NON-COMPACT FORM OF THE ELEMENTARY DISCRETE INVARIANT

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ABSTRACT. We determine the non-compact form of Vishik's Elementary Discrete Invariant for quadrics.

Keywords: Chow groups, quadratic forms, grassmannians.

CONTENTS

1. INTRODUCTION

Let X be a smooth projective quadric of dimension n over a field F associated with a non-degenerate F-quadratic form q. The *splitting pattern* of X is a discrete invariant which measures what are the possible Witt indices of q_E over all field extensions E/F (see [3] and [4]). On the other hand, the *motivic decomposition type* of X is a discrete invariant which measures in what pieces the Chow motive of X can be decomposed. Moreover, Alexander Vishik noticed in [5] that the study of the interaction between these two invariants provides further information about both of them.

For this reason, he introduced the *Generic Discrete Invariant* of quadrics, a bigger discrete invariant containing the splitting pattern and the motivic decomposition type invariants as faces, see [6] and [8]. The Generic Discrete Invariant $GDI(X)$ is defined as follows. Let K/F be a splitting field extension of q. Let us denote $[n/2]$ as d. For any $i \in \{0, \ldots, d\}$, we write G_i for the grassmannian of i-dimensional totally q-isotropic subspaces (in particular G_0 is the quadric X). Then $GDI(X)$ is the collection of the subalgebras of rational elements

$$\overline{\mathrm{Ch}}^*(G_i) := \mathrm{Image}\left(\mathrm{Ch}^*(G_i) \to \mathrm{Ch}^*(G_{iK})\right)$$

for $i \in \{0, \ldots, d\}$, where Ch stands for the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients (an algebraic cycle already defined at the level of the base field F is called *rational*).

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In his paper [9] dedicated to the Kaplansky's conjecture on the u-invariant of a field, A. Vishik used the *Elementary Discrete Invariant* of quadrics, a handier invariant than the GDI as it only deals with some particular cycles in $\mathrm{Ch}^*(G_{iK})$. More precisely, for any $i \in \{0, \ldots, d\}$, we denote by $\mathcal{F}(0,i)$ the partial orthogonal flag variety of q-isotropic lines contained in i-dimensional totally q-isotropic subspaces. One can consider the diagram

$$X \xleftarrow[\pi_{(\underline{0},i)}]{} \mathcal{F}(0,i) \xrightarrow[\pi_{(0,\underline{i})}]{} G_i,$$

given by the natural projections and, for $0 \leq j \leq d$, we set

$$Z^i_{n-i-j} := \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_j) \in \mathrm{CH}^{n-i-j}(G_{iK}),$$

where CH stands for the Chow ring with $\mathbb{Z}$-coefficients and l_j is the class in $\mathrm{CH}_j(X_K)$ of a j-dimensional totally isotropic subspace of $\mathbb{P}((V_q)_K)$ (with V_q the F-vector space associated with q). We set $z^i_{n-i-j} := Z^i_{n-i-j} \pmod 2 \in \mathrm{Ch}^{n-i-j}(G_{iK})$, with Ch the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The cycles z^i_{n-i-j} are the elementary classes defining the Elementary Discrete Invariant $EDI(X)$:

Definition 1.1. The *Elementary Discrete Invariant* $EDI(X)$ is the collection of subsets $EDI(X,i)$ consisting of those integers m such that z^i_m is rational.

Furthermore, for any $r \geq 1$, the Chow motive of X^r with $\mathbb{Z}/2\mathbb{Z}$-coefficients decomposes into a direct sum of shifts of the motive of some G_i, see [1, Corollary 91.8]. Therefore, to know $GDI(X)$ is the same as to know

$$\overline{\mathrm{Ch}}^*(X^r) := \mathrm{Image}\left(\mathrm{Ch}^*(X^r) \to \mathrm{Ch}^*(X^r_K)\right)$$

for all $r \geq 1$. Hence, the collection of the latter subalgebras constitutes a *non-compact* (in the sense that one has to consider infinitely many objects) form of $GDI(X)$. For the same reason, there exists a non-compact form of $EDI(X)$ (with defining cycles living in $\mathrm{Ch}^*(X^r_K)$), which we determine in the current note: for any $i \in \{0, \ldots, d\}$, let us denote by $\mathrm{sym} : \mathrm{CH}^*(X^{i+1}) \to \mathrm{CH}^*(X^{i+1})$ the homomorphism $\Sigma_{s \in S_{i+1}} s_*$, where $s : X^{i+1} \to X^{i+1}$ is the isomorphism associated with a permutation s. For $0 \leq j \leq d$, we set

$$\rho_{i,j} := \mathrm{sym}\left((\times_{k=0}^{i-1} h^k) \times l_j\right) \in \mathrm{CH}^{n-j+i(i-1)/2}(X^{i+1}_K),$$

where $\times$ is the external product and h^k is the k-th power of the hyperplane section class $h \in \mathrm{CH}^1(X)$ (always rational). Note that $\rho_{0,j} = Z^0_{n-j} = l_j$. The symmetric cycles $\rho_{i,j} \pmod 2$ are the classes defining the non-compact form of $EDI(X)$:

Theorem 1.2. *Let $1 \leq i \leq d$ and $0 \leq j \leq d$. The cycle z^i_{n-i-j} is rational if and only if the cycle $\rho_{i,j} \pmod 2$ is rational.*

Theorem 1.2 reduces certain questions about rationality of algebraic cycles on orthogonal grassmannians to the sole level of quadrics. For example, it allows one to reformulate both Vishik's conjecture [7, Conjecture 3.11] and the conjecture [8, Conjecture 0.13] on the *dimensions of Bruno Kahn*.

In Section 2, we introduce some basic tools which are required in Section 3, where we prove Theorem 1.2, using mainly compositions of correspondences and Chern classes of vector bundles over orthogonal grassmannians.

2. Preliminaries

In this section, we continue to use notation introduced in Section 1.

2.1. **Rational cycles on powers of quadrics.** We refer to [1, §68] for an introduction to cycles on powers of quadrics. For any $1 \le i \le d$ and $0 \le j \le i-1$, we set

$$\Delta_{i,j} := \mathrm{sym}\left((\times_{k=0}^{i-1} h^k) \times l_j\right) + \sum_{m=i}^{d} \mathrm{sym}\left((\times_{\substack{k=0\\k\neq j}}^{i-1} h^k) \times h^m \times l_m\right)$$

in $\mathrm{Ch}^{n-j+i(i-1)/2}(X_K^{i+1})$. If $n = 2d$, we choose an orientation l_d of the quadric.

Lemma 2.1. *For any $1 \le i \le d$ and $0 \le j \le i-1$, the cycle $\Delta_{i,j}$ is rational.*

Proof. We proceed by induction on i. In $\mathrm{Ch}^n(X_K^2)$, the cycle $\Delta_{1,0}$ or $\Delta_{1,0} + h^d \times h^d$, depending on whether $l_d^2 = 0$ or not, is the class of the diagonal. Therefore , the cycle $\Delta_{1,0}$ is rational. Let $\sigma \in S_{i+1}$ be a cyclic permutation (with $i \ge 2$). For $0 \le j \le i-2$, the induction hypothesis step is provided by the identity

$$\Delta_{i,j} = \sum_{l=0}^{i} \sigma_*^l(\Delta_{i-1,j} \times h^{i-1}) \text{ in } \mathrm{Ch}(X_K^{i+1}).$$

It remains to show that the cycle $\Delta_{i,i-1}$ is rational to complete the proof. In $\mathrm{Ch}(X_K^{i+1})$, one has

$$\begin{aligned}\Delta_{i,i-1} &= \textstyle\sum_{m=i-1}^{d} \mathrm{sym}\left((\times_{k=0}^{i-2} h^k) \times l_m \times h^m\right)\\ &= \textstyle\sum_{m=0}^{d} \mathrm{sym}\left((\times_{k=0}^{i-2} h^k) \times l_m \times h^m\right)\end{aligned}$$

and the latter sum can be rewritten as

$$\sum_{s \in A_{i+1}} s_*\left((\times_{k=0}^{i-2} h^k) \times \Delta_{1,0}\right).$$

Thus, the cycle $\Delta_{i,i-1}$ is rational.

□

2.2. **Correspondences.** We refer to [1, §62] for an introduction to Chow-correspondences.

For any $1 \le i \le d$, we denote by θ_i the class of the subvariety

$$\{(y, x_1, \ldots, x_{i+1}) \mid x_1, \ldots x_{i+1} \in y\} \subset G_i \times X^{i+1}$$

in $\mathrm{CH}(G_i \times X^{i+1})$ and we view the cycle θ_i as a correspondence $G_i \rightsquigarrow X^{i+1}$.

We set

$$\eta_i := \prod_{k=1}^{i} \left(\mathrm{Id}_{G_i} \times p_{X_k^i}\right)^* ([\mathcal{F}(i,0)]) \in \mathrm{CH}(G_i \times X^i), \tag{2.2}$$

with $p_{X_k^i}$ the projection from X^i to the k-th coordinate. For any integer $i \le s \le d$, we write

$$W_{s-i}^i := \pi_{(0,\underline{i})_*} \circ \pi_{(\underline{0},i)}^*(h^s) \in \mathrm{CH}^{s-i}(G_i),$$

and $w^i_{s-i} := W^i_{s-i} \pmod 2 \in \mathrm{Ch}^{s-i}(G_i)$. Since the variety X_K is cellular, the cycle $[\mathcal{F}(i,0)]$ decomposes as

$$[\mathcal{F}(i,0)] = \sum_{s=0}^{d} z^i_{n-i-s} \times h^s + \sum_{s=i}^{d} w^i_{s-i} \times l_s \text{ in } \mathrm{Ch}(G_{iK} \times X_K), \tag{2.3}$$

where l_d has to be replaced by the other class l'_d of maximal totally isotropic subspaces if $n = 2d$ and l_d^2 is not zero, i.e., if 4 divides n (see [1, Theorem 66.2]).

The two following lemmas, where we write p with underlined target for projections, can be proven the same way [2, Lemmas 3.2 and 3.10] have been proven but with Z^i_{n-i-j} (resp. z^i_{n-i-j}) instead of Z^i_{n-i} (resp. z^i_{n-i}).

Lemma 2.4. *For any $1 \leq i \leq d$, $0 \leq j \leq d$ and $x \in \mathrm{CH}(X_K)$, one has*

$$\left((\theta_i)_*(Z^i_{n-i-j})\right)_*(x) = p_{G_i \times \underline{X^i}}{}_*\left(p^*_{\underline{G_i} \times X^i}\left(\pi_{(0,\underline{i})*} \circ \pi^*_{(\underline{0},i)}(x) \cdot Z^i_{n-i-j}\right) \cdot \eta_i\right),$$

where the cycle $(\theta_i)_*(Z^i_{n-i-j})$ is viewed as a correspondence $X_K \rightsquigarrow X^i_K$.

For any $1 \leq i \leq d$, we write $\mathcal{F}(i-1,i)$ for the partial orthogonal flag variety of $(i-1)$-dimensional totally isotropic subspaces contained in i-dimensional totally isotropic subspaces and we consider the diagram

$$G_{i-1} \xleftarrow[\pi_{(\underline{i-1},i)}]{} \mathcal{F}(i-1,i) \xrightarrow[\pi_{(i-1,\underline{i})}]{} G_i,$$

given by the natural projections.

Lemma 2.5. *For any $2 \leq i \leq d$, $0 \leq j \leq d$ and $i \leq m \leq d$, the cycle*

$$p_{G_i \times \underline{X^i}}{}_*\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \in \mathrm{Ch}(X^i_K),$$

where we abuse notation and write η_i for $\eta_i \pmod 2$, can be rewritten as

$$\sum_{s=0}^{m} \sum_{k=\max(i-s,0)}^{\min(m-s,i)} p_{G_{i-1} \times \underline{X^{i-1}}}{}_*\left(w^{i-1}_{m-s-k} \cdot \sigma^k_{i-1} \cdot z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right) \times h^s,$$

with $\sigma^k_{i-1} = \pi_{(\underline{i-1},i)*} \circ \pi^*_{(i-1,\underline{i})}(z^i_{n-2i+k}) \in \mathrm{Ch}^j(G_{i-1K})$.

3. Equivalence

In this section, we continue to use notation and material introduced in the previous sections and we prove Theorem 1.2.

For $1 \leq i \leq d$ and $0 \leq j \leq i-1$, we set

$$\alpha_{i,j} := (\theta_i)_*(Z^i_{n-i-j}) + \rho_{i,j} \in \mathrm{CH}(X^{i+1}_K),$$

and we view the cycle $\alpha_{i,j}$ as a correspondence $X_K \rightsquigarrow X^i_K$.

Proposition 3.1. *One has*

$$(\alpha_{i,j} \pmod 2)_* (h^m) = \begin{cases} \operatorname{sym}\left(\times_{k=0}^{i-1} h^k\right) & \text{if } m = j; \\ \operatorname{sym}\left((\times_{\substack{k=0\\k\neq j}}^{i-1} h^k) \times h^m \right) & \text{if } i \leq m \leq d; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any $x \in \mathrm{CH}^m(X_K)$ with $m \leq i-1$, the cycle $\pi_{(0,\underline{i})*} \circ \pi^*_{(\underline{0},i)}(x)$ is trivial by dimensional reasons. Thus, by Lemma 2.4, the cycle $\left((\theta_i)_*(Z^i_{n-i-j})\right)_*(x)$ is also trivial. Therefore, since $(\rho_{i,j})_*(h^m) = \operatorname{sym}\left(\times_{k=0}^{i-1} h^k\right)$ if $m = j$ and is trivial otherwise, one get the conclusion of Proposition 3.1 for the cases $m \leq i-1$.

Moreover, for $i \leq m \leq d$, Lemma 2.4 provides the identity

$$(\alpha_{i,j} \pmod 2)_* (h^m) = p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \text{ in } \mathrm{Ch}(X^i_K). \tag{3.2}$$

We prove the cases $i \leq m \leq d$ of Proposition 3.1 by descending induction on i. The base of the descending induction $i = d$ (so $i = m = d$) is obtained by combining the identities (2.2), (2.3) and (3.2) for $i = d$ (recall also that, by [9, Proposition 2.1], one has $W^i_0 = 1$ for any $0 \leq i \leq d$) with the fact that, for any integers $0 \leq a_0 \leq a_1 \leq \cdots \leq a_e \leq d$, with $e \leq d$, one has

$$\deg\left(\prod_{k=0}^{e} z^d_{n-d-a_k}\right) = \begin{cases} 1 & \text{if } \{a_0, a_1, \ldots, a_e\} = \{0, 1, \ldots, d\}\ ; \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg : \mathrm{Ch}(G_{dK}) \to \mathrm{Ch}(\mathrm{Spec}(K)) = \mathbb{Z}/2\mathbb{Z}$ is the homomorphism associated with the push-forward of the structure morphism, see [1, Lemma 87.6].

Let $2 \leq i \leq d$ and $0 \leq j \leq i-2$. On the one hand, by descending induction hypothesis, for any $i \leq m \leq d$, one has

$$p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) = \operatorname{sym}\left((\times_{\substack{k=0\\k\neq j}}^{i-1} h^k) \times h^m \right). \tag{3.3}$$

Therefore, the coordinate of (3.3) on top right h^{i-1}, i.e.,

$$p_{\underline{X^{i-1}} \times X\,*}\left(\left(p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right)\right) \cdot [X^{i-1}] \times l_{i-1}\right)$$

is equal to

$$\operatorname{sym}\left((\times_{\substack{k=0\\k\neq j}}^{i-2} h^k) \times h^m \right). \tag{3.4}$$

On the other hand, by Lemma 2.5, this coordinate is also equal to

$$\sum_{k=1}^{\min(m-i+1,i)} p_{G_{i-1} \times \underline{X^{i-1}}\,*}\left(w^{i-1}_{m-i+1-k} \cdot \sigma^k_{i-1} \cdot z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right). \tag{3.5}$$

Let us denote by T_{i-1} the tautological vector bundle on G_{i-1}, i.e., T_{i-1} is given by the closed subvariety of the trivial bundle $V\mathbb{1} = V_q \times G_{i-1}$ consisting of pairs (u, U) such that $u \in U$. Note that the vector bundle T_{i-1} has rank i. For a vector bundle E over a scheme, we write $c_i(E)$ for the i-th Chern class with value in CH. Since $W^{i-1}_{m-i+1-k} =$

$c_{m-i+1-k}(V\mathbb{1}/T_{i-1})$ (see [9, Proposition 2.1]) and $\sigma^k_{i-1} = c_k(T_{i-1}) \pmod 2$ (see [2, Lemma 2.6]), by Whitney Sum Formula (see [1, Proposition 54.7]), one has

$$(3.6)\qquad \sum_{k=0}^{\min(m-i+1,i)} w^{i-1}_{m-i+1-k}\cdot\sigma^k_{i-1} = \sum_{k=0}^{\min(m-i+1,i)} c_{m-i+1-k}(V\mathbb{1}/T_{i-1})\cdot c_k(T_{i-1}) = c_{m-i+1}(V\mathbb{1}).$$

Moreover, one has $c_{m-i+1}(V\mathbb{1})=0$ because $m-i+1>0$. Consequently, in view of (3.4), (3.5) and (3.6), one get

$$(3.7)\qquad {p_{G_{i-1}\times\underline{X^{i-1}}}}_*\left(w^{i-1}_{m-i+1}\cdot z^{i-1}_{n-i+1-j}\cdot\eta_{i-1}\right) = \mathrm{sym}\left((\times^{i-2}_{\substack{k=0\\k\neq j}}h^k)\times h^m\right).$$

By identities (3.2) and (3.7), it only remains to prove the case $m=i-1$ to complete the descending induction step. On the one hand, by descending induction hypothesis, the coordinate of ${p_{G_i\times\underline{X^i}}}_*\left(z^i_{n-i-j}\cdot\eta_i\right)$ on top right h^i is

$$\mathrm{sym}\left(\times^{i-1}_{\substack{k=0\\k\neq j}}h^k\right)$$

(see (3.3)) and, on the other hand, by Lemma 2.5, it is also equal to

$${p_{G_{i-1}\times\underline{X^{i-1}}}}_*\left(z^{i-1}_{n-i+1-j}\cdot\eta_{i-1}\right).$$

Proposition 3.1 is proven. □

As a consequence of Proposition 3.1, we partially obtain the first part of Theorem 1.2.

Corollary 3.8. *Let $1\le i\le d$ and $0\le j\le i-1$. If the cycle z^i_{n-i-j} is rational then the cycle $\rho_{i,j} \pmod 2$ is also rational.*

Proof. In view of the ring structure of $\mathrm{CH}(X^{i+1}_K)$ (see [1, §68]), and knowing that the cycle $\alpha_{i,j}$ is symmetric, one deduces from Proposition 3.1 that

$$\alpha_{i,j} \pmod 2 = \Delta_{i,j}+\beta$$

with β a sum of nonessential elements (a nonessential element is an external product of powers of the hyperplane class, it is always rational). Since $\alpha_{i,j}=(\theta_i)_*(Z^i_{n-i-j})+\rho_{i,j}$ and $\Delta_{i,j}$ is rational (Lemma 2.1), the corollary is proven. □

Remark 3.9. As a consequence of Proposition 3.1 and its proof, one make the following observation. Let $1\le i\le d-1$, $0\le j\le i-1$, $i+1\le m\le d$ and $s\in\{0,1,\dots,i\}\backslash\{j\}$. For any integers $0\le a_1\le a_2\le\dots\le a_i\le d$, the integer

$$\deg\left(\left(Z^i_{n-i-j}\cdot\prod_{l=1}^{i}Z^i_{n-i-a_l}\right)\cdot\left(\sum_{k=0}^{i-s}W^i_{m-s-k}\cdot c_k(T_i)\right)\right)$$

is congruent to $1 \pmod 2$ if $\{a_1,\dots,a_i\}=\{m\}\cup(\{0,1,\dots,i\}\backslash\{j,s\})$ and to $0 \pmod 2$ otherwise.

The following proposition will complete the first part of Theorem 1.2 (see Corollary 3.12).

Proposition 3.10. *For any $i \leq j \leq d$, one has*

$$\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) = \begin{cases} \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We already know from Lemma 2.4 that $\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) = 0$ for $m \leq i-1$. We prove the cases $i \leq m \leq d$ by descending induction on i. The base of the descending induction (so $i=j=m=d$) is done similarly as the base of the descending induction in the proof of Proposition 3.1.

Let $2 \leq i \leq d$, $i \leq j \leq d$ and $i \leq m \leq d$. On the one hand, by Lemma 2.4 and the descending induction hypothesis, one has

$$(3.11)\qquad \begin{aligned} \left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^m) &= p_{G_i \times \underline{X^i}}{}_*\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \\ &= \begin{cases} \mathrm{sym}\left(\times_{k=0}^{i-1} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, the coordinate of (3.11) on top right h^{i-1} is

$$\begin{cases} \mathrm{sym}\left(\times_{k=0}^{i-2} h^k\right) & \text{if } m=j; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by Lemmas 2.4, 2.5 and identity (3.6), this coordinate is also equal to

$$\left((\theta_{i-1})_*(z^{i-1}_{n-i+1-j})\right)_*(h^m).$$

It remains to consider the cases $i \leq j \leq d$ with $m = i-1$ and $j = i-1$ with $i-1 \leq m \leq d$ to complete the descending induction step. Let $i-1 \leq j \leq d$. By Lemma 2.4, one has

$$\left((\theta_{i-1})_*(z^{i-1}_{n-i+1-j})\right)_*(h^{i-1}) = p_{G_{i-1} \times \underline{X^{i-1}}}{}_*\left(z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right).$$

By Lemma 2.5, the latter cycle is the coordinate on top right h^i of $\left((\theta_i)_*(z^i_{n-i-j})\right)_*(h^i)$. If $j \geq i$ then this coordinate is trivial by the descending induction hypothesis. Otherwise – if $j = i-1$ – then one has

$$\left((\theta_i)_*(z^i_{n-2i+1})\right)_*(h^i) = \rho_{i,i-1}{}_*(h^i) + \left(\alpha_{i,i-1} \pmod 2\right)_*(h^i).$$

By Proposition 3.1, the latter cycle is equal to

$$\mathrm{sym}\left((\times_{k=0}^{i-2} h^k) \times h^i\right),$$

whose coordinate on top right h^i is

$$\mathrm{sym}\left(\times_{k=0}^{i-2} h^k\right).$$

Now suppose that $j = i-1$ and let $i \leq m \leq d$. By Lemmas 2.4, 2.5 and identity (3.6), the cycle $\left((\theta_{i-1})_*(z^{i-1}_{n-2i+2})\right)_*(h^m)$ is the coordinate of $\left((\theta_i)_*(z^i_{n-2i+1})\right)_*(h^m)$ on top right h^{i-1}. Since

$$\left((\theta_i)_*(z^i_{n-2i+1})\right)_*(h^m) = \rho_{i,i-1}{}_*(h^m) + \left(\alpha_{i,i-1} \pmod 2\right)_*(h^m),$$

${\rho_{i,i-1}}_*(h^m) = 0$ (because $m \neq i-1$) and $(\alpha_{i,i-1} \pmod 2)_*(h^m) = \mathrm{sym}\left((\times_{k=0}^{i-2} h^k) \times h^m\right)$ (Proposition 3.1), this coordinate is trivial. This completes the descending induction step. The proposistion is proven. □

Corollary 3.12. *Let $1 \leq i \leq d$ and $i \leq j \leq d$. If the cycle z^i_{n-i-j} is rational then the cycle $\rho_{i,j} \pmod 2$ is also rational.*

Proof. In view of the ring structure of $\mathrm{CH}(X_K^{i+1})$ and knowing that the cycle $(\theta_i)_*(Z^i_{n-i-j})$ is symmetric, one deduces from Proposition 3.10 that

$$(\theta_i)_*(z^i_{n-i-j}) = \rho_{i,j} \pmod 2 + \beta$$

with β a sum of nonessential elements. The corollary is proven. □

The next proposition gives the second part of Theorem 1.2.

Proposition 3.13. *Let $1 \leq i \leq d$ and $0 \leq j \leq d$. If the cycle $\rho_{i,j}$ is rational then the cycle Z^i_{n-i-j} is also rational.*

Proof. Since ${\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)}(h^i) = [G_i]$ in $\mathrm{CH}^0(G_i)$, by dimensional reasons, one has

$$({\pi_{(0,\underline{i})}}_* \circ \pi^*_{(\underline{0},i)})^{\times i+1} \left(h^i \times h^{i-1} \times \cdots \times 1 \cdot \mathrm{sym}\left((\times_{k=0}^{i-1} h^k) \times l_j\right)\right) = [G_i]^{\times i} \times Z^i_{n-i-j}. \tag{3.14}$$

The conclusion follows by taking the image of cycle (3.14) under the pull-back of the diagonal morphism $X \to X^{i+1}$. □

4. Witt index

We continue to use notation and material introduced in the previous sections. In this section, we assume that the F-quadric X is anisotropic and we study some restrictions on the Elementary Discrete Invariant when the first Witt index i_1 of X is sufficiently large, using the non-compact form (Theorem 1.2).

By [1, Lemmas 73.18 and 73.3], there exists a unique minimal rational cycle in $\mathrm{Ch}^{n-i_1+1}(X_K^2)$ containing $1 \times l_{i_1-1}$. This cycle is symmetric ([1, Lemma 73.17]) and is called the 1-*primordial cycle.* We denote it by π.

Proposition 4.1. *Let $i \in \{1, \ldots, d\}$. Suppose that the quadric X is anisotropic with $i_1 > i$. If $m \in EDI(X, i)$ is such that $n - m \notin \{i_1\} \cup \{2i_1, \ldots, d+1\}$ then $m+1 \in EDI(X, i-1)$.*

Proof. We set $j = n - i - m$. Since $m \in EDI(X,i)$, the cycle $\rho_{i,j} \pmod 2$ is rational by Theorem 1.2. We claim that the hypothesis $i_1 > i$ implies that the cycle $\rho_{i-1,j} \pmod 2$ is also rational, which, by Theorem 1.2, gives the conclusion.

The rational cycle π decomposes as

$$\pi = 1 \times l_{i_1-1} + l_{i_1-1} \times 1 + \sum_{k=i_1}^{d-i_1+1} a_k \left(h^k \times l_{k+i_1-1} + l_{k+i_1-1} \times h^k\right)$$

for some $a_k \in \mathbb{Z}/2\mathbb{Z}$. The fact that one can choose to make the previous sum start from $k = i_1$ is due to [1, Proposition 73.27]. Since $i_1 > i$, and $j \notin \{i_1 - i\} \cup \{2i_1 - i, \ldots, d - i + 1\}$, one has

$$(\rho_{i,j} \pmod 2) \circ \left((1 \times h^{i_1 - i}) \cdot \pi\right) = 1 \times (\rho_{i-1,j} \pmod 2),$$

where $\circ$ stands for the composition of correspondences. Therefore, pulling back the latter identity with respect to the diagonal morphism δ_i, one get that the cycle $\rho_{i-1,j} \pmod 2$ is rational. □

The following statement is obtained by applying recursively Proposition 4.1.

Corollary 4.2. *Let $i \in \{1, \ldots, d\}$. Suppose that the quadric X is anisotropic with $i_1 > i$. One has*

(i) *if $m \in EDI(X, i)$ then $n - m \geq i_1$;*
(ii) *if $m \in EDI(X, i)$ and $n - m = i_1 + l$ or $d + 1 + l$ for some $1 \leq l < i$ then $m + l \in EDI(X, i - l)$.*

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