorphism of $T(A) - T(A)$ bimodules $\psi : T(A) \to D(T(A))$ given by $\psi(x_1, f)(x_2, f_2) = f_1(x_2) + f_2(x_1)$, and consider the linear map $F = \psi(1,0) : T(A) \to k$. This functional is handy to describe the constants of $\psi$ in the above theorem. We start by giving an explicit description of it. Let $\{\overline{p_1}, \overline{p_2}, \ldots, \overline{p_i}, \overline{p_{i+1}}, \overline{p_{i+2}}, \ldots, \overline{p_s}\}$ be a basis of $A$, extending the chosen basis $\{\overline{1}, \overline{p_2}, \ldots, \overline{p_i}\}$ of the vector space generating by all the maximal paths in $A$, we will denote by $\{\overline{p_1^*, \overline{p_2^*}, \ldots, \overline{p_s^*}}\}$ the basis of $D(A)$ dual to $\{\overline{p_1}, \overline{p_2}, \ldots, \overline{p_s}\}$. Let $\phi : kQ_T(A) \to T(A)$ be the map defined by $\phi(\alpha) = (\overline{\alpha}, 0)$ for $\alpha \in (Q_A)_1$ and $\phi(\beta_p) = (0, \overline{p^*})$. It is not difficult to see that the induced linear functional $f : kQ_T(A) \to k$ defined by the composition $F\phi$ satisfies the following condition: $f$ is constant and non zero on the support $C$ of each minimal oriented cycle $C$ non zero in $T(A)$ (see Lemma 7 and its remark). Moreover, for the elementary oriented cycle $q\beta_p$, we have that $\overline{q} = f(q\beta_p)\overline{p_i}$. Therefore the condition $iii)$ in Theorem 15 can be changed by $iii)'$ The elements $\delta_1 - f(\delta_1\nu)/(f(\delta_2\nu))^{-1}\delta_2$, where $\delta_1, \delta_2$ are paths in $Q_T(A)$ having the same ending and starting vertices and such that there exists a path $\nu$ with $\delta_1\nu$ and $\delta_2\nu$ elementary oriented cycles.

This last condition will be used in the proof of Theorem 11.

**Proof of Theorem 11:** Let $\Lambda = Q_A/I$ where $I$ is an admissible ideal and denote by $\overline{\gamma}$ the congruence class $\gamma + I$ in $\Lambda$. If $Q_A$ has a loop we get by Lemma 14 that $\Lambda/I_c \simeq k$ and hence $\Lambda \simeq T(\Lambda/I_c)$.

Assume that $Q_A$ has no loops and let $\mathcal{C}(\Lambda) = \{\beta_1, \beta_2, \ldots, \beta_t\}$. Let $A = \Lambda/I_c$, then by Lemma 12 we have that $A$ is schurian and its ordinary quiver is $Q_c$. Now, for each $i = 1, 2, \ldots, t$ we choose a path $p_i$ in $\text{Supp}(\beta_i)$. So, By Lemma 13 the set $\{\pi(\overline{p_i})\} 1 \leq i \leq t$ is a $k$-basis of the vector space generated by all the maximal paths in $A$ where $\pi : \Lambda \to A = \Lambda/I_c$ is the canonical epimorphism. By Theorem 15 we have that $Q_T(A) = Q_A \cup \mathcal{C}(\Lambda) = Q_A$. Hence $\psi : kQ_T(A) \to A$ where $\psi(\gamma) = \overline{\gamma}$ is an epimorphism of $k$-algebras. So, we have to check that $\text{Ker} \psi = I_{T(A)}$ to obtain $T(A) \simeq \Lambda$. We will need the following Lemmas:

**Lemma (A):** Let $C$ be an oriented minimal cycle in $Q_T(A)$. Then $C$ is non zero in $T(A)$ if and only if it is non zero in $\Lambda$.

**Proof:**

($\Rightarrow$) : Assume $C$ is non zero in $T(A)$. Then by Theorem 15 there is an elementary oriented cycle $q\beta_i$ such that $C(j) = q\beta_i$ for some vertex $j$ in $C$. Therefore $C(j)$ is non zero in $\Lambda$ and by Lemma 7, $C$ is non zero in $\Lambda$.

($\Leftarrow$) : Suppose that $C$ is non zero in $\Lambda$. Then by the definition of $\mathcal{C}(\Lambda)$ we get that $C$ contains an arrow from $\mathcal{C}(\Lambda)$. So, $C$ is an elementary oriented cycle and therefore it is non zero in $T(A)$.

Let $\{\overline{C}_1, \ldots, \overline{C}_n\}$ be the set of supports of the non zero minimal oriented cycles in $\Lambda$. Since $\Lambda$ is symmetric we have by Lemma 7 (see also its remark) a linear functional $\varphi : \Lambda \to k$ non zero on the supports $\overline{C}_i$ for all $i = 1, 2, \ldots, m$. Making a change of variables $\beta_i \mapsto a_i\beta_i$ with adequate $a_i \in k - \{0\}$ for all $i = 1, 2, \ldots, t$ we may assume that $\varphi(\overline{p_i}) = 1$ for all $i = 1, 2, \ldots, t$. Now, this functional satisfies the following property.

**Lemma(B):** Let $q$ be a path in $Q_A$ such that $\varphi(\overline{q}) = c_q\pi(\overline{p_i})$ with $c_q \in k - \{0\}$ and
some $i = 1, 2, \cdots, t$. Then $c_q = \varphi(q\beta_i)$.

**Proof:** Since $\overline{q} - c_q\overline{p} \in Ker\pi = I_C$ we get by Lemma 13 a) that $\overline{q} = c_q\overline{p}$. Therefore $c_q = \varphi(q\beta_i)$ since $\varphi(p_i\beta_i) = 1$. □

**Remark:** Let $C$ be a minimal oriented cycle non zero in $T(A)$. Then this lemma give us that $\varphi(C) = f(C)$ where $f : kQ_{T(A)} \to k$ is the linear functional defined above. □

Let us prove that $Ker\psi \supseteq I_{T(A)}$: By Lemma(A) and Lemma 2 we obtain that the relations i) and ii) in Theorem 15 are zero in $\Lambda$. Using Lemma(B) and its remark we obtain that the above relations iii) are zero in $\Lambda$ because it is quasi-schurian. So, $Ker\psi \supseteq I_{T(A)}$.

Finally, we will check that $Ker\psi \subseteq I_{T(A)}$: Let $\gamma \in Ker\psi$. Then $\gamma$ is zero in $A$ and therefore $\gamma$ is zero in $T(A)$ since $A$ is a sub algebra of $T(A)$. So, $\gamma \in I_{T(A)}$ because $kQ_{T(A)}/I_{T(A)} \simeq T(A)$. Hence $Ker\psi \subseteq I_{T(A)}$. □

Now it is easy to obtain the main result of this section.

**Theorem 16** Let $\Lambda$ be basic connected finite dimensional $k$-algebra. The following statements are equivalent

1) There exists a schurian basic triangular algebra $\Lambda'$ such that $\Lambda \simeq T(\Lambda')$.

2) $\Lambda$ is symmetric quasi-schurian, and we can choose a set $\mathcal{C}(\Lambda)$ such that the quiver $Q_C$ has non oriented cycles.

If these conditions hold, then $\Lambda' \simeq \Lambda/I_C$ where $I_C$ is the ideal generated by $\mathcal{C}(\Lambda)$ in $\Lambda$.

**Proof:** Follows from Theorems 11 and 15. □

Another application of Theorem 11 is the following result.

**Theorem 17** Let $Q$ be a quiver without oriented cycles, and $\Lambda$ an iterated tilted algebra of type $Q$. If $\Gamma = T(\Lambda)$ is quasi-schurian then, for each choice $\mathcal{C}(\Gamma)$ as above and such that the quiver $Q_C$ has non oriented cycles, $\Gamma/I_C$ is an iterated tilted algebra of type $Q$.

**Proof:** It follows immediately from Theorem 11 and the next lemma. □

**Lemma 18** Let $Q$ be a quiver without oriented cycles, and $\Lambda$ an iterated tilted algebra of type $Q$. Let $\Lambda'$ be a basic finite dimensional $k$-algebra.

If $T(\Lambda) \simeq T(\Lambda')$ and $\Lambda'$ has finite global dimension then $\Lambda'$ is iterated tilted of type $Q$.

**Proof:** The proof is based on known results about derived categories and repetitive algebras (see [2],[5] and [6]). Since $T(\Lambda) \simeq T(\Lambda')$ we get that the repetitive algebra $\hat{\Lambda}$ of $\Lambda$ is isomorphic to the repetitive algebra $\hat{\Lambda}'$ of $\Lambda'$. In particular, we obtain that the triangulated category $\text{mod}\hat{\Lambda}$ is triangle equivalent to $\text{mod}\hat{\Lambda}'$. Since $\Lambda$ and $\Lambda'$ have finite global dimension we have the diagram

$$D^b(\Lambda) \sim \text{mod}\hat{\Lambda} \sim \text{mod}\hat{\Lambda}' \sim D^b(\Lambda')$$

where $\sim$ denotes a triangle equivalence. Thus $D^b(\Lambda)$ is triangle equivalent to $D^b(\Lambda')$, and therefore $\Lambda'$ is an iterated tilted algebra of type $Q$ (see [5] or [6]). □

We get now a useful approach to obtain iterated tilted algebras of a given tree class, generalizing an analogous result proven in [3] for Dynkin quivers.
Corollary 19 Let $Q$ be an oriented tree and $\Gamma = T(kQ)$. For each choice $C(\Gamma)$ as above with $Q_C$ without oriented cycles we have that $\Gamma/I_C$ is an iterated tilted algebra of type $Q$.

Proof: It follows immediately from Theorem 17. $\square$

Example: Let $Q$ be the following oriented tree

and $\Gamma = T(kQ)$ be the trivial extension of $kQ$. Considering the maximal paths $p_1 = \alpha_3\alpha_2\alpha_1$, $p_2 = \alpha_4\alpha_1$, and $p_3 = \alpha_5\alpha_1$ we obtain by Theorem 15 that $Q_\Gamma$ is

Let $C(\Gamma) = \{\alpha_3, \alpha_4, \alpha_5\}$, and $\Lambda = \Gamma/I_C$. By Lemma 12 we get that $Q_\Lambda$ is

So, by Corollary 19 we have that $\Lambda \simeq kQ_\Lambda/\langle \alpha_2\alpha_1\beta_{p_2}, \alpha_2\alpha_1\beta_{p_3} \rangle$ is an iterated tilted algebra of type $Q$.

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E-mail: omendoza@criba.edu.ar