Abstract

We study a geometric Ramsey type problem where the vertices of the complete graph $K_n$ are placed on a set $S$ of $n$ points in general position in the plane, and edges are drawn as straight-line segments. We define the empty convex polygon Ramsey number $R_{EC}(k, k)$ as the smallest number $n$ such that for every set $S$ of $n$ points and for every two-coloring of the edges of $K_n$ drawn on $S$, at least one color class contains an empty convex $k$-gon. A polygon is empty if it contains no points from $S$ in its interior. We prove $17 \leq R_{EC}(3, 3) \leq 463$ and $57 \leq R_{EC}(4, 4)$. Further, there are three-colorings of the edges of $K_n$ (drawn on a set $S$) without empty monochromatic triangles.

A related Ramsey number for islands in point sets is also studied.

1 Introduction

Ramsey’s theorem ensures that for every two-coloring of the edges of the complete graph $K_n$ on a large enough number $n$ of vertices, at least one of the two color classes contains a clique of a given size. The Ramsey number $R(s, t)$ is the smallest number $n$ such that every two-coloring of the edges of $K_n$ contains a clique on $s$ vertices from the first color class or a clique on $t$ vertices from the other color class. Geometric variants of Ramsey’s theorem have been studied, see e.g. [9]. When the vertices of $K_n$ are drawn on a set of $n$ points in the plane, and edges as straight-line segments, geometry comes into play by considering crossings of edges. Throughout, we only consider point sets $S$ in general position, meaning sets without three collinear points. For example, in [11] it was shown that for every set $S$ of $n$ points and for every two-coloring of the edges of $K_n$ drawn on $S$, one color class has non-crossing cycles of lengths $3, 4, \ldots, \left\lfloor \sqrt{n/2} \right\rfloor$. In this work we consider another geometric constraint, namely emptiness. A simple polygon is empty if it has no points of $S$ in its interior. The number of empty convex polygons in $K_n$ drawn on sets $S$ of $n$ points have been estimated, see e.g. [1, 2, 7, 10]. We define the empty convex polygon Ramsey number $R_{EC}(s, t)$ as the smallest number $n$ such that for every set $S$ of $n$ points and for every two-coloring of the edges of $K_n$ drawn on $S$, the first color class contains an empty convex $s$-gon or the second color class contains an empty convex $t$-gon. For the case of empty triangles, the bounds $17 \leq R_{EC}(3, 3) \leq 463$ are shown. We also prove that there are three-colorings of the edges of $K_n$, drawn on some point set $S$, without empty monochromatic triangles; in other words $R_{EC}(3, 3, 3) = 0$. For the case of empty convex quadrilaterals we can show the lower bound $R_{EC}(4, 4) \geq 57$. We were not able to prove an upper bound. Finally we consider a Ramsey number for islands in point sets. An island of a point set $S$ is a subset $I$ of $S$ such that $Conv(I) \cap S = I$. Islands in point sets were also studied in [3, 4, 6]. In our context, an island is a clique formed by a subset of vertices of $K_n$ drawn on $S$ which contains no further point of $S$ in its interior. We remark that the Ramsey number $R(s, t)$ equals the smallest number $n$ such that every two-coloring of the edges of $K_n$ drawn on a set of $n$ points in convex position contains an island on $s$ points in one color class or an island on $t$ points in the other color class. This is, because there, all islands are in convex position. In [13] it was shown that for every set $S$ of $n$ points, the edges of $K_n$, drawn on $S$, can be two-colored such that there is no monochromatic island on four points with triangular convex hull. We prove that there are point sets $S$ and a two-coloring of the edges of $K_n$, drawn on $S$, such that there is no monochromatic island on four points (regardless of the form of the convex hull). That is, the island Ramsey number for four points $R(I(4, 4))$ is zero.

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2 The empty triangle Ramsey number

Theorem 1 The empty triangle Ramsey number satisfies $17 \leq R_{EC}(3,3) \leq 463$.

Proof. For the upper bound, we use the fact that every sufficiently large point set in general position contains an empty convex hexagon [8, 14]. Koshelev obtained the current best bound, 463, on the number of points needed to guarantee such an empty convex hexagon [12]. Consider only the complete graph on six vertices $K_6$ formed by the vertices of this hexagon. Ramsey’s theorem tells us that every two-coloring of $K_6$ contains a monochromatic triangle. Since the hexagon is empty, the monochromatic triangle is so as well. For the lower bound, a two-colored complete geometric graph on 16 vertices without an empty monochromatic triangle is shown in Figure 1.

Figure 1: A two-coloring of the edges of $K_{16}$ without an empty monochromatic triangle. Only the edges of one color class are drawn.

Theorem 2 The empty triangle Ramsey number for three-colored complete graphs $R_{EC}(3,3,3)$ is zero.

Proof. We have to present a three-coloring of the edges of the complete geometric graph $K_n$ drawn on a set $S$ of $n$ points. The point set $S$ is the so-called Horton set $H(n)$, see e.g. [1, 2, 5, 10], defined recursively as follows: $H(1) = \{(1,1)\}$ and $H(2) = \{(1,1), (2,2)\}$. When $H(n)$ is defined, set

$$H(2n) = \{(2x - 1, y) \mid (x, y) \in H(n)\} \cup \{(2x, y + 3^n) \mid (x, y) \in H(n)\}.$$  

In this construction $H(2n)$ is obtained by taking $H(n)$ and a copy of $H(n)$ which is slightly shifted to the right and placed far above the other set $H(n)$. To define an edge-coloring of the complete graph drawn on $H(n)$ we use an auxiliary three-coloring of the vertices of $H(n)$: vertex $(x, y)$ gets color $x \mod 3$. This three-coloring for $H(8)$ is shown in Figure 2. In [5], Theorem 3.3, it was proved that this coloring admits no empty triangles with its three vertices from the same color class. The three-coloring for the edges of $K_n$ is now defined as follows: an edge connecting points $(x_1, y_1)$ and $(x_2, y_2)$ gets color $x_1 + x_2 \mod 3$. Then, a triangle formed by points $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is monochromatic if and only if $x_1, x_2$ and $x_3$ belong to the same congruence class modulo three. Thus, the vertices of a monochromatic triangle have the same color and from [5] we know that these triangles are not empty.

3 The empty convex quadrilateral Ramsey number

Theorem 3 The empty convex quadrilateral Ramsey number satisfies $57 \leq R_{EC}(4,4)$.

Proof. Figure 3 shows a two-coloring of the edges of $K_{11}$ in convex position without an empty convex monochromatic quadrilateral. A drawing of $K_{56}$ (indicated in Figure 4) and a two-coloring of its edges without an empty convex monochromatic quadrilateral is obtained by placing five groups of 11 points (with two-coloring as in Figure 3) in such a way that the 55 points lie on five small semi-circles with centers the vertices of a regular pentagon. Then the last point is placed in the center of this pentagon and connected to the 55 points with the same color as the drawn
Figure 3: A two-coloring of the edges of $K_{11}$ without an empty convex monochromatic quadrilateral. Only the edges of one color class are drawn.

Figure 4: Schematic drawing of $K_{56}$ without an empty convex monochromatic quadrilateral. Only the edges of one color class are indicated.

edges in Figure 3. □

4 The Ramsey number for islands

**Theorem 4** The island Ramsey number $R_I(4,4)$ is zero.

**Proof.** We present a two-coloring of the edges of $K_n$ drawn on the Horton set $H(n)$ without an empty monochromatic $K_4$. As in the proof of Theorem 2, we start with the auxiliary three-coloring of the vertices of $H(n)$ where vertex $(x,y)$ gets color $x$ mod 3. Now we define a two-coloring for the edges of $K_n$ as follows: an edge connecting points $(x_1,y_1)$ and $(x_2,y_2)$ gets color 0 if $x_1 - x_2$ mod 3 = 0 and gets color 1 otherwise. In other words, an edge gets color 0 if and only if its two vertices have the same color in the auxiliary vertex coloring. Then, a complete subgraph $K_4$ is monochromatic if and only if its four vertices have the same color in the auxiliary vertex coloring. Thus, if a $K_4$ is monochromatic, then from [5] Theorem 3.3, we know that none of its triangles is empty, which implies that this $K_4$ is not an island. □

5 Concluding Remarks

An obvious problem left open is to close the gap between lower and upper bound for $R_{EC}(3,3)$. Very interesting would be to prove an upper bound on the empty convex quadrilateral Ramsey number. Computer experiments suggest that it is finite and probably not too large.

References