# AROUND RATIONALITY OF CYCLES 

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#### Abstract

In this article we prove certain results comparing rationality of algebraic cycles over the function field of a quadric and over the base field. Those results have already been proved by Alexander Vishik in the case of characteristic 0, which allowed him to work with algebraic cobordism theory. Our proofs use the modulo 2 Steenrod operations in the Chow theory and work in any characteristic $\neq 2$.


Keywords: Chow groups, quadrics, Steenrod operations.

In many situations it can be important to know, if an element of the modulo 2 Chow group (denoted as $C h$ ) of some variety considered over an algebraic closure $\bar{F}$ of its base field $F$, is actually defined over the base field itself.

Given a smooth quasi-projective $F$-variety $Y$, A. Vishik has proved a few years ago that if a cycle $y \in C h^{m}\left(Y_{\bar{F}}\right)$ is defined over the function field $F(Q)$ of a projective quadric $Q$ of dimension $n$, then $S^{j}(y)$ is defined over $F$ for all $j$ such that $j>m-n / 2$ (where $S^{j}$ is the $j$-th Steenrod operation of cohomological type, see [5, Theorem 3.1(1)]). This technical result, which A. Vishik called the Main Tool Lemma, plays the crucial role in his construction of fields with $u$-invariant $2^{r}+1$ (for $r \geq 3$ ), see [7]. A. Vishik's proof of the Main Tool Lemma uses symmetric operations in the algebraic cobordism theory constructed in [6], which requires to work with fields of characteristic 0 .

Recently, N. Karpenko has proved a weaker version of the Main Tool Lemma (see [4, Theorem 2.1]) saying that if a cycle $y \in C h^{m}\left(Y_{\bar{F}}\right)$, with $m<n / 2$, is defined over $F(Q)$, then $y$ is the sum of a rational element and the class modulo 2 of an exponent 2 element in the integral Chow group $C H^{m}\left(Y_{\bar{F}}\right)$ (this corresponds to the case $j=0$ of the previously mentioned A. Vishik's result). Since his proof only uses Steenrod operations on modulo 2 Chow groups, it works for all fields of characteristc $\neq 2$.

In the first part of this paper, we continue the work of N. Karpenko by showing that for any cycle $y \in C h^{m}\left(Y_{\bar{F}}\right)$ defined over $F(Q)$, the element $S^{j}(y)$ is the sum of a rational element and the class modulo 2 of an exponent 2 element in $C H^{m+j}\left(Y_{\bar{F}}\right)$ for all $j$ such that $j>m-n / 2$ (see Theorem 1.1). Since we use N. Karpenko's method, it works for all fields of characteristic $\neq 2$. Furthermore, the only use of the Steenrod operations allows one to get rid of the assumption of quasi-projectivity for $Y$.

In the second part of this paper, we prove some other technical results around rationality of cycles using the same methods (Proposition 2.1 and Theorem 2.4). Those results are weaker versions of some proved by A. Vishik in [5] (Proposition 3.3(2) and Theorem $3.1(2))$ over fields of characteristic 0 .

We refer to [5] and [4] for an introduction into the subject. Notation is introduced in the beginning of Section 1.

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## 1. Main Result

Let $F$ be a field of characteristic $\neq 2, Q$ a smooth projective quadric over $F$ of dimension $n>0$ and $Y$ a smooth $F$-variety (a variety is a separated scheme of finite type over a field).

We write $C H(Y)$ for the integral Chow group of $Y$ (see [2, Chapter X]) and we write $C h(Y)$ for $C H(Y)$ modulo 2. We write $\bar{Y}:=Y_{\bar{F}}$ where $\bar{F}$ is an algebraic closure of $F$. Let $X$ be a geometrically integral variety over $F$ and let us denote its function field as $F(X)$. An element $\bar{y}$ of $C h(\bar{Y})$ (or of $C H(\bar{Y})$ ) is $F(X)$-rational if its image $\bar{y}_{\bar{F}(X)}$ under $\operatorname{Ch}(\bar{Y}) \rightarrow C h\left(Y_{\bar{F}(X)}\right)$ (resp. $\left.C H(\bar{Y}) \rightarrow C H\left(Y_{\bar{F}(X)}\right)\right)$ is in the image of $C h\left(Y_{F(X)}\right) \rightarrow C h\left(Y_{\bar{F}(X)}\right)$ (resp. $\left.C H\left(Y_{F(X)}\right) \rightarrow C H\left(Y_{\bar{F}(X)}\right)\right)$. Finally, an element $\bar{y}$ of $C h(\bar{Y})$ (or of $C H(\bar{Y})$ ) is called rational if it is in the image of $C h(Y) \rightarrow C h(\bar{Y})$ (resp. $\mathrm{CH}(Y) \rightarrow \mathrm{CH}(\bar{Y}))$.

We refer to [2, Chapter XI] for an introduction to the Steenrod operations. We just recall here that for a smooth variety $Y$ over a field $F$ (of characteristic $\neq 2$ ) and for any integer $j$, there is a certain homomorphism $S^{j}: C h^{*}(Y) \rightarrow C h^{*+j}(Y)$ called the $j$-th Steenrod operation on $Y$ of cohomological type. Although the operation $S^{j}$ is constructed only for quasi-projective $Y$ in [2], Patrick Brosnan has extended the operation $S^{j}$ to any smooth variety $Y$ in $[1, \S 10]$.

Theorem 1.1. Assume that $m<n / 2+j$. Let $\bar{y}$ be an $F(Q)$-rational element of $C h^{m}(\bar{Y})$. Then $S^{j}(\bar{y})$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2.

Proof. We assume that $0 \leq j \leq m$ (otherwise we get $S^{j}(\bar{y})=0$, see [2, Theorem 61.13]). The element $\bar{y}$ being $F(Q)$-rational, there exists $y \in C h^{m}\left(Y_{F(Q)}\right)$ mapped to $\bar{y}_{\bar{F}(Q)}$ under the homomorphism

$$
C h^{m}\left(Y_{F(Q)}\right) \rightarrow C h^{m}\left(Y_{\bar{F}(Q)}\right) .
$$

Let us fix an element $x \in C h^{m}(Q \times Y)$ mapped to $y$ under the surjection (see [2, Corollary 57.11])

$$
C h^{m}(Q \times Y) \rightarrow C h^{m}\left(Y_{F(Q)}\right) .
$$

Since over $\bar{F}$ the variety $Q$ becomes completly split (i.e the Witt index $i_{0}(\bar{Q})$ has maximal value $[n / 2]+1$ ), the image $\bar{x} \in C h^{m}(\bar{Q} \times \bar{Y})$ of $x$ decomposes as (see [3, §1])

$$
\begin{equation*}
\bar{x}=h^{0} \times y^{m}+\cdots+h^{\left[\frac{n}{2}\right]} \times y^{m-\left[\frac{n}{2}\right]}+l_{\left[\frac{n}{2}\right]} \times z^{m+\left[\frac{n}{2}\right]-n}+\cdots+l_{\left[\frac{n}{2}\right]-j} \times z^{m+\left[\frac{n}{2}\right]-j-n} \tag{1}
\end{equation*}
$$

with some $y^{i} \in C h^{i}(\bar{Y})$ and some $z^{i} \in C h^{i}(\bar{Y})$, where $y^{m}=\bar{y}$, and where $h^{i} \in C h^{i}(\bar{Q})$ is the $i$ th power of the hyperplane section class while $l_{i} \in C h_{i}(\bar{Q})$ is the class of an $i$ dimensional subspace of $\mathbb{P}(W)$, where $W$ is a maximal totally isotropic subspace associated with the quadric $\bar{Q}$ (see [2, §68]).

For every $i=0, \ldots, m$, let $s^{i}$ be the image in $C H^{m+i}(\bar{Q} \times \bar{Y})$ of an element in $C H^{m+i}(Q \times$ $Y$ ) representing $S^{i}(x) \in C h^{m+i}(Q \times Y)$. We also set $s^{i}:=0$ for $i>m$.

The integer $n$ can be uniquely written in the form $n=2^{t}-1+s$, where $t$ is a nonnegative integer and $0 \leq s<2^{t}$. Let us denote $2^{t}-1$ as $d$. Since $d \leq n$, we can fix a
smooth subquadric $P$ of $Q$ of dimension $d$; we write in for the imbedding

$$
(P \hookrightarrow Q) \times i d_{Y}: P \times Y \hookrightarrow Q \times Y .
$$

Lemma 1.2. For any integer $r$, one has

$$
S^{r} p r_{*} i n^{*} x=\sum_{i=0}^{r} p r_{*}\left(c_{i}\left(-T_{P}\right) \cdot i n^{*} S^{r-i}(x)\right) \quad \text { in } C h^{r+m-d}(Y)
$$

(where $T_{P}$ is the tangent bundle of $P, c_{i}$ are the Chern classes, and $p r$ is the projection $P \times Y \rightarrow Y)$.

Proof. The morphism pr: P×Y $\quad$ Y is a smooth projective morphism between smooth schemes. Thus, for any integer $r$, we have by [2, Proposition 61.10],

$$
S^{r} \circ p r_{*}=\sum_{i=0}^{r} p r_{*}\left(c_{i}\left(-T_{p r}\right) \cdot S^{r-i}\right)
$$

where $T_{p r}$ is the relative tangent bundle of $p r$ over $P \times Y$. Furthermore, since $p r$ is the projection $P \times Y \rightarrow Y$, one has $T_{p r}=T_{P}$. Hence, we get

$$
S^{r} p r_{*} i n^{*} x=\sum_{i=0}^{r} p r_{*}\left(c_{i}\left(-T_{P}\right) \cdot S^{r-i}\left(i n^{*} x\right)\right)
$$

Finally, since in : P×Y $\hookrightarrow Q \times Y$ is a morphism between smooth schemes, the Steenrod operations of cohomological type commute with $i n^{*}$ (see [2, Theorem 61.9]), we are done.

We apply Lemma 1.2 taking $r=d+j$. Since $p r_{*} i n^{*} x \in C h^{m-d}(Y)$ and $m-d<d+j$ (indeed, $m-d<n / 2+j-d$ by assumption, and $n / 2<2 d$ thanks to our choice of $d$ ), we have $S^{d+j} p r_{*} i n^{*} x=0$.

Hence, we have by Lemma 1.2,

$$
\sum_{i=0}^{d+j} p r_{*}\left(c_{i}\left(-T_{P}\right) \cdot i n^{*} S^{d+j-i}(x)\right)=0 \quad \text { in } C h^{m+j}(Y)
$$

In addition, for any $i=0, \ldots, d$, by [2, Lemma 78.1] we have $c_{i}\left(-T_{P}\right)=\binom{d+i+1}{i} \cdot h^{i}$, where $h^{i} \in C h^{i}(P)$ is the $i$ th power of the hyperplane section class, and where the binomial coefficient is considered modulo 2 . Furthermore, for any $i=0, \ldots, d$, the binomial coefficient $\binom{d+i+1}{i}$ is odd (because $d$ is a power of 2 minus 1 , see [2, Lemma 78.6]). Moreover, for $i>d$, we have $c_{i}\left(-T_{P}\right)=0$ because $c_{i}\left(-T_{P}\right) \in C H^{i}(P)$ by definition of Chern classes and $C H^{i}(P)=0$ by dimensional reasons. Thus, we get

$$
\sum_{i=0}^{d} p r_{*}\left(h^{i} \cdot i n^{*} S^{d+j-i}(x)\right)=0 \quad \text { in } C h^{m+j}(Y)
$$

Therefore, the element

$$
\sum_{i=0}^{d} p r_{*}\left(h^{i} \cdot i n^{*} s^{d+j-i}\right) \in C H^{m+j}(\bar{Y})
$$

is twice a rational element.
Furthermore, for any $i=0, \ldots, d$, we have

$$
p r_{*}\left(h^{i} \cdot i n^{*} s^{d+j-i}\right)=p r_{*}\left(i n_{*}\left(h^{i} \cdot i n^{*} s^{d+j-i}\right)\right)
$$

(the first $p r$ is the projection $P \times Y \rightarrow Y$ while the second $p r$ is the projection $Q \times Y \rightarrow Y$ ). Since $i n$ is a proper morphism between smooth schemes, we have by the projection formula (see [2, Proposition 56.9]),

$$
i n_{*}\left(h^{i} \cdot i n^{*} s^{d+j-i}\right)=i n_{*}\left(h^{i}\right) \cdot s^{d+j-i}=h^{n-d+i} \cdot s^{d+j-i}
$$

and we finally get

$$
p r_{*}\left(h^{i} \cdot i n^{*} s^{d+j-i}\right)=p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) .
$$

Hence, we get that the element

$$
\sum_{i=0}^{d} p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) \in C H^{m+j}(\bar{Y})
$$

is twice a rational element.
We would like to compute the sum obtained modulo 4. Since $s^{d+j-i}=0$ if $d+j-i>m$, the $i$ th summand is 0 for any $i<d+j-m((j-m) \leq 0$ by assumption). Otherwise - if $i \geq d+j-m$ - the factor $h^{n-d+i}$ is divisible by 2 (indeed, we have $h^{n-d+i}=2 l_{d-i}$ because $n-d+i \geq n+j-m>n / 2$, see $[2, \S 68])$ and in order to compute the $i$ th summand modulo 4 it suffices to compute $s^{d+j-i}$ modulo 2 , that is, to compute $S^{d+j-i}(\bar{x})$.

According to the decomposition (1), we have

$$
S^{d+j-i}(\bar{x})=\sum_{k=0}^{\left[\frac{n}{2}\right]} S^{d+j-i}\left(h^{k} \times y^{m-k}\right)+\sum_{k=0}^{j} S^{d+j-i}\left(l_{\left[\frac{n}{2}\right]-k} \times z^{m+\left[\frac{n}{2}\right]-k-n}\right)
$$

And we set

$$
A_{i}:=\sum_{k=0}^{\left[\frac{n}{2}\right]} S^{d+j-i}\left(h^{k} \times y^{m-k}\right) \text { and } B_{i}:=\sum_{k=0}^{j} S^{d+j-i}\left(l_{\left[\frac{n}{2}\right]-k} \times z^{m+\left[\frac{n}{2}\right]-k-n}\right)
$$

For any $k=0, \ldots,\left[\frac{n}{2}\right]$, we have by [2, Theorem 61.14],

$$
S^{d+j-i}\left(h^{k} \times y^{m-k}\right)=\sum_{l=0}^{d+j-i} S^{d+j-i-l}\left(h^{k}\right) \times S^{l}\left(y^{m-k}\right)
$$

Moreover, for any $l=0, \ldots, d+j-i$, we have by [1, Corollary 78.5],

$$
S^{d+j-i-l}\left(h^{k}\right)=\binom{k}{d+j-i-l} h^{d+j+k-i-l} .
$$

Thus, choosing an integral representative $\varepsilon_{k, l} \in C H^{m-k+l}(\bar{Y})$ of $S^{l}\left(y^{m-k}\right)$ (we choose $\varepsilon_{k, l}=0$ if $\left.l>m-k\right)$, we get that the element

$$
\sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(h^{d+j+k-i-l} \times \varepsilon_{k, l}\right) \in C H^{m+d+j-i}(\bar{Q} \times \bar{Y})
$$

is an integral representative of $A_{i}$.
Therefore, for any $i \geq d+j-m$, choosing an integral representative $\tilde{B}_{i}$ of $B_{i}$, there exists $\gamma_{i} \in C H^{m+d+j-i}(\overline{\bar{Q}} \times \bar{Y})$ such that

$$
s^{d+j-i}=\sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(h^{d+j+k-i-l} \times \varepsilon_{k, l}\right)+\tilde{B}_{i}+2 \gamma_{i} .
$$

Hence, according to the multiplication rules in the ring $C H(\bar{Q})$ described in [2, Proposition 68.1], for any $i \geq d+j-m$, we have

$$
h^{n-d+i} \cdot s^{d+j-i}=2 \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(l_{l-j-k} \times \varepsilon_{k, l}\right)+h^{n-d+i} \cdot \tilde{B}_{i}+4 l_{d-i} \cdot \gamma_{i} .
$$

If $k \leq d-i$, one has $j+k \leq d+j-i$, and for any $0 \leq l \leq d+j-i$, we have by dimensional reasons,

$$
p r_{*}\left(l_{l-j-k} \times \varepsilon_{k, l}\right)=\left\{\begin{array}{cl}
\varepsilon_{k, l} & \text { if } l=j+k \\
0 & \text { otherwise } .
\end{array}\right.
$$

Otherwise $k>d-i$, and $p r_{*}\left(l_{l-j-k} \times \varepsilon_{k, l}\right)=0$ for any $0 \leq l \leq d+j-i$. Moreover, for $k>d-i$, one has $j+k>j+d-i \geq m>m-k$, therefore $\varepsilon_{k, j+k}=0$.

Thus we deduce the identity

$$
p r_{*}\left(2 \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(l_{l-j-k} \times \varepsilon_{k, l}\right)\right)=2 \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k} .
$$

Then,

$$
\left.\begin{array}{rl}
\sum_{i=d+j-m}^{d} p r_{*}\left(2 \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(l_{l-j-k} \times \varepsilon_{k, l}\right)\right.
\end{array}\right) \quad \begin{aligned}
& =2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k}
\end{aligned}
$$

In the latest expression, for every $k=0, \ldots,\left[\frac{n}{2}\right]$, the total coefficient at $\varepsilon_{k, j+k}$ is

$$
2 \sum_{i=d+j-m}^{d}\binom{k}{d-i-k}=2 \sum_{i=d-2 k}^{d-k}\binom{k}{d-i-k}=2 \sum_{s=0}^{k}\binom{k}{s}=2^{k+1}
$$

which is divisible by 4 for $k \geq 1$.
Therefore, the cycle $\sum_{i=d+j-m}^{d} p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) \in C H^{m+j}(\bar{Y})$ is congruent modulo 4 to

$$
2 \varepsilon_{0, j}+\sum_{i=d+j-m}^{d} p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)
$$

Thus, the cycle $2 \varepsilon_{0, j}+\sum_{i=d+j-m}^{d} p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)$ is congruent modulo 4 to twice a rational element.

Finally, the following lemma will lead to the conclusion.
Lemma 1.3. For any $d+j-m \leq i \leq d$, one can choose an integral representative $\tilde{B}_{i}$ of $B_{i}$ so that

$$
p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)=0
$$

Proof. We recall that $B_{i}:=\sum_{k=0}^{j} S^{d+j-i}\left(l_{\left[\frac{n}{2}\right]-k} \times z^{m+\left[\frac{n}{2}\right]-k-n}\right)$. For any $k=0, \ldots, j$, we have by [2, Theorem 61.14],

$$
S^{d+j-i}\left(l_{\left[\frac{n}{2}\right]-k} \times z^{m+\left[\frac{n}{2}\right]-k-n}\right)=\sum_{l=0}^{d+j-i} S^{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k}\right) \times S^{l}\left(z^{m+\left[\frac{n}{2}\right]-k-n}\right)
$$

And for any $l=0, \ldots, d+j-i$, we have by [2, Corollary 78.5],

$$
S^{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k}\right)=\binom{n+1-\left[\frac{n}{2}\right]+k}{d+j-i-l} l_{\left[\frac{n}{2}\right]-k-d-j+i+l}
$$

Thus, choosing an integral representative $\delta_{k, l} \in C H^{m-k+l}(\bar{Y})$ of $S^{l}\left(z^{m+\left[\frac{n}{2}\right]-k-n}\right)$ (we choose $\delta_{k, l}=0$ if $l>m+\left[\frac{n}{2}\right]-k-n$ ), we get that the element

$$
\sum_{k=0}^{j} \sum_{l=0}^{d+j-i}\binom{n+1-\left[\frac{n}{2}\right]+k}{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k-d-j+i+l} \times \delta_{k, l}\right) \in C H^{m+d+j-i}(\bar{Q} \times \bar{Y})
$$

is an integral representative of $B_{i}$. Let us denote it $\tilde{B}_{i}$.
Hence, we have

$$
h^{n-d+i} \cdot \tilde{B}_{i}=\sum_{k=0}^{j} \sum_{l=0}^{d+j-i}\binom{n+1-\left[\frac{n}{2}\right]+k}{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right) .
$$

Moreover, we have

$$
p r_{*}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right) \neq 0 \Longrightarrow l=j+k+n-\left[\frac{n}{2}\right] .
$$

Furthermore, for any $0 \leq k \leq j$, we have $d+j-i \leq m<j+\frac{n}{2} \leq j+n-\left[\frac{n}{2}\right] \leq$ $j+k+n-\left[\frac{n}{2}\right]$. Thus, for any $0 \leq l \leq d+j-i$ and for any $0 \leq k \leq j$, we have $p r_{*}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right)=0$. It follows that $p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)=0$ and we are done.

We deduce from Lemma 1.3 that the cycle $2 \varepsilon_{0, j} \in C H^{m+j}(\bar{Y})$ is congruent modulo 4 to twice a rational cycle. Therefore, there exist a cycle $\gamma \in C H^{m+j}(\bar{Y})$ and a rational cycle $\alpha \in C H^{m+j}(\bar{Y})$ so that

$$
2 \varepsilon_{0, j}=2 \alpha+4 \gamma,
$$

hence, there exists an exponent 2 element $\delta \in C H^{m+j}(\bar{Y})$ so that

$$
\varepsilon_{0, j}=\alpha+2 \gamma+\delta .
$$

Finally, since $\varepsilon_{0, j}$ is an integral representative of $S^{j}(\bar{y})$, we get that $S^{j}(\bar{y})$ is the sum of a rational element and the class modulo 2 of an integral element of exponent 2. We are done with the proof of Theorem 1.1.

## 2. Other results

In this section we continue to use notation introduced in the beginning of Section 1.
Proposition 2.1. Let $x \in C h^{m}(Q \times Y)$ be some element, and $y^{i}, z^{i} \in C h^{i}(\bar{Y})$ be the coordinates of $\bar{x}$ as in the beginning of proof of Theorem 1.1. Assume that $m=\left[\frac{n+1}{2}\right]+j$. Then $S^{j}\left(y^{m}\right)+y^{m} \cdot z^{j}$ differs from a rational element by the class of an exponent 2 element of $C H^{m+j}(\bar{Y})$.

Proof. We assume that $0 \leq j \leq m$. The image $\bar{x} \in C h^{m}(\bar{Q} \times \bar{Y})$ of $x$ decomposes as in (1). Let $\mathbf{x} \in C H^{m}(Q \times Y)$ be an integral representative of $x$. The image $\overline{\mathbf{x}} \in C H^{m}(\bar{Q} \times \bar{Y})$ decomposes as

$$
\overline{\mathbf{x}}=h^{0} \times \mathbf{y}^{m}+\cdots+h^{\left[\frac{n}{2}\right]} \times \mathbf{y}^{m-\left[\frac{n}{2}\right]}+l_{\left[\frac{n}{2}\right]} \times \mathbf{z}^{m+\left[\frac{n}{2}\right]-n}+\cdots+l_{\left[\frac{n}{2}\right]-j-1} \times \mathbf{z}^{m+\left[\frac{n}{2}\right]-j-n}
$$

where the elements $\mathbf{y}^{i} \in C H^{i}(\bar{Y})$ (resp. $\mathbf{z}^{i} \in C H^{i}(\bar{Y})$ ) are some integral representatives of the elements $y^{i}$ (resp. $z^{i}$ ) appearing in (1).

For every $i=0, \ldots, m-1$, let $s^{i}$ be the image in $C H^{m+i}(\bar{Q} \times \bar{Y})$ of an element in $C H^{m+i}(Q \times Y)$ representing $S^{i}(x) \in C h^{m+i}(Q \times Y)$. We also set $s^{i}:=0$ for $i>m$. Finally, we set $s^{0}:=\overline{\mathbf{x}}$ and $s^{m}:=\left(s^{0}\right)^{2}$ (because $S^{m}(x)=x^{2}$, see [2, Theorem 61.13]). Therefore, for any nonnegative integer $i, s^{i}$ is the image in $C H^{m+i}(\bar{Q} \times \bar{Y})$ of an integral representative of $S^{i}(x)$.

The integer $n$ can be uniquely written in the form $n=2^{t}-1+s$, where $t$ is a nonnegative integer and $0 \leq s<2^{t}$. Let us denote $2^{t}-1$ as $d$.

We would like to use again Lemma 1.2 to get that the sum

$$
\begin{equation*}
\sum_{i=d+j-m}^{d} p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) \in C H^{m+j}(\bar{Y}) \tag{2}
\end{equation*}
$$

is twice a rational element. To do this, it suffices to check that $m-d<d+j$. Then the same reasoning as the one used during the proof of Theorem 1.1 gives us the desired result.

We have $m-d=\left[\frac{n+1}{2}\right]+j-d=d+j+\left(\left[\frac{n+1}{2}\right]-2 d\right)$, and since our choice of $d$ and the assumption $n>0$, one can easily check that $2 d>\left[\frac{n+1}{2}\right]$. Thus we do get that the sum (2) is twice a rational element. We would like to compute that sum modulo 4.

For any $i \geq d+j-m$, the factor $s^{d+j-i}$ present in the $i$ th summand is congruent modulo 2 to $S^{d+j-i}(\bar{x})$, which is represented by $\tilde{A}_{i}+\tilde{B}_{i}$, where

$$
\tilde{A}_{i}:=\sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{d+j-i}\binom{k}{d+j-i-l}\left(h^{d+j+k-i-l} \times \varepsilon_{k, l}\right)
$$

and

$$
\tilde{B}_{i}:=\sum_{k=0}^{j} \sum_{l=0}^{d+j-i}\binom{n+1-\left[\frac{n}{2}\right]+k}{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k-d-j+i+l} \times \delta_{k, l}\right)
$$

where $\varepsilon_{k, l} \in C H^{m-k+l}(\bar{Y})$ (resp. $\delta_{k, l} \in C H^{m-k+l}(\bar{Y})$ ) is an integral representative of $S^{l}\left(y^{m-k}\right)$ (resp. of $S^{l}\left(z^{m+\left[\frac{n}{2}\right]-k-n}\right)$ ), and we choose $\varepsilon_{k, l}=0$ if $l>m-k$ (resp. $\delta_{k, l}=0$ if $\left.l>m+\left[\frac{n}{2}\right]-k-n\right)$. Finally, in the case of even $m-j$, we choose $\varepsilon_{\frac{m-j}{2}, \frac{m+j}{2}}=\left(\mathbf{y}^{\frac{m+j}{2}}\right)^{2}$.

Furthermore, for any $i \geq d+j-m$, we have

$$
h^{n-d+i} \cdot \tilde{B}_{i}=\sum_{k=0}^{j} \sum_{l=0}^{d+j-i}\binom{n+1-\left[\frac{n}{2}\right]+k}{d+j-i-l}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right) .
$$

And we have

$$
p r_{*}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right) \neq 0 \Longrightarrow l=j+k+n-\left[\frac{n}{2}\right] .
$$

On the one hand, for any $i>d+j-m$, we have $d+j-i<m=n-\left[\frac{n}{2}\right]+j \leq j+k+n-\left[\frac{n}{2}\right]$. Hence, for any $0 \leq l \leq d+j-i$ and for any $0 \leq k \leq j$, we have $p r_{*}\left(l_{\left[\frac{n}{2}\right]-k-n-j+l} \times \delta_{k, l}\right)=0$. Then, for any $i>d+j-m$, we get that $p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)=0$.

On the other hand, for $i=d+j-m$, we have $d+j-i=j+n-[n / 2]$ and

$$
l=j+k+n-\left[\frac{n}{2}\right] \Longleftrightarrow k=0 \text { and } l=d+j-i .
$$

Thus, we have

$$
p r_{*}\left(h^{n+j-m} \cdot \tilde{B}_{d+j-m}\right)=\delta_{0, m} .
$$

Since $m>m+[n / 2]-n$, we get that $\delta_{0, m}=0$.
Therefore, for any $i \geq d+j-m$, we have

$$
p r_{*}\left(h^{n-d+i} \cdot \tilde{B}_{i}\right)=0 .
$$

Then, for any $i>d+j-m$, the cycle $h^{n-d+i}$ is divisible by 2 . Hence, according to the multiplication rules in the ring $C H(\bar{Q})$ described in [2, Proposition 68.1] and by doing the same computations as those done during the proof of Theorem 1.1, for any $i>d+j-m$, we get the congruence

$$
p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) \equiv 2 \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k}(\bmod 4) .
$$

Moreover, since $d-i-k \leq k$ if and only if $k \leq\left[\frac{m-j}{2}\right]$, for any $i>d+j-m$, we have the congruence

$$
\begin{equation*}
p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right) \equiv 2 \sum_{k=0}^{\left[\frac{m-j}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k}(\bmod 4) . \tag{3}
\end{equation*}
$$

Now, we would like to study the $(d+j-m)$ th summand, that is to say the cycle $p r_{*}\left(h^{n+j-m} \cdot s^{m}\right)$ modulo 4 . That is the purpose of the following lemma.

Lemma 2.2. One has

$$
p r_{*}\left(h^{n+j-m} \cdot s^{m}\right) \equiv\left\{\begin{array}{cl}
2 \varepsilon_{\frac{m-j}{2}, \frac{m+j}{2}+2 \boldsymbol{y}^{m} \cdot \boldsymbol{z}^{j}(\bmod 4)}^{2 \boldsymbol{y}^{m} \cdot \boldsymbol{z}^{j}(\bmod 4)} & \text { if } m-j \text { is even } \\
\text { if } m-j \text { is odd. } .
\end{array}\right.
$$

Proof. We recall that $s^{m}=(\overline{\mathbf{x}})^{2}$. Thus, we have

$$
h^{n+j-m} \cdot s^{m}=h^{n+j-m} \cdot(A+B+C)
$$

where

$$
\begin{gathered}
A:=\sum_{0 \leq i, l \leq\left[\frac{n}{2}\right]} h^{i+l} \times\left(\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}\right), \\
B:=\sum_{0 \leq i, l \leq j}\left(l_{\left[\frac{n}{2}\right]-i} \cdot l_{\left[\frac{n}{2}\right]-l}\right) \times\left(\mathbf{z}^{j-i} \cdot \mathbf{z}^{j-l}\right)
\end{gathered}
$$

and

$$
C:=2 \sum_{i=0}^{\left[\frac{n}{2}\right]} h^{i} \times \mathbf{y}^{m-i} \cdot \sum_{l=0}^{j} l_{\left[\frac{n}{2}\right]-l} \times \mathbf{z}^{j-l} .
$$

First of all, we have

$$
h^{n+j-m} \cdot A=\sum_{0 \leq i, l \leq\left\lfloor\frac{n}{2}\right]} h^{n+j-m+i+l} \times\left(\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}\right)
$$

Now we have $m=\left[\frac{n+1}{2}\right]+j$, so $n+j-m+i+l=\left[\frac{n}{2}\right]+i+l$. Thus, if $i \geq 1$ or $l \geq 1$, we have $n+j-m+i+l>\left[\frac{n}{2}\right]$, and in this case we have $h^{n+j-m+i+l}=2 l_{m-i-l-j}$. Therefore, the cycle $h^{n+j-m} \cdot A$ is equal to

$$
h^{n+j-m} \times\left(\mathbf{y}^{m}\right)^{2}+4 \sum_{1 \leq i, l \leq\left[\frac{n}{2}\right]} l_{m-i-l-j} \times\left(\mathbf{y}^{m-i} \cdot \mathbf{y}^{m-l}\right)+2 \sum_{i=1}^{\left[\frac{n}{2}\right]} l_{m-j-2 i} \times\left(\mathbf{y}^{m-i}\right)^{2}
$$

Then, since $n \geq 1$, we have $n+j-m \neq n$. It follows that $p r_{*}\left(h^{n+j-m} \times\left(\mathbf{y}^{m}\right)^{2}\right)=0$.
Furthermore, we have

$$
p r_{*}\left(\sum_{i=1}^{\left[\frac{n}{2}\right]} l_{m-j-2 i} \times\left(\mathbf{y}^{m-i}\right)^{2}\right)=\left\{\begin{array}{cl}
\left(\mathbf{y}^{\frac{m+j}{2}}\right)^{2} & \text { if } m-j \text { is even } \\
0 & \text { if } m-j \text { is odd. }
\end{array}\right.
$$

Therefore, $p r_{*}\left(h^{n+j-m} \cdot A\right)$ is congruent modulo 4 to $2 \varepsilon_{\frac{m-j}{2}, \frac{m+j}{2}}$ if $m-j$ is even, and to 0 if $m-j$ is odd.

Then, by dimensional reasons, we have $l_{\left[\frac{n}{2}\right]-i} \cdot l_{\left[\frac{n}{2}\right]-l}=0$ if $i \geq 1$ or if $l \geq 1$. Hence, we have $B=\left(l_{\left[\frac{n}{2}\right]} \cdot l_{\left[\frac{n}{2}\right]}\right) \times\left(\mathbf{z}^{j}\right)^{2}$. It follows that

$$
h^{n+j-m} \cdot B=\left(l_{0} \cdot l_{\left[\frac{n}{2}\right]}\right) \times\left(\mathbf{z}^{j}\right)^{2}
$$

and $l_{0} \cdot l_{\left[\frac{n}{2}\right]}=0$ by dimensional reasons. Therefore, we get that $h^{n+j-m} \cdot B=0$.
Finally, we have

$$
h^{n+j-m} \cdot C=2 \sum_{i=0}^{\left[\frac{n}{2}\right]} h^{n+j-m+i} \times \mathbf{y}^{m-i} \cdot \sum_{l=0}^{j} l_{\left[\frac{n}{2}\right]-l} \times \mathbf{z}^{j-l} .
$$

Now for any $i \geq 1$, we have $n+j-m+i>\left[\frac{n}{2}\right]$, and in this case the cycle $h^{n+j-m+i}$ is divisible by 2 . Thus, the element $h^{n+j-m} \cdot C$ is congruent modulo 4 to

$$
2 \sum_{l=0}^{j}\left(h^{\left[\frac{n}{2}\right]} \cdot l_{\left[\frac{n}{2}\right]-l}\right) \times\left(\mathbf{y}^{m} \cdot \mathbf{z}^{j-l}\right)
$$

and, by dimensional reasons, in the latest sum, each summand is 0 except the one corresponding to $l=0$. Therefore, the cycle $h^{n+j-m} \cdot C$ is congruent modulo 4 to $2 l_{0} \times\left(\mathbf{y}^{m} \cdot \mathbf{z}^{j}\right)$. It follows that $p r_{*}\left(h^{n+j-m} \cdot C\right)$ is congruent modulo 4 to $2 \mathbf{y}^{m} \cdot \mathbf{z}^{j}$. We are done.

By the congruence (3) and Lemma 2.2, we deduce that the cycle

$$
\sum_{i=d+j-m}^{d} p r_{*}\left(h^{n-d+i} \cdot s^{d+j-i}\right)
$$

is congruent modulo 4 to

$$
2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{\left[\frac{m-j}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k}+2 \mathbf{y}^{m} \cdot \mathbf{z}^{j}
$$

It follows that the cycle

$$
2 \sum_{i=d+j-m}^{d} \sum_{k=0}^{\left[\frac{m-j}{2}\right]}\binom{k}{d-i-k} \varepsilon_{k, j+k}+2 \mathbf{y}^{m} \cdot \mathbf{z}^{j}
$$

is congruent modulo 4 to twice a rational element $\alpha \in C H^{m+j}(\bar{Y})$. Then, we finish as in the proof of Theorem 1.1. For every $k=0, \ldots,[(m-j) / 2]$, the total coefficient at $\varepsilon_{k, j+k}$ is $2^{k+1}$, which is divisible by 4 for $k \geq 1$. Therefore, there exists a cycle $\gamma \in C H^{m+j}(\bar{Y})$ such that

$$
2 \varepsilon_{0, j}+2 \mathbf{y}^{m} \cdot \mathbf{z}^{j}=2 \alpha+4 \gamma,
$$

hence, there exists an exponent 2 element $\delta \in C H^{m+j}(\bar{Y})$ so that

$$
\varepsilon_{0, j}+\mathbf{y}^{m} \cdot \mathbf{z}^{j}=\alpha+2 \gamma+\delta
$$

Finally, since $\varepsilon_{0, j}$ is an integral representative of $S^{j}\left(y^{m}\right)$ and $\mathbf{y}^{m}$ (resp. $\mathbf{z}^{j}$ ) is an integral representative of $y^{m}$ (resp. of $z^{j}$ ), we get that $S^{j}\left(y^{m}\right)+y^{m} \cdot z^{j}$ differs from a rational element by the class of an exponent 2 element of $\mathrm{CH}^{m+j}(\bar{Y})$. We are done with the proof of Proposition 2.1.

Remark 2.3. In the case of $j=0$, and if we make the extra assumption that the image of $x$ under the composition

$$
C h^{m}(Q \times Y) \rightarrow C h^{m}\left(Q_{F(Y)}\right) \rightarrow C h^{m}\left(Q_{\bar{F}(Y)}\right) \rightarrow C h^{m}(\bar{Q})
$$

(the last passage is given by the inverse of the change of field isomorphism) is rational, then we get the stronger result that the cycle $y^{m}$ differs from a rational element by the class of an exponent 2 element of $C H^{m}(\bar{Y})$. That is the subject of [4, Proposition 4.1].

Finally, the following theorem is a consequence of Proposition 2.1.

Theorem 2.4. Assume that $m=\left[\frac{n+1}{2}\right]+j$. Let $\bar{y}$ be an $F(Q)$-rational element of $C h^{m}(\bar{Y})$. Then there exists a rational element $z \in C h^{j}(\bar{Y})$ such that $S^{j}(\bar{y})+\bar{y} \cdot z$ is the sum of $a$ rational element and the class modulo 2 of an integral element of exponent 2.
Proof. The element $\bar{y}$ being $F(Q)$-rational, there exists $x \in C h^{m}(Q \times Y)$ mapped to $\bar{y}_{\bar{F}(Q)}$ under the composition

$$
C h^{m}(Q \times Y) \rightarrow C h^{m}\left(Y_{F(Q)}\right) \rightarrow C h^{m}\left(Y_{\bar{F}(Q)}\right) .
$$

Moreover, the image $\bar{x} \in C h^{m}(\bar{Q} \times \bar{Y})$ of $x$ decomposes as in (1). Thus, by Proposition 2.1, the cycle $S^{j}(\bar{y})+\bar{y} \cdot z^{j}$ is the sum of a rational element and the class of an element of exponent 2 .

Finally, we have by [2, Proposition 49.20],

$$
\overline{(p r)_{*}\left(x \cdot h^{\left[\frac{n}{2}\right]}\right)}=p r_{*}\left(\bar{x} \cdot h^{\left[\frac{n}{2}\right]}\right)=z^{j} .
$$

We are done with the proof of Theorem 2.4.

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