Introduction to Moduli Spaces Associated to Quivers

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Introduction

Moduli are traditionally used in order to classify "continuous" families of geometric objects in a natural way. Usually, one can not hope, to classify in that way all objects, even after fixing some discrete invariants. In algebraic geometry, for example the classification of vector-bundles with fixed rank and degree on a smooth curve leads after some work to the problem of classifying the orbits of a reductive group on a *projective* variety. Here, after linearizing that action, Mumford's notion of (semi-) stability surges and leads to the notion of stable vector-bundles, where in fact a moduli space exists.

In turn, in representation theory one would like to classify representations of a quiver $Q = (Q_0, Q_1, s, t)$ for a fixed dimension-vector $\mathbf{v} \in \mathbb{N}^{Q_0}$. Here, it is quite obvious to find a locally universal family of such representations:

$$\mathrm{Rep}_Q^{\mathbf{v}} := \prod_{\alpha \in Q_1} \mathrm{Hom}_{\mathrm{k}}(\mathrm{k}^{\mathbf{v}(s\alpha)}, \mathrm{k}^{\mathbf{v}(t\alpha)})$$

with the action of $\mathrm{Gl}_{\mathbf{v}} := \prod_{i \in Q_0} \mathrm{Gl}_{\mathbf{v}(i)}$ by conjugation:

$$g \cdot (f_\alpha)_{\alpha \in Q_1} := (g_{t\alpha} f_\alpha g_{s\alpha}^{-1})_{\alpha \in Q_1}$$

The $\mathrm{Gl}_{\mathbf{v}}$-orbits are precisely the isoclasses of representations of Q of dimension $\mathbf{v}$. Now, this variety is an affine space, so apparently there is no need to linearize the action of G. On the other hand, if Q has no oriented cycles, classical invariant theory produces no interesting results, since there is exactly one closed orbit, corresponding to the unique semi-simple representation of dimension $\mathbf{v}$ for Q.

Recently, this problem has been tackled by an useful version of classical GIT for the action of a reductive group G on an *affine* space X by King [8], which includes a notion of stability with respect to a character χ of G. This leads to interesting moduli spaces for χ-semistable representations, which are projective over the usual categorical quotient. Some recent work on the structure of these Moduli-spaces includes [1],[6], [9],[16].

The same method applies for a related problem - the framed moduli introduced by H. Nakajima [13]: Take an additional dimension-vector $\mathbf{w}$, and consider

$$\mathrm{Rep}_Q^{\mathbf{v},\mathbf{w}} := \mathrm{Rep}_Q^{\mathbf{v}} \times \prod_{i\in Q_0} \mathrm{Hom}_{\mathrm{k}}(\mathrm{k}^{\mathbf{w}(i)}, \mathrm{k}^{\mathbf{v}(i)})$$

with the action of G by

$$g\,.\,((f_\alpha)_{\alpha\in Q_1}, (h_i)_{i\in Q_0}) := (g\,.\,(f_\alpha)_{\alpha\in Q_1}, (g_i h_i)_{i\in Q_0})$$

If we take $\chi(g) := \prod_{i\in Q_0} \det(g_i)$, it turns out, that

$$((f_\alpha)_{\alpha\in Q_1}, (h_i)_{i\in Q_0}) \in \mathrm{Rep}_Q^{\mathbf{v},\mathbf{w}}$$

is χ-stable if and only if it is χ-semistable if and only if the subrepresentation of $f = (f_\alpha)_{\alpha\in Q_1}$, generated by the images of the (h_i) is in fact f [13, 3.4].

A special case of this problem - when Q consists of just one loop has been independently of interest in control theory for quite a while, see for example [19], or [10, I.3] for a short review. In fact, there one studies the affine space $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$ with the action of Gl_n via $g\,.\,(A,B) := (gAg^{-1}, gB)$; in particular, one is interested in *completely reachable sytems* $(A,B) \in \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$, where by definition the column vectors of B form a system of generators for the $\mathrm{k}[x]$-module structure on k^n defined by A. By the above result of Nakajima, the completely reachable sytems are precisely the det-stable points on $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$, and we can conclude, that there exists a fine moduli space for that problem.

We want to work out in these notes that last aspect, and compare the result with the information which can be obtained in this special situation by elementary methods. In fact, quite a lot can be done here without GIT, due to the fact that one can "guess" local normal forms for completely reachable sytems. Finally, by putting the two aspects together, we find an explicit system of generators for the algebra of relative invariants for our action of Gl_n on $\mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m}$.

Finally, let me point out, that most of the material we present here on completely reachable sytems can be found in some form in [19]; we just use different normal forms, and relative invariants are not treated explicitly there.

I thank L. Hille for several helpful discussions on moduli spaces, and in particular for pointing out the connection between the above problem from controll-theory with framed moduli.

1 Preliminaries

Let k be an algebraically closed field of arbitrary characteristic. We will use the name "variety" for abstract varieties, though in most cases we can think just of quasi-projective varieties.

In the following discussion on quotients and reductive groups I took great advantege from the exposition of K. Bongartz in [3]. The exposition on families and moduli-spaces is essentially a compressed version of [14, 1]. We do not discuss the étale topology, see for example [18, §1.] for an introduction.

1.1 (Quotients) Let an algebraic group G act on a variety X, i.e. the action is given by a morphism $G \times X \longrightarrow X, (g, x) \mapsto g \,.\, x$. And let $\varphi : X \longrightarrow M$ be a G-invariant morphism.

Definition (φ, M) is called a *categorical quotient*, if it is universal in the sense, that each other G-invariant morphism $\varphi' : X \longrightarrow M'$ factors over φ. Write in this case also $X//G := M$.

A categorical quotient (φ, M) is called *orbit space* if the fibers of φ are precisely the G-orbits on X.

(φ, M) is called a *geometric quotient*, if φ is open, if its fibers are the orbits, and for any open subset $U \subset M$ the restriction of φ to $\varphi^{-1}(U) \longrightarrow U$ induces an isomorphism between the algebra of G-invariant functions $\mathrm{k}[\varphi^{-1}(U)]^G$ and $\mathrm{k}[U]$.

Remark: (1) A geometric quotient is easily seen to be a categorical quotient [12, Proposition 0.1], thus both are unique, if they exist.

(2) Suppose, that a G-invariant morphism $\varphi : X \longrightarrow M$ admits a section σ, that meets each G-orbit on M, then (φ, M) is an orbit space. In fact, the hypothesis implies, that the fibers of φ are precisely the orbits, thus $\varphi' = \varphi' \sigma \varphi$ for each G-invariant morphism φ'.

(3) If an orbit-space for the action of G on X exists, this action is necessarily *separated*, i.e. the image Γ of the morphism

$$\Psi : G \times X \longrightarrow X \times X, (g, x) \mapsto (g \,.\, x, x)$$

is closed in $X \times X$. If moreover Ψ induces an isomorphism of $G \times X$ with Γ, the action is called *free.*

1.2 (Reductive Groups) A linear algebraic group is called *reductive* if its radical (the unique maximal connected normal solvable subgroup) is isomorphic to a direct product of copies of k^*. Important examples are the groups $\mathrm{Gl}_n, \mathrm{PGl}_n$ and Sl_n.

We have the following fundamental result, which contains the work of many Mathematicians, among them W.J. Haboush, D. Hilbert, D. Mumford, M. Nagata. For a proof (except the result of Haboush [4]) and remarks on the history of the theorem consult the textbook [14]. Note moreover, that the result is much easier to prove if $\operatorname{char} \mathrm{k} = 0$.

Theorem *Let X be an affine variety ($X = \operatorname{Spec}(R)$, where $R := \mathrm{k}[X]$ is the ring of regular functions on X) with the action of a reductive group G, and let $R^G \subseteq R$ be the ring of invariant functions on X. Then R^G is a finitely generated* k*-algebra, and the invariant morphism $\varphi : X \longrightarrow \operatorname{Spec}(R^G) = X//G$ is a categorical quotient. Moreover, φ sends disjoint G-invariant closed sets to disjoint closed sets in* $\operatorname{Spec}(R^G)$.

There exists an (possibly empty) open subset $U \subseteq X//G$, such that $\varphi^{-1}(U)$ consists of the orbits of maximal dimension which are also closed. The corresponding restriction of φ is a geometric quotient.

Remark: (1) If G is not reductive, we can not expect the quotient to be affine for example take the action of B the subgroup of lower triangular matrices in Gl_n, then the quotient $\mathrm{Gl}_n //B$ exists and can be identified with a complete flag variety, which is projective. Note, that Gl_n is affine.

(2) If X is not affine, even for a free action of G reductive on X, a quotient needs not to exist, see [3, 6.3] for an elementary example.

If X can be covered by open affine and G-invariant pieces, one can try to glue the local quotients together, but the result can be non separated (a very "pre" pre-scheme) [12, p.38]. For example take $X = \mathrm{k} \times \mathrm{k} \setminus \{(0,0)\}$ with the action of k^* by $\lambda \,.\, (a,b) := (\lambda a, \lambda^{-1} b)$. The result of the gluing is the affine line with the origin doubled.

1.3 (Families) In this situation we can define a family F of G-orbits over a variety S by the following data: An open covering $(U_i)_{i\in I}$ of S and a a "compatible" family $\varphi_i : U_i \longrightarrow X$, where the "compatibility" is given by a family of morphisms $\gamma_{i,j} : U_i \cap U_j \longrightarrow G$, such that $\varphi_i(x) = \gamma_{i,j}(x) \,.\, \varphi_j(x)$ for all $x \in U_i \cap U_j$. Moreover the $\gamma_{i,j}$ should fulfill the usual cocycle conditions:

$$\begin{aligned} \gamma_{i,i}(x) &= 1_G & &\text{for all } x \in U_i \\ \gamma_{i,j}(x) \cdot \gamma_{j,k}(x) &= \gamma_{i,k}(x) & &\text{for all } x \in U_i \cap U_j \cap U_k \end{aligned}$$

Note, that each family F on S defines a map (of sets)

$$\nu_F : S \longrightarrow \{G\text{-orbits on } X\}$$

We have several natural choices (which we call (a), (b) and (c) below) to define an equivalence relation on the families over a given variety S. Two families F and F' on S are *equivalent*, if

(a) $\nu_F = \nu_{F'}$.

(b) for each pair $(i, i') \in I \times I'$ there exists a morphism $\delta_{i,i'} : U_i \cap U'_{i'} \longrightarrow G$, such that $\varphi_i(x) = \delta_{i,i'}(x) \,.\, \varphi_{i'}(x)$ for $x \in U_i \cap U_{i'}$.

(c) for each pair $(i, i') \in I \times I'$ there exists a morphism $\delta_{i,i'} : U_i \cap U'_{i'} \longrightarrow G$, such that $\varphi_i(x) = \delta_{i,i'}(x) \,.\, \varphi_{i'}(x)$ for $x \in U_i \cap U_{i'}$, *and* $\delta_{i,i'}(x)\gamma_{i',j'}(x) = \gamma_{i,j}(x)\delta_{j,j'}(x)$ for $x \in U_i \cap U_j \cap U_{i'} \cap U_{j'}$.

We have obviously $[F]_c \subseteq [F]_b \subseteq [F]_a$ for the respective equivalence classes of a family F.

This setup fulfills the usual requirements for a moduli problem:

- The equivalence classes of families on the trivial variety $\{\mathrm{pt}\}$ are identified with the G-orbits on X.
- We have an obvious pull-back construction: If $\psi : T \longrightarrow S$ is a morphism of varieties, the family $\psi^* F$ on T is given by the data: $(\psi^{-1}(U_i))_{i \in I}$ and $\psi \circ \varphi_i : \psi^{-1}(U_i) \longrightarrow X$ etc., which is compatible with each of our equivalence relations.

Moreover, the tautological family $\tilde{U}$ on X given by $\mathbf{1} : X \longrightarrow X$ has the local universal property: For $x \in S$ we can find $i \in I$ such that $x \in U_i$, and $F \mid_{U_i} = \varphi_i^* \tilde{U}$ trivially.

Thus, we obtain a contravariant functor $\mathcal{F}_i$: varieties $\longrightarrow$ Sets, $i \in \{a, b, c\}$, which assigns to each variety S the set of equivalence classes of families on S. For $[X] \in \mathcal{F}(S)$, we get a map $\bar{\nu}_X : S \longrightarrow \mathcal{F}(\{\mathrm{pt}\}), s \mapsto [X_s]$, where X_s is the restriction of X to the point $s \in S$.

Example If we take $X = \mathrm{k}^{n \times n} \times \mathrm{k}^{n \times m}$ as in the introduction, then a family of systems on a variety S can be described as a triple (F, α, β), where

- F is a rank n vector-bundle on S,
- α is a (vector-bundle) endomorphism of F
- $\beta\colon S \times \mathrm{k}^m \longrightarrow F$ is a homomorphism of vector-bundles.

Naturally, we will consider two families (F, α, β) and (F', α', β') *equivalent*, if there is an isomorphism $\delta\colon F \longrightarrow F'$ of vector-bundles such that $\delta\alpha = \alpha'\delta$ and $\beta' = \delta\beta$. This is exactly the relation of type (c), discussed above.

We have an obvious restriction to families of completely reachable systems, and we will see in 2.4, that for families of completely reachable systems in fact the three above defined equivalence relations coincide. For general families of systems however, this is not true, compare [14, 2.4].

1.4 Definition A *fine moduli space* for $\mathcal{F}$, consists of a variety M, and a natural transformation $\Phi : \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, M)$, which represent the functor $\mathcal{F}$.

A *coarse moduli space* for $\mathcal{F}$ consists of a variety M, and a natural transformation $\Phi\colon \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, M)$, such that

(i) $\Phi_{\{\mathrm{pt}\}}$ is bijective

(ii) Φ is universal, i.e. if $\Psi : \mathcal{F}(-) \longrightarrow \mathrm{Hom}(-, N)$ is an other natural transformation, then there exists an *unique* natural transformation $\Omega\colon \mathrm{Hom}(-, M) \longrightarrow \mathrm{Hom}(-, N)$ such that $\Psi = \Omega \circ \Phi$

Remarks: (1) A fine moduli space is also coarse, and both are unique, if they exist. The existence of a coarse moduli space does *not* depend on the choice of the equivalence relation on families (as long as it fulfills the usual requirements).

(2) A coarse moduli space M is fine, if

(i) it admits a *universal family* U, i.e. $\nu_U\colon M \longrightarrow \mathcal{F}(\{\mathrm{pt}\})$ is bijective,

(ii) For families X, X' on a variety S, we have $\nu_X = \nu_{X'} \Longrightarrow [X] = [X']$.

Note, that condition (ii) is on the equivalence relation, and not on M; in our situation the equivalence relation (a) fulfills by construction condition (ii), but this might be not the most natural choice.

(3) In our situation (where a semi-universal family on X exists), it follows basically from abstract nonsense, that a coarse moduli space "is the same" as an orbit space for the action of G on X.

1.5 (Stability) We present a version of classical GIT, which was adapted by A. King [8] to the linear action of a reductive group on a *affine* space.

Let $G \times \mathbb{V} \longrightarrow \mathbb{V}, (g, v) \mapsto g \,.\, v$ be a linear representation of G, and denote by K its kernel. Moreover, let $\chi \colon G \longrightarrow \mathrm{k}^*$ be a character of G. We write

$$\mathrm{k}[\mathbb{V}]^{G,\chi} := \{f \in \mathrm{k}[\mathbb{V}] \mid f(g \,.\, v) = \chi(g) f(x) \ \forall g \in G, v \in \mathbb{V}\},$$

the space of relatively invariant functions of weight χ.

Definition (i) A point $v \in \mathbb{V}$ is χ-*semistable*, if there exists a relative invariant $f \in \mathrm{k}[\mathbb{V}]^{G,\chi^n}$ with $n \geq 1$, such that $f(x) \neq 0$. Write $\mathbb{V}^{\mathrm{ss}}$ for the open set of semistable point in $\mathbb{V}$.

(ii) A point $v \in \mathbb{V}$ is χ-*stable*, if there exists a relative invariant $f \in \mathrm{k}[\mathbb{V}]^{G,\chi^n}$ with $n \geq 1$, such that $f(x) \neq 0$, and, $\dim G \,.\, x = \dim G/K$ and the action on the affine variety $\mathbb{V}_f$ is separated. Write $\mathbb{V}^{\mathrm{s}}$ for the open set of stable points.

By a standard construction (compare for example [14, 3.14]) together with the fundamental properties of reductive groups 1.2, we obtain a categorical quotient

$$\phi \colon \mathbb{V}^{\mathrm{ss}} \longrightarrow \mathbb{V}/\!/(G, \chi) := \mathrm{Proj}(\bigoplus_{n \geq 0} \mathrm{k}[\mathbb{V}]^{G,\chi^n})$$

This contains an open set $S \subset \mathbb{V}/\!/(G, \chi)$ with $\phi^{-1}(S) = \mathbb{V}^{\mathrm{s}}$, and the restriction to this is a geometric quotient. Note moreover, that $\mathbb{V}/\!/(G, \chi)$ is by construction projective over the usual algebraic quotient $\mathbb{V}/\!/G = \mathrm{Spec}\, \mathrm{k}[\mathbb{V}]^G$.

Usually, it is difficult, to identify stable and semistable points. However, we have the following version of Mumford's "numerical criterion".

In this context, we need the definition of a *one-parameter subgroup* (1-PSG) of G, i.e. an morphism of algebraic groups $\lambda \colon \mathrm{k}^* \longrightarrow G$. Given such a group, we obtain for any $v \in \mathbb{V}$ a morphism $\lambda_v \colon \mathrm{k}^* \longrightarrow \mathbb{V}, t \mapsto \lambda(t) \,.\, v$. If λ_v can be extended to a morphism $\mathrm{k} \longrightarrow \mathbb{V}$, then this is unique, and we write $\lim_{t \to 0} \lambda(t) \,.\, v$ for the corresponding element of $\mathbb{V}$.

Note moreover, that the composition $\chi \circ \lambda$ is an automorphism of the multiplicative group k^*, i.e. it is necessarily of the form $t \mapsto t^n$ for some $n \in \mathbb{Z}$; we express this situation by writing $\langle \chi, \lambda \rangle := n$.

Proposition [8, 2.5] *A point $v \in \mathbb{V}$ is χ-semistable if and only if $\chi(K) = \{1\}$ and every 1-PSG λ, for which $\lim_{t \to 0} \lambda(t) \,.\, v$ exists, satisfies $\langle \chi, \lambda \rangle \geq 0$.*

Such a point is χ-stable if and only if the only 1-PSG subgroups λ of G, for which $\lim_{t \to 0} \lambda(t) \,.\, v$ exists, and $\langle \chi, \lambda \rangle = 0$, are in K.

Roughly speaking, the proof is based on the following construction: Consider the "lift" of the G-action on $\mathbb{V} \times \mathrm{k}$ by $g\,.\,(v,t) := (g\,.\,v, \chi(g)^{-1}t)$; now, by the result 1.2 on reductive groups, semistability of x is equivalent to saying that the closure of $G\,.\,(x,1)$ contains no point of $\mathbb{V} \times \{0\}$. This can be translated back into the original action by the 'fundamental theorem' [7, 1.4], that any closed G-invariant set, that meets the closure of an orbit contains a point in the closure of some 1-PSG orbit.

2 Basic Results

Define for $(A,B) \in \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m} =: \mathbb{A}^{(n,m)}$ the bloc-matrices $\tilde{R}(A,B) := (B, AB, A^2B, \cdots A^nB) \in \mathrm{k}^{n\times(n+1)m}$ and $R(A,B) := (B, AB, \ldots, A^{n-1}B) \in \mathrm{k}^{n\times nm}$. We will write for example $\tilde{R}(A,B)_i$ for the i-th column-vector of $\tilde{R}(A,B)$, and $\tilde{R}(A,B)^j$ for the j-th row-vector, and $\tilde{R}(A,B)_{i_1,\ldots,i_n}$ will be the submatrix of $\tilde{R}(A,B)$ formed from the columns $i_1,\ldots,i_n$ in that order. Finally, we denote by $\mathbf{E} \in \mathrm{Gl}_n$ the unit-matrix, and consequently $\mathbf{E}_i$ is the i-th unit-vector of k^n.

2.1 Remark: (1) If we consider $\mathrm{k}^{n\times(n+1)m}$ as a Gl_n-variety with the natural action by left-multiplication of matrices, then $\tilde{R} :: \mathrm{k}^{n\times n} \times \mathrm{k}^{n\times m} \longrightarrow \mathrm{k}^{n\times(n+1)m}$ is a Gl_n-morphism of varieties.

(2) If $j_1, j_j, \ldots j_n$ is a sequence in $\{1,\ldots,(n+1)m\}$, then we define

$$d_{j_1,\ldots,j_n}(A,B) := \det(\tilde{R}(A,B)_{j_1,\ldots,j_n})$$

and clearly $\det(g) d_{j_i,\ldots,j_n}(A,B) = d_{j_i,\ldots,j_n}(g\,.\,(A,B))$, i.e. we have found some relative invariants. We will see later, that we get in this way a system of generators for the algebra of relative invariants.

2.2 Lemma *Let $(A,B) \in \mathbb{A}^{(n,m)}$, then (A,B) is completely reachable iff* $\operatorname{rank} R(A,B) = n$ *iff* $\operatorname{rank} \tilde{R}(A,B) = n$

Proof: Consider k^n as $\mathrm{k}[x]$-module via multiplication by A; then (A,B) is completely reachable iff the submodule generated by the columns of B is k^n, but by the Cayley-Hamilton theorem this submodule is precisely the k-span of the columns of $R(A,B)$. □

2.3 Corollary *If $(A,B) \in \mathbb{A}^{(n,m)}$ is completely reachable, then the stabilizer $\mathrm{Stab}_{\mathrm{Gl}_n}(A,B)$ is (even scheme-theoretically) trivial. In particular, the orbit map $\mathrm{Gl}_n \longrightarrow \mathrm{Gl}_n\,.\,(A,B), g \mapsto g\,.\,(A,B)$ is separated.*

Proof: Note first, that for arbitrary $(A,B) \in \mathbb{A}^{(n,m)}$ the stabilizer is defined by *linear* equations, thus it is (scheme-theoretically) reduced. Thus, the differential of the orbit map is surjective. Now, let $g\,.\,(A,B) = (A,B)$, for (A,B) completely reachable, thus by the above remark $R(A,B) = R(g\,.\,(A,B)) = gR(A,B)$. By the lemma, $R(A,B)$ has rank n, implying $g = \mathbf{E}$. □

2.4 Remark: If (F,α,β) is a family of completely reachable systems on S, as discussed in the example of 1.3, then by the above lemma

$$(\beta, \alpha\beta, \dots, \alpha^{n-1}\beta) \colon s \times \mathrm{k}^{nm} \longrightarrow F)$$

is an epimorphism of vector-bundles. It follows, that the isomorphism class of F is already determined by the morphisms $\gamma_i \colon U_i \longrightarrow \mathbb{A}^{(n,m)}$; this implies that the equivalence relations (b) and (c), defined in 1.3, coincide. Since we will see later, that the open subset of $\mathbb{A}^{(n,m)}$ of completely reachable sytems is a Gl_n-principal bundle in the Zariski topology, even the equivalence relations (a) and (b) coincide.

2.5 Proposition *(a) The subset $\mathbb{A}^{(n,m)}_{\text{c.r.}}$ of completely reachable sytems in $\mathbb{A}^{(n,m)}$ is open in the Zariski topology and Gl_n-invariant.*

(b) The Gl_n-equivariant restriction to the open subsets

$$\tilde{R} \colon \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathrm{Mat}(n,(n+1)m)_{reg} := \{M \in \mathrm{k}^{n \times (n+1)m} \mid \operatorname{rank} M = n\}$$

is injective.

(c) The action of Gl_n on $\mathbb{A}^{(n,m)}_{\text{c.r.}}$ is separated.

Proof: (a) is clear. For (b) suppose $\tilde{R}(A,B) = \tilde{R}(A',B')$, then already $B = B'$ and $R(A,B) = R(A',B')$ has rank n. Now $\tilde{R}(A,B) = (B, AR(A,B))$ implies $A'R(A,B) = AR(A,B)$ and consequently $A = A'$.

c) It is well-known, that the action of Gl_n on $\mathrm{Mat}(n,(n+1)m)_{\text{reg}}$ admits as geometric quotient the Grassmaninan $\gamma \colon \mathrm{Mat}(n,(n+1)m)_{\text{reg}} \longrightarrow \mathrm{Gr}_n^{(n+1)m}$, thus $\psi := \gamma\tilde{R}$ is a Gl_n invariant morphism, with fibers the Gl_n-orbits on $\mathbb{A}^{(n,m)}_{\text{c.r.}}$. Thus, $\Gamma := \{(g\,.\,v, v) \in \mathbb{A}^{(n,m)}_{\text{c.r.}} \times \mathbb{A}^{(n,m)}_{\text{c.r.}} \mid g \in \mathrm{Gl}_n, v \in \mathbb{A}^{(n,m)}_{\text{c.r.}}\}$ coincides with $(\psi \times \psi)^{-1}(\Delta)$, where Δ is the diagonal in $\mathrm{Gr}_n^{(n+1)m} \times \mathrm{Gr}_n^{(n+1)m}$, thus it is closed. □

Remark: One would like to have the differential of $\tilde{R} : \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathrm{Mat}(n,(n+1)m)_{\text{reg}}$ be injective in order to get the image of $\tilde{R}$ locally closed by the étale version of the implicit function theorem. Unfortunately, this differential is quite complicated, due to the powers of A appearing in the definition of $\tilde{R}$.

3 Elementary Methods

Consider $X_{n,m} := \{1,\dots,m\} \times \{1,\dots,n\}$ with the usual lexicographical order. We say, that a sequence $I = ((i_1(1), i_2(1)), \dots, (i_1(n), i_2(n)))$ in $X_{n,m}$ is *nice* if it is strictly monotonous ascending, and if $i_2(j) > 1$, then $(i_1(j-1), i_2(j-1)) = (i_1(j), i_2(j) - 1)$.

Given a nice sequence I, we find two sequences $j_I(1), \dots, j_I(k)$ and $p_I(1), \dots, p_I(k)$ such that

$$I = ((j_I(1), 1), \dots, (j_I(1), p_I(1)), (j_I(2), 1), \dots (j_I(k), p_I(k))),$$

and we define

$$h_I(t) := \sum_{s=1}^{t} p_I(s); \quad h_I(0) := 0$$

For $\mathbf{i} := (i_1, i_2) \in X_{n,m}$, we define $R(A,B)_{\mathbf{i}} := R(A,B)_{i_1 + m(i_2 - 1)}$. Finally, for a sequence $I = (\mathbf{i}(1), \dots, \mathbf{i}(n))$ in $X_{n,m}$ we write $d_I(A,B) := d_{i_1(1)+m(i_2(1)-1), \dots, i_1(n)+m(i_2(n)-1)}(A,B)$, as defined in 2.1 (2), thus the principal open sets $(\mathbb{A}^{(n,m)})_{d_I} := \{(A,B) \in \mathbb{A}^{(n,m)} \mid d_I(A,B) \neq 0\}$ are Gl_n-invariant.

3.1 Lemma (Kalman) *$(A,B) \in \mathbb{A}^{(n,m)}$ is completely reachable if and only if there exists a <u>nice</u> sequence I in $X_{n,m}$, such that $d_I(A,B) \neq 0$. Thus $\mathbb{A}^{(n,m)}_{\text{c.r.}}$ admits an open covering by Gl_n-invariant affine sets*

$$\mathbb{A}^{(n,m)}_{\text{c.r.}} = \bigcup_{I \subset X_{n,m} nice} (\mathbb{A}^{(n,m)})_{d_I}$$

3.2 Lemma *(a) For (A,B) in the open set $(\mathbb{A}^{(n,m)})_{d_I}$ with I nice, there exists a unique $g \in \mathrm{Gl}_n(\mathrm{k})$, such that $g\,.\,(A,B) := (A',B')$ is of the form:*

$$\begin{aligned} A'_i &= \mathbf{E}_{i+1} && \text{if } i \notin \{h_I(1), h_I(2), \dots, h_I(k)\}, \\ B'_{j_I(i)} &= \mathbf{E}_{h_I(i-1)+1} && \text{for } i = 1, \dots, k \end{aligned}$$

In particular, $(\mathbb{A}^{(n,m)})_{d_I}$ is a trivial Gl_n bundle over $\mathrm{k}^{n \times m}$.

(b) The non-constant components of (A', B') can be expressed in terms of invariant functions:

$$A'^{i}_{h(t)} = \det \tilde{R}(A,B)_{\mathbf{i}(1),\dots,\mathbf{i}(i-1),(j_I(t)+mp_I(t)),\mathbf{i}(i+1),\dots,\mathbf{i}(n)}/d_I(A,B)$$
$$B'^{i}_{t} = \det \tilde{R}(A,B)_{\mathbf{i}(1),\dots,\mathbf{i}(i-1),t,\mathbf{i}(i+1),\dots,\mathbf{i}(n)}/d_I(A,B)$$

Proof: (a) Note first, that since I is nice, $(A', B') \in \mathbb{A}^{(n,m)}$ is of the form described in the Lemma if and only if $R(A', B')_I = \mathbf{E}$. In particular, all these matrices lie in $(\mathbb{A}^{(n,m)})_{d_I}$.

For $(A,B) \in (\mathbb{A}^{(n,m)})_{d_I}$ we have by definition $g(A,B) := \tilde{R}(A,B)_I \in \mathrm{Gl}_n(\mathrm{k})$, thus we have uniquely $g(A,B)^{-1}\tilde{R}(A,B)_I = \mathbf{E}$. Since in general $\tilde{R}(g\,.\,(A,B)) = g\tilde{R}(A,B)$, we conclude, that $g(A,B)^{-1}\,.\,(A,B)$ is of the desired form.

Next, note, that the space of matrices in that form, $(\mathbb{A}^{(n,m)})'_I$, is naturally isomorphic to $\mathrm{k}^{n\times m}$, and that

$$\phi\colon (\mathbb{A}^{(n,m)})_{d_I} \to \mathrm{Gl}_n \times (\mathbb{A}^{(n,m)})'_I,\ (A,B) \mapsto (g(A,B)^{-1}, g(A,B)^{-1}\,.\,(A,B))$$

is a Gl_n-morphism of varieties with an obvious inverse.

(b) The functions in question, being quotients of relative invariants of weight det are clearly Gl_n-invariant on $(\mathbb{A}^{(n,m)})_{d_I}$, thus we have to verify the equalities only on matrices in $(\mathbb{A}^{(n,m)})'_I$, where the result is immediate from the fact that $\tilde{R}(A', B')_{(j_I(t),p_I(t))} = A'_{h_I(t)}$ for $t = 1, \dots, k$. $\square$

3.3 Theorem *There exists a geometric quotient $\varphi\colon \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathcal{M}_{n,m}$ with an open covering $\mathcal{M}_{n,m} = \cup_{I nice} U_I$ such that $U_I \cong \mathrm{k}^{n\times m}$ and $\phi^{-1}(U_I) = (\mathbb{A}^{(n,m)})_{d_I}$. In particular, ϕ admits local sections and $\mathcal{M}_{n,m}$ is a smooth rational variety.*

Proof: Clearly, we can glue together the quotients for the $(\mathbb{A}^{(n,m)})_{d_I}$ to obtain a *pre*-variety $\mathcal{M}_{n,m}$ with the desired properties [5, II, exercise 2.12]. However, we have to make sure that this is a variety indeed [12], i.e. that the diagonal is closed. This follows easily from the fact, that the action of Gl_n on $\mathbb{A}^{(n,m)}_{\text{c.r.}}$ is closed 2.5, and the fact, that each of the local quotients admits a section [3, Lemma 5.5]. $\square$

3.4 Corollary *$\mathcal{M}_{n,m}$ is a fine moduli space for (algebraic) families of completely observable systems.*

Proof: By construction, the tautological family on $\mathbb{A}^{(n,m)}_{\text{c.r.}}$ has the local semi-universal property for families of completely observable systems. Thus, the geometric quotient $\mathcal{M}_{n,m}$ is a coarse moduli space for this problem. Since $\varphi\colon \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathcal{M}_{n,m}$ is a Gl_n-principal bundle in the Zariski topology (see construction in Theorem 3.3 and Lemma 3.2), we obtain from the local sections the required (1.4, remark (2)) universal family on $\mathcal{M}_{n,m}$. Finally, we need not worry about the equivalence relation by 2.4. $\square$

4 Geometric Invariant Theory

Let $\mathrm{Gl}_n \longrightarrow \mathrm{k}^*, g \mapsto \det(g)$ be the natural character of Gl_n. We want to interpret the completely reachable tuples $(A,B) \in \mathbb{A}^{(n,m)}$ as det-stable points, using a version of classical GIT, adapted by A. King [8] to the linear action of a reductive group G on a *affine* space $\mathbb{V}$, see 1.5.

4.1 Proposition *Let* $\det\colon \mathrm{Gl} \longrightarrow \mathrm{k}^*$ *be the "standard" character of* Gl_n. *With the action* $\mathrm{Gl}_n \times \mathbb{A}^{(n,m)} \longrightarrow \mathbb{A}^{(n,m)}, (g,(A,B)) \mapsto (gAg^{-1}, gB)$ *a point* $(A,B) \in \mathbb{A}^{(n,m)}$ *is completely reachable if and only if it is* det*-semistable if and only if it is* det*-stable.*

Proof: Recall, that (A,B) completely reachable implies that $\det R(A,B)_I \neq 0$ for some nice selection I of columns of $R(A,B)$ and that the stabilizer of (A,B) is trivial 2.3. Thus, (A,B) is stable (just use the definition), since $d_I : \mathbb{A}^{(n,m)} \longrightarrow \mathrm{k}, (A',B') \mapsto \det R(A',B')_I$ is a relative invariant of weight det. Finally, the action of Gl_n on $(\mathbb{A}^{(n,m)})_{d_I}$ is separated by 2.5 (c).

Thus, it remains to show that if $(A,B) \in \mathbb{A}^{(n,m)}$ is not completely reachable, then it is not semistable. In this situation, $r := \operatorname{rank} R(A,B) < n$, and we can suppose (up to the action of Gl_n), that

$$A^i_j = 0 \text{ if } i \leq n-r \text{ and } j > n-r$$
$$B^i_j = 0 \text{ if } i \leq n-r.$$

Now consider the 1-PSG $\lambda(t) := \operatorname{diag}(t^{-1}, t^{-1}, \ldots, t^{-1}, 1, \ldots, 1)$ of Gl_n ($n-r$-times t^{-1}), then clearly $\lim_{t\to 0} \lambda(t) \,.\, (A,B)$ exists, and $\langle \lambda, \det \rangle = r-n < 0$; thus (A,B) is not det-semistable by the numerical criterion . $\square$

Thus, we obtain with

$$\mathcal{M}_{n,m} := \operatorname{Proj}\Big(\bigoplus_{i\geq 0} \mathrm{k}[\mathbb{A}^{(n,m)}]^{\mathrm{Gl}_n, \det^i}\Big)$$

a geometric quotient $\phi : \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathcal{M}_{n,m}$. By remark (3) in 1.4 this is also a coarse moduli space for families of completely reachable systems. Moreover, we get directly from a Corollary of Luna's 'slice theorem', that this is a Gl_n-principal bundle in the étale topology if $\operatorname{char} \mathrm{k} = 0$ (see for example [18, §5, Korollar 1]). Since by 2.3 the orbit map $\mathrm{Gl}_n \longrightarrow \mathbb{A}^{(n,m)}_{\text{c.r.}}, g \mapsto g\,.(A,B)$ is separated for each $(A,B) \in \mathbb{A}^{(n,m)}_{\text{c.r.}}$, we get the same result also for $\operatorname{char} \mathrm{k} \neq 0$ by the characteristic-p version of the slice-theorem [2]. Finally, by an old result of Serre [17], Each Gl_n-principal bundle in the étale topology is also locally trivial in the Zariski topology (Warning: This is not true for PGl-bundles, e.g. the PGl_2-bundle of 2-dimensional simple $\mathrm{k}\langle x,y\rangle$-representations, see [3, Prp. 6.7] for more details). Thus we get without invoking the results from our explicit calculations in section 3:

4.2 Theorem $\phi\colon \mathbb{A}^{(n,m)}_{\text{c.r.}} \longrightarrow \mathcal{M}_{n,m}$ *is a* Gl_n *principal bundle in the Zariski topology, thus* $\mathcal{M}_{n,m}$ *is a fine moduli space for completely reachable systems of type* (n,m)*. Moreover,* $\mathcal{M}_{n,m}$ *is for* $m \geq 2$ *a stably rational quasi-projective variety which is not affine.*

Remark: For the open, affine Gl_n-invariant subsets $(\mathbb{A}^{(n,m)})_{d_I}$ there exists by 1.2 an affine geometric quotient $\psi\colon (\mathbb{A}^{(n,m)})_{d_I} \to (\mathbb{A}^{(n,m)})_{d_I} /\!/ \mathrm{Gl}_n$, which is by similar arguments as above, a Gl_n-principal bundle even in the Zariski topology. Clearly, $(\mathbb{A}^{(n,m)})_{d_I} /\!/ \mathrm{Gl}_n$ is a smooth variety, but we will need the explicit calculations from 3.2, to identify it with $\mathrm{k}^{n\times m}$ in case I is nice; compare with [19, p.52].

4.3 (Relative invariants) Let $R := \oplus_{i\geq 0} \mathrm{k}[\mathbb{A}^{(n,m)}]^{\mathrm{Gl}_n, \det^i}$ be the graded ring of relative invariants.

$R_0 = \mathrm{k}[s_1, \ldots, s_n]$ is a polynomial ring in n-variables, with the functions s_i given by $s_i(A,B) = s_i(A)$ and $\det(t\,.E - A) = t^n + s_1(A)t^{n-1} + \cdots + s_n(A)$ the usual invariant functions of $n \times n$-matrices under conjugation. Indeed, R_0 contains clearly this ring. On the the other hand, since Gl_n is reductive and $\mathbb{A}^{(n,m)}$ is affine, $\operatorname{Spec}(R_0)$ is the categorical quotient for the action of Gl_n on $\mathbb{A}^{(n,m)}$, thus $f(A,B) = f(A',B')$ for $f \in R_0$ and (A',B') in the Zariski closure of $\mathrm{Gl}_n\,.(A,B)$. In particular $(A,0) = \lim_{t\to 0} tE\,.(A,B)$, i.e. we are reduced to the well-known case $m = 0$, in particular $\operatorname{Spec} R_0$ is the affine n-space; see for example [14].

Next, we claim that R is generated as R_0 algebra by the elements d_I where $d_I(A,B) := \det R(A,B)_I$, with $I = (i_1, i_2, \ldots i_n)$ is a strictly monotonous sequence of integers in $\{1, 2, \ldots, nm\}$. In fact, as we already observed, $d_I(g\,.\,(A,B)) = \det(g)d_I(A,B)$, i.e. $d_I \in R_1$, thus we can consider the graded R_0-subalgebra of R, generated by these d_I. Note first, that by Cayley-Hamilton $\tilde{d}_I \in R'_1$, where $\tilde{d}_I(A,B) := \det \tilde{R}(A,B)_I$ for some $I = (i_1, \dots i_n)$ in $\{1, \dots, (n+1)m\}$. Thus for I nice, $(R'_{d_I})_0$ contains the invariant functions on $(\mathbb{A}^{(n,m)})_{d_I}$ constructed in Lemma 3.2 (b). Since we had obtained in this way a geometric quotient for $(\mathbb{A}^{(n,m)})_{d_I}$ we conclude

$$\mathrm{k}[(\mathbb{A}^{(n,m)})_{d_I}]^{\mathrm{Gl}_n} \subseteq (R'_{d_I})_0 \subseteq (R_{d_I})_0 = \mathrm{k}[(\mathbb{A}^{(n,m)})_{d_I}]^{\mathrm{Gl}_n}$$

i.e. we have equality everywhere. Since the $(d_I \mid I \text{ nice}) = R_+$ we conclude $R' = R$.

4.4 (The projection $\mathcal{M}_{n,m} \longrightarrow \mathbb{A}^n$) $\mathrm{k}[X_1, \dots, X_n] \cong R_0 \subset R$ induces the projection $\pi\colon \mathrm{Proj}(R) = \mathcal{M}_{n,m} \longrightarrow \mathrm{Spec}\, R_0 \cong \mathbb{A}^n$.

Lemma *The projection π is a flat morphism. Moreover, for general $a = (a_1, \dots, a_n) \in \mathbb{A}^n(\mathrm{k})$, we have $\pi^{-1}(a) \cong (\mathbb{P}^{(m-1)})^n$*

Proof: Since both varieties are smooth, we have to show for the flatness of π only $\dim \pi^{-1}(v) = n(m-1)$ for all $a \in \mathbb{A}^n$, see [5, III, Ex. 10.9] or [EGA IV 6.1.5]. Clearly, the dimension of the fiber is always at least $n(m-1)$, since π is projective, thus in particular closed and surjective. Now, we have

$$\pi^{-1}(a_1, \dots, a_n) = \varphi(\{(A,B) \in \mathbb{A}^{(n,m)}_{\text{c.r.}} \mid A^n + a_1 A^{n-1} + \dots + a_n \mathbf{E} = 0\}),$$

thus we can stratify $\pi^{-1}(a_1, \dots, a_n)$ along the possible Jordan normal forms $A_{(1)}, \dots, A_{(t)}$ for the given characteristic polynomial, i.e.

$$\pi^{-1}(a) = \dot{\bigcup_{i=1,\dots,t}} \{B \in \mathrm{k}^{n\times m} \mid \mathrm{rank}\, R(A_{(i)}, B) = n\} /\!/ \mathrm{Stab}_{\mathrm{Gl}_n}(A_{(i)})$$

via the corresponding associated fiber-bundles. Now the dimension of each of these stabilizers is at least n, acting on an open subset of $\mathrm{k}^{n\times m}$, and thus leading to a space of dimension at most $n(m-1)$.

Observe, that in the generic case, when the discriminant of $(a_1, \dots, a_n)$ does not vanish, we have only one normal form $A_{(1)} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with stabilizer $\mathrm{k}^* \times \dots \times \mathrm{k}^*$ and

$$\{B \in \mathrm{k}^{n\times m} \mid \mathrm{rank}\, R(A_{(1)}, B) = n\} = \{B \in \mathrm{k}^{n\times m} \mid B^i \neq 0 \text{ for } i = 1, \dots n\}$$

thus we obtain $\pi^{-1}(a) = (\mathbb{P}^{(m-1)})^n$ generically. $\square$

Let us recall finally, that the flatness of $\pi : \operatorname{Proj}(R) \longrightarrow \operatorname{Spec} R_0 \cong \mathbb{A}^n$ implies, that the Hilbert polynomial of $\pi^{(-1)}(a)$ for $a \in \mathbb{A}^n$ is constant for every closed embedding $\operatorname{Proj}(R) \subset \mathbb{P}^s_{\mathbb{A}^n}$, see [5, III 9.9].

4.5 (Representations of Quivers) In this situation, also mentioned in the introduction, the characters of $\mathrm{Gl}_{\mathbf{v}}$ are of the form

$$\chi_\theta \colon \mathrm{Gl}_{\mathbf{v}} \longrightarrow \mathrm{k}^*, g \mapsto \prod_{i \in Q_0} (\det g_i)^{\theta(i)}$$

for $\theta \in \mathbb{Z}^{Q_0}$. Since $\mathrm{Gl}_{\mathbf{v}}$ acts on $\mathrm{Rep}_Q^{\mathbf{v}}$ with kernel $\Delta = \{(\lambda e_i)_{i \in Q_0} \mid \lambda \in \mathrm{k}^*\}$, we can expect χ_θ-semistable points only for $\sum_{i \in Q_0} \theta(i) = 0$, see 1.5. By the work of Schofield [15], one finds even stable points if $\mathbf{v}$ is a Schur-root (i.e. if there exists a representation of dimension $\mathbf{v}$ with trivial endomorphism ring). The corresponding geometric quotient for χ_θ-stable points

$$\varphi \colon (\mathrm{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \longrightarrow \mathcal{M}(Q, \mathbf{v}, \chi)$$

is by the same argument as in the proof of 4.2 a $\mathrm{PGl}_{\mathbf{v}} := \mathrm{Gl}_{\mathbf{v}} / \Delta$-principal bundle in the étale topology. In order to find local sections for φ, (which would define a universal family on $\mathcal{M}(Q, \mathbf{v}, \chi)$), we should know, that this is locally trivial in the Zariski topology. This is the case if $\mathbf{v}$ is moreover indivisible, i.e. $\gcd(\mathbf{v}(i)_{i \in Q_0}) = 1$, since then the action of $\mathrm{Gl}_{\mathbf{v}}$ on $(\mathrm{Rep}_Q^{\mathbf{v}})^{\mathrm{s}}$ can be lifted to an action on $(\mathrm{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \times \mathrm{k}^*$ with trivial stabilizer, and we obtain now a $\mathrm{Gl}_{\mathbf{v}}$-principal bundle

$$\tilde{\varphi} \colon (\mathrm{Rep}_Q^{\mathbf{v}})^{\mathrm{s}} \times \mathrm{k}^* \longrightarrow \mathcal{M}(Q, \mathbf{v}, \chi)$$

by the result of Serre in [17]. This is our interpretation of [8, 5.3].

The notion of a family of representations of Q on a variety S is just a representation of Q in the category of vector-bundles on S, i.e. a family F of representations of dimension-vector E on S consists of a family of vector-bundles $(F_i)_{i \in Q_0}$ on S with $\operatorname{rank} F_i = E(i)$, and a family of morphism of vector-bundles $(f_\alpha : F_{s\alpha} \longrightarrow F_{t\alpha})_{\alpha \in Q_1}$. This can be obviously restricted to families of χ-stable representations. Unfortunately, the natural notion of equivalence for such families, induced by isomorphisms of vector-bundles, will never admit a fine moduli-space for the same reasons as in [14, 2.1]. Thus, we have to stick to a equivalence relation of type (b) as in 1.3, in order to obtain a fine moduli space in case of a indivisible Schur-root $\mathbf{v}$.

4.6 (Problems) (1) Are there besides the completely reachable sytems problem and the linear quiver discussed in [13], other quivers, where one can "guess" normal forms for the framed moduli? One will need a version of the Kalman-Lemma 3.1.

(2) Understand the special fibers of the projection π discussed above 4.4. The normal forms from 3.2 suggest, that the singularities appearing there, should be similar to the singularities found in [11].

(3) Understand the type of varieties, which can appear as fine moduli spaces for stable representations of dimension $\mathbf{v}$, where $\mathbf{v}$ is a indivisble Schur-root. From [16] it is known, that these spaces are allways rational, and in the case $\mathbf{v} = (1, 1, \dots, 1)$ it is known, that the moduli are toric varieites [6].

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