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***J*-INVARIANT OF HERMITIAN FORMS OVER  
QUADRATIC EXTENSIONS**

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# **$J$ -INVARIANT OF HERMITIAN FORMS OVER QUADRATIC EXTENSIONS**

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**We develop a version of the  $J$ -invariant for hermitian forms over quadratic extensions using a similar method to Alexander Vishik's approach using quadratic forms. This discrete invariant contains information about rationality of algebraic cycles on the maximal unitary grassmannian associated with a hermitian form over a quadratic extension. The computation of the canonical 2-dimension of this grassmannian in terms of the  $J$ -invariant is provided, as well as a complete motivic decomposition.**

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## **1. Introduction**

Let  $F$  be an arbitrary field and  $K/F$  a quadratic separable field extension. In this article, we define a new discrete invariant  $J(h)$  for a nondegenerate  $K/F$ -hermitian form  $h : V \times V \rightarrow K$ . This invariant is developed on the model of the  $J$ -invariant for quadratic forms, due to Alexander Vishik [2005], and later generalized to an arbitrary semisimple algebraic group of inner type by V. Petrov, N. Semenov and

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K. Zainoulline in [Petrov et al. 2008]. Let  $X$  denote the  $F$ -variety of maximal totally  $h$ -isotropic subspaces of  $V$ . The invariant  $J(h)$  contains information about rationality of algebraic cycles on  $X$  over a splitting field of  $h$ . In the same way as it was obtained by Nikita A. Karpenko and Alexander S. Merkurjev for maximal orthogonal grassmannians in the case of quadratic forms (see [Elman et al. 2008, Theorem 90.3]), the invariant  $J(h)$  notably allows one to recover the canonical 2-dimension of the maximal unitary grassmannian  $X$  (Theorem 8.2).

In general, the  $J$ -invariant has several important applications. For example, A. Vishik used it in his refutation of Kaplansky's conjecture on the  $u$ -invariant of a field (see [Vishik 2009]), as did Semenov when he answered a question by J-P. Serre about groups of type  $E_8$  (see [Semenov 2016]).

In the case of quadratic forms, the Chow motive of the maximal orthogonal grassmannian associated with a quadratic form splits as a sum of Tate motives over a splitting field of the quadratic form; the reason being that there is a nice filtration of the maximal orthogonal grassmannian by affine bundles. Because this does not stand in the case of hermitian forms, we use [Karpenko 2000, Theorem 15.8] and the modified Chow ring  $\mathrm{Ch}_K(X) := \mathrm{Ch}(X)/\mathrm{Im}(\mathrm{Ch}(X_K) \rightarrow \mathrm{Ch}(X))$ , with  $\mathrm{Ch}(X)$  the integral Chow ring  $\mathrm{CH}(X)$  modulo 2. These considerations allow one to follow the method introduced by Vishik for quadratic forms to describe completely the ring  $\mathrm{Ch}_K(X)$  when  $h$  is split (equivalently, when  $X$  has a rational point) and the subring  $\mathrm{Im}(\mathrm{Ch}_K(X) \rightarrow \mathrm{Ch}_K(X_{F(X)}))$  of rational elements for arbitrary  $h$ , where  $F(X)$  is the function field of  $X$ . We also work with the category of  $\mathrm{Ch}_K$ -motives, defined from  $\mathrm{Ch}_K$ , and provide a complete motivic decomposition of the  $\mathrm{Ch}_K$ -motive  $M^K(X)$  of  $X$  in terms of the  $J$ -invariant  $J(h)$  (Theorem 9.4). The  $\mathrm{Ch}_K$ -motive  $M^K(X)$  is related to the *essential motive* of  $X$  (see Remark 9.7).

By a theorem of Jacobson (see [Karpenko 2012, Corollary 9.2]), the nondegenerate  $K/F$ -hermitian form  $h$  is entirely determined by the associated  $F$ -quadratic form  $q : v \mapsto h(v, v)$ , with  $V$  considered as an  $F$ -vector space. Moreover, the  $F$ -quadratic forms arising this way from  $K/F$ -hermitian forms can be described as the tensor product of a nondegenerate bilinear form by the norm form of  $K/F$ , which is an anisotropic binary quadratic form. Conversely, an  $F$ -quadratic form defined by such a tensor product is isomorphic to the quadratic form arising from the hermitian form induced by the bilinear form and the quadratic separable field extension  $K/F$  given by the discriminant of the binary quadratic form. As explained by Karpenko in the introduction of [Karpenko 2012], although these observations show that the study of  $K/F$ -hermitian forms is equivalent to the study of binary divisible quadratic forms over  $F$ , this does not show that the hermitian forms are not worthy of interest. Indeed, on one hand, it shows that the class of binary divisible quadratic forms is quite important. On the other hand, it provides the opportunity to use the world of hermitian forms to study such quadratic forms, which can be more

appropriate than staying exclusively at the level of quadratic forms, as illustrated by [Proposition 10.1](#).

The paper is organized as follows. In [Section 3](#), we use the relative cellular space structure on  $X$  given by [\[Karpenko 2000, Theorem 15.8\]](#) to get the relation of [Proposition 3.4](#) between Chow rings  $\mathrm{Ch}_K$  (defined in [Section 2](#)) associated with the maximal unitary grassmannian of a hermitian subform of an isotropic  $K/F$ -hermitian form  $h$ . From [Section 4](#) to [Section 8](#), we literally follow the thread of [\[Elman et al. 2008, §86 to §90\]](#). In this part of the article, we first use the previously mentioned relation to get a complete description of  $\mathrm{Ch}_K(X)$  in the split case in terms of generators and relations ([Theorem 4.9](#) and [Proposition 4.15](#)), from which we deduce a description of the subring of rational elements in the general case in terms of those generators ([Theorem 5.7](#)). The  $J$ -invariant  $J(h)$  is then defined from the latter description. We also compute some Steenrod operations of cohomological type on  $\mathrm{Ch}(X)$  in the split case ([Theorem 7.2](#)). In [Theorem 8.2](#), we obtain the canonical 2-dimension of  $X$  in terms of  $J(h)$ , on the model of [\[Elman et al. 2008, Theorem 90.3\]](#). In [Section 9](#), using Rost nilpotence, we provide the complete motivic decomposition of  $M^K(X)$  in terms of  $J(h)$  ([Theorem 9.4](#)), in the spirit of [\[Petrov et al. 2008, Theorem 5.13\]](#). In [Section 10](#), we compare the  $J$ -invariant  $J(h)$  of a nondegenerate  $K/F$ -hermitian form  $h$  with the  $J$ -invariant  $J(q)$  of the associated quadratic form  $q$  ([Proposition 10.1](#)).

## 2. $K$ -Chow rings

Let  $F$  be an arbitrary field,  $K/F$  a quadratic separable field extension and  $X$  an  $F$ -variety (i.e., a separated  $F$ -scheme of finite type). We denote by  $\mathrm{Ch}(X)$  the integral Chow ring  $\mathrm{CH}(X)$  modulo 2.

We set

$$\mathrm{Ch}_K(X) := \mathrm{Ch}(X) / \mathrm{Im}(\mathrm{Ch}(X_K) \rightarrow \mathrm{Ch}(X)),$$

where the homomorphism  $\mathrm{Ch}(X_K) \rightarrow \mathrm{Ch}(X)$  is the push-forward of the projection  $X_K \rightarrow X$ .

Note that  $\mathrm{Im}(\mathrm{Ch}(X_K) \rightarrow \mathrm{Ch}(X))$  is an ideal by the projection formula ([\[Elman et al. 2008, Proposition 56.9\]](#)), called the *norm ideal*, so that  $\mathrm{Ch}_K(X)$  inherits the ring structure of the initial Chow ring. For example, one has

$$\mathrm{Ch}_K(\mathrm{Spec}(F)) = \mathbb{Z}/2\mathbb{Z},$$

and for any  $F$ -variety  $X$ , the ring  $\mathrm{Ch}_K(X_K)$  is trivial. We write  $(\varphi)_K$  for the  $K$ -Chow group homomorphism associated with a Chow group homomorphism  $\varphi$  which preserves norm ideals.

Since the norm ideal is preserved by pull-backs and push-forwards, one can define the additive category of  $\mathrm{Ch}_K$ -motives the same way as the category of Chow

motives (see [Elman et al. 2008, Chapter XII]) but using the Chow rings  $\mathrm{Ch}_K$  instead of the usual Chow rings  $\mathrm{CH}$ . For a smooth proper  $F$ -variety  $X$ , we write  $M^K(X)$  for the associated  $\mathrm{Ch}_K$ -motive.

**Remark 2.1.** For a field extension  $E/F$  and  $j \geq 0$ , let us denote by  $N_j(E)$  the subgroup of the Milnor group  $K_j^M(E)$  generated by the norms from finite field extensions of  $E$  that split the extension  $K/F$ . Then the cycle module  $E \mapsto K_*^M(E)$  over  $F$  gives rise to an assignment  $E \mapsto K_*^M(E)/N_*(E)$ . One can check that the latter is also a cycle module over  $F$ , in particular, the fact that residue maps are well defined comes from the rule [Rost 1996, R3b]. Hence, one can consider the cohomology theory associated with this cycle module (which contained the  $K$ -Chow groups) instead of the cohomology theory of the Milnor cycle module and thus obtain some  $\mathrm{Ch}_K$ -versions of results for classical Chow groups (see Propositions 6.5, 8.1 and 9.2).

### 3. Isotropic hermitian forms

**3A. Relative cellular spaces.** Let  $F$  be a field.

**Definition 3.1.** Let  $X$  be a smooth proper  $F$ -variety supplied with a filtration  $\mathcal{F}$  by closed subvarieties

$$\emptyset = X_{(-1)} \subset X_{(0)} \subset \cdots \subset X_{(n)} = X.$$

The variety  $X$  is a *relative cellular space* over a smooth proper  $F$ -variety  $Y$  if the associated *adjoint* variety

$$\mathrm{Gr}_{\mathcal{F}} X = \bigsqcup_{k=0}^n X_{(k)} \setminus X_{(k-1)}$$

is a vector bundle over  $Y$ . The variety  $Y$  is called the *base* of  $X$ .

For  $V$  a finite-dimensional  $F$ -vector space, we denote by  $\Gamma(V)$  the full grassmannian of  $F$ -subspaces of  $V$ . To an epimorphism  $p : V \rightarrow V'$  of  $F$ -vector spaces, one can associate the filtration

$$\emptyset = \Gamma(V)_{(-1)} \subset \Gamma(V)_{(0)} \subset \cdots \subset \Gamma(V)_{(\dim V')} = \Gamma(V)$$

on  $\Gamma(V)$  defined as follows: for any local commutative  $F$ -algebra  $R$  and  $0 \leq k \leq \dim V'$ , one has

$$\Gamma(V)_{(k)}(R) = \{N \in \Gamma(V)(R) \mid \Lambda^{k+1}(p_R(N)) = 0\},$$

where  $p_R : V_R \rightarrow V'_R$  is induced by  $p$  and  $\Lambda^{k+1}$  stands for the  $(k+1)$ -th exterior power.

Let  $0 \rightarrow V'' \rightarrow V \rightarrow V' \rightarrow 0$  be an exact sequence of  $F$ -vector spaces. The result [Karpenko 2000, Corollary 9.11] by Karpenko asserts that  $\Gamma(V)$  supplied with the filtration associated with  $V \rightarrow V'$  is a relative cellular space over  $\Gamma(V'') \times \Gamma(V')$ .

Moreover, let  $K/F$  be a quadratic separable field extension. Suppose that  $V$ ,  $V'$  and  $V''$  are  $K$ -vector spaces and that the short sequence is an exact sequence of  $K$ -vector spaces. Then the previous relative cellular structure on  $\Gamma(V)$  induces a relative cellular structure on the Weil restriction  $\Gamma^K(V)$  of the full grassmannian of  $K$ -subspaces with respect to the extension  $K/F$ :  $\Gamma^K(V)$  is a relative cellular space over  $\Gamma^K(V'') \times \Gamma^K(V')$ ; see [Karpenko 2000, Theorem 10.9]. The associated filtration is the restriction of the previous one by  $K$ -subspaces.

Suppose that the  $K$ -vector space  $V$  is isomorphic to a sum of  $K$ -subspaces  $V' \oplus V'' \oplus \tilde{V}$ . Using the exact sequences

$$0 \rightarrow V'' \oplus \tilde{V} \rightarrow V \rightarrow V' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow V'' \rightarrow V'' \oplus \tilde{V} \rightarrow \tilde{V} \rightarrow 0,$$

and composing the relative structures, the variety  $\Gamma^K(V)$  is turned into a relative cellular space over  $\Gamma^K(V') \times \Gamma^K(V'') \times \Gamma^K(\tilde{V})$ , as described in [Karpenko 2000, Example 10.14].

Let  $h : V \times V \rightarrow K$  be a nondegenerate isotropic  $K/F$ -hermitian form on  $V$ . Denote by  $L \subset V$  an isotropic line and set  $\tilde{V} = L^\perp/L$ . Let  $L^*$  be an arbitrary splitting of  $V \rightarrow V/L^\perp$ . Applying the observation of the previous paragraph to the decomposition  $V \simeq L \oplus L^* \oplus \tilde{V}$ , one obtains that  $\Gamma^K(V)$  is a relative cellular space over  $\Gamma^K(L) \times \Gamma^K(L^*) \times \Gamma^K(\tilde{V})$ .

Furthermore, by [Karpenko 2000, Theorem 15.8], the latter relative cellular structure restricts to the  $h$ -isotropic subspaces in the following way. Let  $\tilde{h}$  be the  $K/F$ -hermitian form on  $\tilde{V}$  induced by  $h$ . Let  $Y$  and  $\tilde{Y}$  be the  $F$ -varieties of totally isotropic subspaces of  $h$  and  $\tilde{h}$  respectively. Let  $Z$  be the Weil transfer of the  $K$ -variety of 2-flags of  $K$ -vector subspaces of  $L$  with respect to the extension  $K/F$  (note that the 0-dimensional  $F$ -variety  $Z$  is the disjoint union of three copies of  $\text{Spec}(F)$  and three copies of  $\text{Spec}(K)$ ). Then  $Y$  is a relative cellular space over  $Z \times \tilde{Y}$ . The associated filtration is the restriction of the filtration associated with the relative cellular space structure of  $\Gamma^K(V)$  over  $\Gamma^K(L) \times \Gamma^K(L^*) \times \Gamma^K(\tilde{V})$  by  $h$ -isotropic subspaces.

In [Karpenko 2000], the author proves that, in general, the Chow motive of a relative cellular space is isomorphic to the Chow motive of its base ([Karpenko 2000, Theorem 6.5]) and he describes in [Karpenko 2000, Corollary 6.11] how this isomorphism restricts to the irreducible components of the relative cellular space.

Applied to the previous situation, this gives the following. Let  $X$  be the  $F$ -variety of maximal totally isotropic subspaces in  $h$ . The dimension of such a subspace is  $r := \lfloor \dim(h)/2 \rfloor$ . Note that  $X$  is an irreducible component of  $Y$ . Besides, the unitary grassmannian  $X$  is a projective homogeneous variety under a projective unitary

group of outer type (see the introduction of [Karpenko 2012]). We write  $\tilde{X}$  for the maximal unitary grassmannian associated with  $\tilde{h}$ . Since maximal totally isotropic subspaces of  $\tilde{V}$  are in one-to-one correspondence with those of  $V$  containing  $L$ , one can view  $\tilde{X}$  as a closed subvariety of  $X$ . Let  $i : \tilde{X} \hookrightarrow X$  denote the closed embedding. Let  $\beta : \tilde{X} \rightsquigarrow X$  be the correspondence given by the scheme of pairs  $(W/L, U)$ , where  $U$  is a totally isotropic  $r$ -dimensional  $K$ -subspace of  $V$ ,  $W$  is a totally isotropic  $r$ -dimensional subspace of  $L^\perp$  containing  $L$ , and  $\dim_K(U + W) \leq r + 1$  (correspondences are defined in [Elman et al. 2008, §62]). Then one has the Chow motivic decomposition

$$(3.2) \quad M(X) \simeq M(\tilde{X})\{d\} \oplus M(\tilde{X}) \oplus M$$

with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients, where  $d = \dim(X) - \dim(\tilde{X})$ , the morphism  $M(\tilde{X})\{d\} \rightarrow M(X)$  is given by  $\beta$ , the morphism  $M(\tilde{X}) \rightarrow M(X)$  is given by the class in  $\text{Ch}(\tilde{X} \times X)$  of the graph of  $i$  and  $M$  is a sum of shifts of  $M(\text{Spec}(K))$ .

At the level of  $K$ -Chow groups (introduced in the previous section), decomposition (3.2) implies that

$$(3.3) \quad \text{Ch}_K^*(X) \simeq \text{Ch}_K^*(\tilde{X}) \oplus \text{Ch}_K^{*-d}(\tilde{X}),$$

where the injection  $\text{Ch}^*(\tilde{X}) \hookrightarrow \text{Ch}^*(X)$  coincides with  $\beta_*$  and  $\text{Ch}^{*-d}(\tilde{X}) \hookrightarrow \text{Ch}^*(X)$  coincides with  $i_*$ . In particular, if  $h$  is split (i.e., if the Witt index  $i_0(h)$  of  $h$  is equal to  $r$ ), one deduces by induction that  $\text{Ch}_K(X)$  is a free  $\mathbb{Z}/2\mathbb{Z}$ -module of rank  $2^r$ .

We write  $j$  for the open embedding  $X \setminus \tilde{X} \hookrightarrow X$  and we set

$$f := \beta^t \circ j : X \setminus \tilde{X} \rightsquigarrow \tilde{X},$$

with  $\beta^t$  the transpose of  $\beta$ .

Since  $\text{Im}(i_*) = \text{Ker}(j^*)$  by the localization exact sequence (see [Elman et al. 2008, §52.D]), it follows from (3.3) that

$$\text{Ch}_K^*(X) = \text{Im}((\beta_*)_K) \oplus \text{Ker}((j^*)_K).$$

Hence, since  $j^*$  is surjective (see [loc. cit.]), we have obtained the following statement.

**Proposition 3.4.** *The homomorphism*

$$(f^*)_K : \text{Ch}_K^*(\tilde{X}) \rightarrow \text{Ch}_K^*(X \setminus \tilde{X})$$

*is an isomorphism.*

The above proposition is crucial for the induction in the proof of Theorem 4.9 in the next section.

**3B. Associated quadrics.** We use notation introduced in [Section 3A](#). Let  $q : V \rightarrow F$ ,  $v \mapsto h(v, v)$  be the nondegenerate  $F$ -quadratic form associated with  $h$ , where  $V$  is considered as an  $F$ -vector space. Note that  $\dim(q) = 2 \dim(h)$ . We denote by  $Q$  the smooth projective quadric of  $q$ . Similarly, let  $\tilde{q} : \tilde{V} \rightarrow F$  be the nondegenerate  $F$ -quadratic form associated with the hermitian form  $\tilde{h}$  and let us denote by  $\tilde{Q}$  the smooth projective quadric of  $\tilde{q}$ . Note that since  $\tilde{q}$  is also the form induced by  $q$  on  $P^\perp/P$ , with  $P$  the  $q$ -isotropic  $F$ -plane corresponding to  $L$ , it is Witt-equivalent to  $q$ .

The *incidence correspondence*  $\alpha : \tilde{Q} \rightsquigarrow Q$  is given by the scheme of pairs  $(B/P, A)$  of isotropic  $F$ -lines in  $P^\perp/P$  and  $V$ , respectively, with  $A \subset B$ . By [\[Elman et al. 2008, Lemma 72.3\]](#), for  $k < i_0(q)$ , one has  $\alpha_*(\tilde{l}_{k-2}) = l_k$  and  $\alpha^*(l_k) = \tilde{l}_{k-2}$ , where  $l_k$  (resp.  $\tilde{l}_k$ ) is the class in  $\text{CH}_k(Q)$  (resp.  $\text{CH}_k(\tilde{Q})$ ) of a  $k$ -dimensional totally  $q$ -isotropic (resp.  $\tilde{q}$ -isotropic) subspace of  $\mathbb{P}_F(V)$  (resp.  $\mathbb{P}_F(\tilde{V})$ ). If  $q$  is split, given an orientation of  $Q$  (i.e., the choice of one of the two classes in  $\text{CH}(Q)$  of maximal isotropic subspaces), we choose an orientation of  $\tilde{Q}$  so that the previous formulas hold for the classes of the respective maximal isotropic subspaces.

We write  $E$  for the vector bundle of rank  $2r$  over  $X$  given by the closed subvariety of the trivial bundle  $V \mathbb{1} = V \times X$  consisting of pairs  $(u, U)$  such that  $u \in U$  (with  $V$  viewed as an  $F$ -vector space). Let us denote as  $E^\perp$  the kernel of the natural morphism  $V \mathbb{1} \rightarrow E^\vee$  given by the polar bilinear form associated with the quadratic form  $q$ , where  $E^\vee$  is the dual bundle of  $E$ . If the dimension of  $h$  is even, one has  $E^\perp = E$ . Otherwise,  $E$  is a subbundle of  $E^\perp$  of corank 2. For our purpose, the vector bundle  $E$  is the appropriate hermitian version of the vector bundle used in [\[Elman et al. 2008, §86\]](#) for the case of quadratic forms.

The associated projective bundle  $\mathbb{P}(E)$  is a closed subvariety of codimension

$$2\lfloor(\dim(h) + 1)/2\rfloor - 1$$

of  $Q \times X$  and we denote by  $in$  the associated closed embedding. We write  $\gamma$  for the class of  $\mathbb{P}(E)$  in  $\text{CH}(Q \times X)$  and view it as a correspondence  $Q \rightsquigarrow X$ . Similarly, one can consider the analogous correspondence  $\tilde{\gamma} : \tilde{Q} \rightsquigarrow \tilde{X}$ .

**Lemma 3.5.** *One has  $\gamma \circ \alpha = \beta \circ \tilde{\gamma}$  and  $\tilde{\gamma} \circ \alpha^t = i^t \circ \gamma$ .*

*Proof.* The proof is almost the same as in the case of quadratic forms; see [\[Elman et al. 2008, Lemma 86.7\]](#). By [\[Elman et al. 2008, Corollary 57.22\]](#), it suffices to check the required identities at the level of cycles representing the correspondences. By definition of the composition of correspondences, the composition  $\gamma \circ \alpha$  and  $\beta \circ \tilde{\gamma}$  coincide with the cycle of the subscheme of  $\tilde{Q} \times X$  consisting of all pairs  $(B/P, U)$  with  $\dim_F(B + U) \leq 2r + 2$  and the compositions  $\tilde{\gamma} \circ \alpha^t$  and  $i^t \circ \gamma$  coincide with the cycle of the subscheme of  $Q \times \tilde{X}$  consisting of all pairs  $(A, W/L)$  with  $A \subset W$ .  $\square$



#### 4. Split maximal unitary grassmannian

We use notation introduced in [Section 3](#).

In this section, we make the assumption that the nondegenerate  $K/F$ -hermitian form  $h$  is split and we provide a description of the Chow ring  $\mathrm{Ch}_K(X)$  of the maximal unitary grassmannian in terms of generators and relations ([Theorem 4.9](#) and [Proposition 4.15](#)).

The method is the one introduced by Vishik [\[2005\]](#) to get the description of the Chow ring modulo 2 of the maximal orthogonal grassmannian in the split case, except that we work with the Chow rings  $\mathrm{Ch}_K$  and use [Proposition 3.4](#) in place of the filtration by affine bundles on the maximal orthogonal grassmannian.

The exposition below closely follows the one given in the corresponding part of [\[Elman et al. 2008, §86\]](#) in the case of quadratic forms. The proofs are very similar to the original ones in [\[Elman et al. 2008, §86\]](#).

Let us write  $\dim(h) = 2n + 2$  or  $2n + 1$ , with  $n \geq 0$ . Note that since  $i_0(q) = 2i_0(h)$  (see [\[Karpenko 2012, Lemma 9.1\]](#)), the quadratic form  $q$  is split if and only if  $h$  is of even dimension. In both cases,  $\mathbb{P}(E)$  has codimension  $2n + 1$  in  $Q \times X$ .

If  $\dim(h) = 2n + 2$ , the cycle  $\gamma$  decomposes as

$$(4.1) \quad \gamma = l_{2n+1} \times e_0 + l'_{2n+1} \times e'_0 + \sum_{k=1}^{2n+1} h^{2n+1-k} \times e_k \quad \text{in } \mathrm{CH}^{2n+1}(Q \times X)$$

for some unique elements  $e_k \in \mathrm{CH}^k(X)$ ,  $k \in [0, 2n + 1]$  and  $e'_0 \in \mathrm{CH}^0(X)$ , with  $l_{2n+1}$  and  $l'_{2n+1}$  the two different classes of  $(2n+1)$ -dimensional totally  $q$ -isotropic subspaces of  $\mathbb{P}_F(V)$  and  $h^i$  the  $i$ -th power of the pull-back  $h^1 \in \mathrm{CH}^1(Q)$  of the hyperplane class  $H \in \mathrm{CH}^1(\mathbb{P}_F(V))$  under the closed embedding  $em : Q \hookrightarrow \mathbb{P}_F(V)$  (see [\[Elman et al. 2008, Propositions 64.3, 68.1 and 68.2\]](#)). The variety  $X$  is connected because the algebraic unitary group  $\mathrm{U}(h)$  is connected and acts transitively on it. Hence, pulling [\(4.1\)](#) back with respect to the canonical morphism  $Q_{F(X)} \rightarrow Q \times X$ , we see that one can choose an orientation of  $Q$  such that  $e_0 = 1$  and  $e'_0 = 0$ .

Otherwise – if  $\dim(h) = 2n + 1$  – the cycle  $\gamma$  decomposes as

$$(4.2) \quad \gamma = \gamma' + l_{2n-1} \times e_0 + \sum_{k=2}^{2n+1} h^{2n+1-k} \times e_k \quad \text{in } \mathrm{CH}^{2n+1}(Q \times X)$$

for some unique elements  $e_k \in \mathrm{CH}^k(X)$ ,  $k \in [2, 2n + 1]$  and  $e_0 \in \mathrm{CH}^0(X)$ , with  $\gamma'$  a cycle such that, by denoting by  $p_X$  the projection  $Q \times X \rightarrow X$ , one has  $p_{X*}((l_{2n+1-k} \times 1) \cdot \gamma') = 0$  for any  $k \in [2, 2n + 1]$  and  $p_{X*}((h^{2n-1} \times 1) \cdot \gamma') = 0$ . Pulling [\(4.2\)](#) back with respect to  $Q_{F(X)} \rightarrow Q \times X$ , one gets that  $e_0 = 1$ .

The multiplication rules in the ring  $\mathrm{CH}(Q)$  (see [\[Elman et al. 2008, Proposition 68.1\]](#)) gives that  $e_k = p_{X*}((l_{2n+1-k} \times 1) \cdot \gamma)$ , for  $k \in [2, 2n + 1]$  and also for  $k = 1$

if  $\dim(h)$  is even. In other words, the correspondence  $\gamma$  satisfies  $\gamma_*(l_{2n+1-k}) = e_k$  for  $k \in [2, 2n+1]$  and also for  $k = 1$  for  $h$  of even dimension. Note also that, in the even-dimensional case, one has  $\gamma_*(h^{2n+1}) = 1$  and that, in the odd-dimensional case, one has  $\gamma_*(h^{2n-1}) = 1$ .

The relations of the previous paragraph between the correspondence  $\gamma$  and the elements  $e_k$  can be rewritten as

$$(4.3) \quad e_k = (p_X \circ \text{in})_* \circ (p_Q \circ \text{in})^*(l_{2n+1-k}),$$

for  $k \in [2, 2n+1]$  and also for  $k = 1$  for  $h$  of even dimension, with  $p_Q$  the projection  $Q \times X \rightarrow Q$  (see [Elman et al. 2008, Proposition 62.7]).

**Lemma 4.4.** *One has  $e_{2n+1} = [\tilde{X}]$  in  $\text{CH}^{2n+1}(X)$ .*

*Proof.* On the one hand, by (4.3) one has  $(p_X \circ \text{in})_* \circ (p_Q \circ \text{in})^*(l_0) = e_{2n+1}$ . On the other hand, by denoting by  $A$  a closed point of  $Q$  of degree 1, the cycle  $(p_X \circ \text{in})_* \circ (p_Q \circ \text{in})^*(l_0)$  is identified with  $[\{U \mid A \subset U\}] = [\{U \mid A \otimes_F K \subset U\}]$  in  $\text{CH}^{2n+1}(X)$ . The latter algebraic cycle is equal to  $[\{U \mid L \subset U\}] = [\tilde{X}]$ .  $\square$

Let us denote by  $\tilde{e}_k \in \text{CH}^k(\tilde{X})$  the elements given by (4.1) or (4.2) for  $\tilde{X}$ . Similarly, the correspondence  $\tilde{\gamma}$  satisfies  $\tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \tilde{e}_k$  for the appropriate integers  $k$  with respect to the parity of  $\dim(h)$ .

The following statement is a direct consequence of Lemma 3.5.

**Lemma 4.5.** *For all  $k \in [0, 2n-1] \setminus \{1\}$  and also for  $k = 1$  for  $h$  of even dimension, one has*

- (i)  $\beta_*(\tilde{e}_k) = e_k$  in  $\text{CH}(X)$ ;
- (ii)  $i^*(e_k) = \tilde{e}_k$  in  $\text{CH}(\tilde{X})$ .

*Proof.* (i) For  $k \in [2, 2n-1]$  and also for  $k = 1$  for  $h$  of even dimension, one has

$$\beta_*(\tilde{e}_k) = \beta_* \circ \tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \gamma_* \circ \alpha_*(\tilde{l}_{2n-1-k}) = \gamma_*(l_{2n+1-k}) = e_k.$$

For  $k = 0$ , one can make the same computation with  $\tilde{h}^{2n-1}$  and  $h^{2n+1}$  if  $h$  is even-dimensional and with  $\tilde{h}^{2n-3}$  and  $h^{2n-1}$  if  $h$  is odd-dimensional (the incidence correspondence  $\alpha$  also satisfies  $\alpha_*(\tilde{h}^i) = h^{i+2}$  for  $0 \leq i \leq \lfloor \dim(Q)/2 \rfloor - 2$ , where  $\tilde{h}^i$  is the  $i$ -th power of the hyperplane class  $\tilde{h}^1 \in \text{CH}^1(\tilde{Q})$ ).

(ii) For  $k \in [2, 2n-1]$  and also for  $k = 1$  for  $h$  of even dimension, one has

$$\begin{aligned} i^*(e_k) &= i'_*(e_k) = i'_* \circ \gamma_*(l_{2n+1-k}) = \tilde{\gamma}_* \circ \alpha'_*(l_{2n+1-k}) = \tilde{\gamma}_* \circ \alpha^*(l_{2n+1-k}) \\ &= \tilde{\gamma}_*(\tilde{l}_{2n-1-k}) = \tilde{e}_k. \quad \square \end{aligned}$$

For  $I \subset [0, 2n+1]$  or  $I \subset [0, 2n+1] \setminus \{1\}$ , depending on whether the dimension of  $h$  is respectively even or odd, we write  $e_I$  for the product of  $e_k$  for all  $k \in I$ . Similarly, one defines the elements  $\tilde{e}_J$  for  $J \subset [0, 2n-1]$  or  $J \subset [0, 2n-1] \setminus \{1\}$ .

The following statement is obtained by combining [Lemma 4.5\(ii\)](#) with the projection formula and [Lemma 4.4](#).

**Corollary 4.6.** *One has  $i_*(\tilde{e}_J) = e_J \cdot e_{2n+1} = e_{J \cup \{2n+1\}}$  for every  $J \subset [0, 2n-1]$  or for every  $J \subset [0, 2n-1] \setminus \{1\}$ , depending on whether the dimension of  $h$  is, respectively, even or odd.*

Let us denote by  $I_{\text{od}}$  the odd part of a set of integers  $I$ .

**Corollary 4.7.** *For  $h$  of even dimension, the monomial  $e_{[1, 2n+1]_{\text{od}}} = e_1 e_3 \cdots e_{2n+1}$  is the class of a rational point in  $\text{CH}_0(X)$ . For  $h$  of odd dimension, the monomial  $e_{[3, 2n+1]_{\text{od}}} = e_3 e_5 \cdots e_{2n+1}$  is the class of a rational point in  $\text{CH}_0(X)$ .*

*Proof.* We induct on  $n$ . If  $\dim(h) = 1$ , one has  $X = \text{Spec}(F)$ . If  $\dim(h) = 2$ , the cycle  $e_1$  is the class of a rational point on the curve  $X$ ; see [Lemma 4.4](#). The conclusion follows from the formulas  $e_{[1, 2n+1]_{\text{od}}} = i_*(\tilde{e}_{[1, 2n-1]_{\text{od}}})$  for  $h$  of even dimension and  $e_{[3, 2n+1]_{\text{od}}} = i_*(\tilde{e}_{[3, 2n-1]_{\text{od}}})$  for  $h$  of odd dimension; see [Corollary 4.6](#).  $\square$

**Lemma 4.8.** *One has  $(f^*)_K(\tilde{e}_J) = (j^*)_K(e_J)$  for any  $J \subset [0, 2n-1]$  or for any  $J \subset [0, 2n-1] \setminus \{1\}$ , depending on whether the dimension of  $h$  is, respectively, even or odd.*

*Proof.* By [Lemma 4.5\(i\)](#), it suffices to prove that  $(f^*)_K$  preserves products to get the conclusion. Since the composition of correspondences  $i^* \circ \beta$  coincides with the class of the diagonal in  $\text{CH}(\tilde{X}^2)$ , one has  $i^* \circ \beta_* = \text{Id}_{\text{CH}(\tilde{X})}$ . Let  $\pi$  denote the quotient map  $\text{Ch}_K(X) \rightarrow \text{Ch}_K(X) / \text{Im}((i_*)_K)$  and let  $\overline{(i^*)_K}$  and  $\overline{(j^*)_K}$  be the homomorphisms such that  $\overline{(i^*)_K} \circ \pi = (i^*)_K$  and  $\overline{(j^*)_K} \circ \pi = (j^*)_K$  (note that  $(i^*)_K \circ (i_*)_K = 0$  so  $\overline{(i^*)_K}$  is well defined). One has

$$\overline{(i^*)_K} \circ (\overline{(j^*)_K})^{-1} \circ (j^*)_K \circ (\beta_*)_K = \overline{(i^*)_K} \circ \pi \circ (\beta_*)_K = \text{Id}_{\text{Ch}_K(\tilde{X})};$$

that is, the morphism  $\overline{(i^*)_K} \circ (\overline{(j^*)_K})^{-1}$  is the left inverse of  $(f^*)_K$ . Let  $x, y \in \text{Ch}_K(\tilde{X})$  and let  $z$  be the element of  $\text{Ch}_K(\tilde{X})$  such that

$$(f^*)_K(z) = (f^*)_K(x) \cdot (f^*)_K(y).$$

Then, composing the previous identity by  $\overline{(i^*)_K} \circ (\overline{(j^*)_K})^{-1}$ , one gets

$$z = \overline{(i^*)_K} \circ (\overline{(j^*)_K})^{-1} \circ (j^*)_K((\beta_*)_K(x) \cdot (\beta_*)_K(y)),$$

that is,

$$z = (i^*)_K \circ (\beta_*)_K(x) \cdot (i^*)_K \circ (\beta_*)_K(y),$$

i.e.,  $z = x \cdot y$ . Therefore, the morphism  $(f^*)_K$  preserves products. The lemma is proven.  $\square$

Now, one can prove the  $K/F$ -hermitian analogue of [Vishik 2005, Proposition 2.4(1)] the same way [Elman et al. 2008, Theorem 86.12] has been proven for the case of quadratic forms. We still write  $e_I$  for the class of  $e_I$  in  $\mathrm{Ch}_K(X)$ .

**Theorem 4.9.** *Let  $K/F$  be a quadratic separable field extension and  $h$  a split nondegenerate  $K/F$ -hermitian form. Let  $X$  be the variety of maximal totally isotropic subspaces in  $h$ . If  $h$  is of even dimension  $2n + 2$  then the set of all  $2^{n+1}$  monomials  $e_I$  for all subsets  $I \subset [0, 2n + 1]_{\mathrm{od}}$  is a basis of the  $\mathbb{Z}/2\mathbb{Z}$ -module  $\mathrm{Ch}_K(X)$ . Otherwise — if  $h$  is of odd dimension  $2n + 1$  — the set of all  $2^n$  monomials  $e_I$  for all subsets  $I \subset [0, 2n + 1]_{\mathrm{od}} \setminus \{1\}$  is a basis of the  $\mathbb{Z}/2\mathbb{Z}$ -module  $\mathrm{Ch}_K(X)$ .*

*Proof.* We use an induction on  $n$ . If  $\dim(h) = 2$ , the  $\mathbb{Z}/2\mathbb{Z}$ -module  $\mathrm{Ch}_K^1(X)$  is generated by the class  $e_1$  of a rational point (see decomposition (3.3) and Lemma 4.4). If  $\dim(h) = 1$ , the statement is obviously true since  $X = \mathrm{Spec}(F)$ .

Consider the exact sequence

$$0 \rightarrow \mathrm{Ch}_K(\tilde{X}) \xrightarrow{(i_*)_K} \mathrm{Ch}_K(X) \xrightarrow{(j^*)_K} \mathrm{Ch}_K(X \setminus \tilde{X}) \rightarrow 0$$

associated with the localization exact sequence for classical Chow groups. In both cases, the induction hypothesis and Corollary 4.6 imply that the set of monomials  $e_I$  for all  $I$  containing  $2n + 1$  is a basis of the image of  $(i_*)_K$ . On the other hand, Lemma 4.8, Proposition 3.4 and the induction hypothesis imply that, in both cases, the set of all the elements  $(j^*)_K(e_I)$  with  $2n + 1 \notin I$  is a basis of  $\mathrm{Ch}_K(X \setminus \tilde{X})$ . The conclusion follows.  $\square$

Also, one has  $e_{2k} = 0$  in  $\mathrm{Ch}_K(X)$  for all  $k \geq 2$  and also for  $k = 1$  if  $\dim(h)$  is even; see (4.13) and Corollary 4.14.

**Proposition 4.10.** *In  $\mathrm{CH}(X)$ , one has  $c_k(V\mathbb{1}/E) = 2e_k$  for all  $k \in [2, 2n + 1]$ ,  $c_{2n+2}(V\mathbb{1}/E) = 0$  and  $c_1(V\mathbb{1}/E) = 2e_1$  if the dimension of  $h$  is even.*

*Proof.* We apply  $(em \times \mathrm{Id}_X)_*$  to the cycle  $\gamma$ . Assume first that  $\dim(h) = 2n + 2$ . Since  $em_*(h^k) = 2H^{k+1}$  for all  $k \geq 0$ , decomposition (4.1) gives that

$$[\mathbb{P}(E)] = em_*(l_{2n+1}) \times 1 + \sum_{k=1}^{2n+1} 2H^{2n+2-k} \times e_k \quad \text{in } \mathrm{CH}^{2n+2}(\mathbb{P}_F(V) \times X).$$

On the other hand, by [Elman et al. 2008, Proposition 58.10], one has

$$[\mathbb{P}(E)] = \sum_{k=0}^{2n+2} H^{2n+2-k} \times c_k(V\mathbb{1}/E).$$

Hence, it follows from the projective bundle theorem (see [Elman et al. 2008, Theorem 53.10]) that  $c_k(V\mathbb{1}/E) = 2e_k$  for all  $k \in [1, 2n + 1]$  and  $c_{2n+2}(V\mathbb{1}/E) = 0$  in  $\mathrm{CH}(X)$ .

Assume that  $\dim(h) = 2n + 1$ . Decomposition (4.2) gives the identity

$$[\mathbb{P}(E)] = (em \times \text{Id}_X)_*(\gamma') + em_*(l_{2n-1}) \times 1 + \sum_{k=1}^{2n+1} 2H^{2n+2-k} \times e_k$$

in  $\text{CH}^{2n+2}(\mathbb{P}_F(V) \times X)$ . Moreover, it follows from the conditions defining the cycle  $\gamma'$  in (4.2) that  $(em \times \text{Id}_X)_*(\gamma')$  has the form  $H^{2n+1} \times a$  for some  $a \in \text{CH}^1(X)$ . Consequently, by using [Elman et al. 2008, Proposition 58.10] and the projective bundle theorem as has been done in the even case, one gets that  $c_k(V \mathbb{I}/E) = 2e_k$  for all  $k \in [2, 2n + 1]$  and  $c_{2n+2}(V \mathbb{I}/E) = 0$  in  $\text{CH}(X)$ .  $\square$

**Remark 4.11.** The above proposition shows in particular that for any nondegenerate  $K/F$ -hermitian form  $h$ , the algebraic cycles  $2e_k$ , for  $k \in [2, 2n + 1]$  and also for  $k = 1$  if  $\dim(h)$  is even, are defined over  $F$ , i.e., these are always rational cycles (the cycles  $e_k$  are defined over any splitting field of  $h$ ).

**Remark 4.12.** Suppose  $\dim(h) = 2n + 2$ . Let  $Y$  be the maximal orthogonal grassmannian of  $q$ . Since a totally  $h$ -isotropic  $K$ -subspace is also a totally  $q$ -isotropic  $F$ -subspace, we have a natural closed embedding  $im : X \hookrightarrow Y$ . Let us denote by  $z_k \in \text{CH}^k(Y)$ ,  $1 \leq k \leq 2n + 1$ , the generators of  $\text{CH}(Y)$  introduced by Vishik [2005], quadratic analogues of the elements  $e_k$  (recall that  $q$  is split). We write  $\mathcal{E}$  for the subbundle of the trivial bundle  $V \times Y$  over  $Y$  given by pairs  $(u, U)$  such that  $u \in U$ . Note that one has  $E = im^*(\mathcal{E})$ . Hence, since  $z_k$  is half the  $k$ -th Chern class of the vector bundle  $(V \times Y)/\mathcal{E}$  (see [Elman et al. 2008, Proposition 86.13] or [Vishik 2009, Proposition 2.1]) and  $\text{CH}(X)$  is torsion-free (since the  $K/F$ -hermitian form  $h$  is split; see [Köck 1991] for example), it follows from Proposition 4.10 that

$$e_k = im^*(z_k)$$

in  $\text{CH}(X)$  for all  $1 \leq k \leq 2n + 1$  (this can also be obtained by comparing the decompositions (4.1) and [Elman et al. 2008, (86.4)]). In the odd-dimensional case, the variety  $X$  is naturally a closed subvariety of the second to last grassmannian of  $q$ .

We set  $e_k = 0$  for  $k > 2n + 1$  (which makes sense since by Proposition 4.10 the vector bundle  $V \mathbb{I}/E$  is in both cases of rank  $2n + 2$  with trivial top Chern class). By duality, one has  $V \mathbb{I}/E^\perp \simeq E^\vee$ . Hence, in the Grothendieck ring  $K(X)$ , one has

$$[E^\perp/E] = [E^\perp] - [E] = -[E^\vee] - [E] = [(V \mathbb{I}/E)^\vee] + [V \mathbb{I}/E]$$

modulo the trivial bundles. Therefore, by the Whitney sum formula ([Elman et al. 2008, Proposition 54.7]), one has  $c(E^\perp/E) = c((V \mathbb{I}/E)^\vee) \circ c(V \mathbb{I}/E)$ . Thus, since  $\text{CH}(X)$  is torsion-free, it follows from Proposition 4.10 that, for even-dimensional  $h$ , one has

$$e_k^2 - 2e_{k-1}e_{k+1} + 2e_{k-2}e_{k+2} - \cdots + (-1)^{k-1}2e_1e_{2k-1} + (-1)^k e_{2k} = 0$$

in  $\text{CH}(X)$ , for all  $k \geq 1$ , and for odd-dimensional  $h$ , one has

$$e_k^2 - 2e_{k-1}e_{k+1} + 2e_{k-2}e_{k+2} - \cdots + (-1)^{k-1}c_1(V\mathbb{1}/E) \cdot e_{2k-1} + (-1)^k e_{2k} = 0$$

in  $\text{CH}(X)$ , for all  $k \geq 2$  (recall that  $E^\perp = E$  if  $\dim(h)$  is even and  $E^\perp/E$  has rank 2 otherwise). Moreover, by using decomposition (3.3) and an induction on  $n$ , one obtains that, for odd-dimensional  $h$ , the group  $\text{Ch}_K^1(X)$  is trivial. Consequently, in both even and odd situations, one has

$$(4.13) \quad e_k^2 = e_{2k}$$

in  $\text{Ch}_K(X)$ , for all  $k \geq 2$ , and also for  $k = 1$  if  $\dim(h)$  is even. The relations (4.13) are also a consequence of the respective result for the  $z_k$ 's, see Remark 4.12 and [Elman et al. 2008, (86.15)].

**Corollary 4.14.** *One has  $e_k^2 = 0$  in  $\text{Ch}_K(X)$  for all  $k \geq 2$ , and also for  $k = 1$  if  $\dim(h)$  is even.*

*Proof.* We proceed by induction on  $n$ . For  $n = 0$ , the conclusion is true by dimensional reasons. By (4.13), the conclusion is true for  $k > n$ . Let  $1 \leq k \leq n$  (or  $2 \leq k \leq n$  if  $\dim(h)$  is odd). By codimensional reasons, it suffices to prove that  $(j^*)_K(e_k^2) = 0$  in  $\text{Ch}_K^{2k}(X \setminus \tilde{X})$  to get that  $e_k^2 = 0$  in  $\text{Ch}_K^{2k}(X)$  (see the exact sequence in the proof of Theorem 4.9). Moreover, one has  $(j^*)_K(e_k^2) = (f^*)_K(\tilde{e}_k^2)$  (see Lemma 4.8 and its proof). Since  $\tilde{e}_k^2 = 0$  in  $\text{Ch}_K(\tilde{X})$  by the induction hypothesis, the proof is complete.  $\square$

We have obtained that  $e_{2k} = 0$  in  $\text{Ch}_K(X)$  for all  $k \geq 2$  and also for  $k = 1$  if  $\dim(h)$  is even. We are now able to determine the multiplicative structure for the ring  $\text{Ch}_K(X)$  the same way it has been done for the case of quadratic forms.

**Proposition 4.15.** *There is a ring isomorphism*

$$\text{Ch}_K(X) \simeq \bigotimes_{m \leq k \leq n} (\mathbb{Z}/2\mathbb{Z}[e_{2k+1}] / (e_{2k+1}^2)),$$

with  $m = 0$  if the dimension of  $h$  is even and  $m = 1$  otherwise.

*Proof.* Assume first that  $h$  is of even dimension. Let us denote by  $\mathcal{R}$  the factor ring of the polynomial ring  $\mathbb{Z}/2\mathbb{Z}[t_0, t_1, \dots, t_n]$  by the ideal generated by the monomials  $t_k^2$ , for  $0 \leq k \leq n$ . Then, by Corollary 4.14, the ring homomorphism

$$\varphi : \mathcal{R} \rightarrow \text{Ch}_K(X), \quad t_k \mapsto e_{2k+1}$$

is well defined. Furthermore, it follows from Theorem 4.9 that  $\varphi$  is surjective. Since the classes of the monomials  $t_0^{r_0} t_1^{r_1} \cdots t_n^{r_n}$ , with  $r_k = 0$  or  $1$  for every  $k$ , generate  $\mathcal{R}$ , the ring homomorphism  $\varphi$  is also injective by Theorem 4.9. One proceeds the same way for  $h$  of odd dimension, except there is no variable  $t_0$ .  $\square$

## 5. General maximal unitary grassmannian

In this section, we use notation introduced in the previous sections. We do not make any assumption on the isotropy of the nondegenerate  $K/F$ -hermitian form  $h$ .

For any scheme  $Y$  over  $F$ , we write  $\bar{Y}$  for  $Y \times \operatorname{Spec} F(X)$ , where  $F(X)$  is the function field of the maximal unitary grassmannian  $X$  (note that  $K \otimes_F F(X)$  is still a field). Following the path set by Vishik [2005], we describe the subring

$$\overline{\operatorname{Ch}}_K(X) := \operatorname{Im}(\operatorname{Ch}_K(X) \rightarrow \operatorname{Ch}_K(\bar{X}))$$

of *rational* elements; see Theorem 5.7 (actually, this description does not depend on the choice of a splitting field of  $h$  which does not split  $K$ ). We use similar notation and vocabulary for classical Chow rings and certain products of  $F$ -varieties.

The exposition in this section follows the thread of [Elman et al. 2008, §87]. The proofs are sometimes very similar (the proof of Theorem 5.7 compared to the original [Elman et al. 2008, Theorem 87.7], for example).

We consider the elements  $e_k \in \operatorname{CH}^k(\bar{X})$  introduced in the previous section.

**Proposition 5.1.** *If  $\dim(h) = 2n + 2$  then the elements  $(e_k \times 1) + (1 \times e_k)$  in  $\operatorname{CH}(\bar{X}^2)$  are rational for all  $k \in [1, 2n + 1]$ .*

*If  $\dim(h) = 2n + 1$  then the elements  $(e_k \times 1) + (1 \times e_k)$  in  $\operatorname{Ch}_K(\bar{X}^2)$  are rational for all  $k \in [2, 2n + 1]$ .*

*Proof.* Suppose first that  $\dim(h) = 2n + 2$ . Here we use notation and material introduced in Remark 4.12. Let  $k \in [1, 2n + 1]$ . Since the element  $(z_k \times 1) + (1 \times z_k) \in \operatorname{CH}(\bar{Y}^2)$  is rational (see [Elman et al. 2008, Proposition 86.17 and Corollary 87.3]) and one has  $\operatorname{im}^*(z_k) = e_k$ , the element  $(e_k \times 1) + (1 \times e_k) \in \operatorname{CH}(\bar{X}^2)$  is also rational.

Suppose now that  $\dim(h) = 2n + 1$ . In that case, let us denote by  $\operatorname{im}$  the natural closed embedding  $X \hookrightarrow Y_{2n}$  of the variety  $X$  into the second to last grassmannian of  $q$ . Let us denote by  $z_k^{-1} \in \operatorname{CH}^k(\bar{Y}_{2n})$ ,  $1 \leq k \leq 2n + 1$ , the  $Z$ -type generators of  $\operatorname{CH}^k(\bar{Y}_{2n})$  (see [Vishik 2009, §2]). Note that it follows from [Vishik 2009, Proposition 2.1] and Proposition 4.10 that, for any  $2 \leq k \leq 2n + 1$ , one has

$$\operatorname{im}^*(z_k^{-1}) = e_k$$

in  $\operatorname{CH}(\bar{X})$ . Moreover, we write  $Y$  for the maximal grassmannian of  $q$  and  $z_k \in \operatorname{CH}^k(Y)$ ,  $1 \leq k \leq 2n$ , for the generators of  $\operatorname{CH}(\bar{Y})$  introduced by Vishik [2005]. We consider the diagram

$$Y_{2n} \xleftarrow{f} Y_{2n, 2n+1} \xrightarrow{g} Y,$$

where  $Y_{2n, 2n+1}$  is the partial orthogonal flag variety of totally  $q$ -isotropic  $F$ -vector subspaces of dimension  $2n$  and  $2n + 1$ , and the morphisms  $f$  and  $g$  are the natural projections. By [Vishik 2009, Lemma 2.1], there is an element  $c_1 \in \operatorname{CH}^1(Y_{2n, 2n+1})$

such that, for any  $2 \leq k \leq 2n + 1$ , one has

$$f^*(z_k^{-1}) = c_1 \cdot g^*(z_{k-1}) + g^*(z_k)$$

(with  $z_{2n+1} = 0$ ). Let  $e \in \text{CH}(Y_{2n, 2n+1})$  be the class of a generic point. Using the previous identity and the projection formula, one get that

$$(5.2) \quad z_k^{-1} = f_*(e \cdot c_1 \cdot g^*(z_{k-1})) + f_*(e \cdot g^*(z_k))$$

for any  $2 \leq k \leq 2n + 1$ . Since the element  $(z_k \times 1) + (1 \times z_k) \in \text{CH}(\bar{Y}^2)$  is rational for any  $1 \leq k \leq 2n + 1$ , the element

$$(f_*)^{\times 2}((e \times e) \cdot (c_1 \times 1 + 1 \times c_1) \cdot (g^*)^{\times 2}((z_{k-1} \times 1) + (1 \times z_{k-1}))) \\ + (f_*)^{\times 2}((e \times e) \cdot (g^*)^{\times 2}((z_k \times 1) + (1 \times z_k)))$$

is rational for any  $2 \leq k \leq 2n + 1$ . Furthermore, by (5.2), the latter cycle can be rewritten as

$$(5.3) \quad (z_k^{-1} \times 1) + (1 \times z_k^{-1}) + f_*(e \cdot g^*(z_{k-1})) \times f_*(e \cdot c_1) + f_*(e \cdot c_1) \times f_*(e \cdot g^*(z_{k-1})).$$

Since  $f_*(e \cdot c_1) \in \text{CH}^1(Y_{2n})$  and  $\text{Ch}_K^1(\bar{X})$  is trivial (see paragraph right before Corollary 4.14), the conclusion is obtained by taking the image of the rational cycle (5.3) under  $im^*$  and then projecting to  $\text{Ch}_K(\bar{X})$ .  $\square$

For every subset  $I \subset [1, 2n + 1]_{\text{od}}$ , with  $1 \notin I$  if  $\dim(h) = 2n + 1$ , we set

$$(5.4) \quad x_I := \prod_{k \in I} ((e_k \times 1) + (1 \times e_k)) \in \overline{\text{Ch}}_K(X^2).$$

**Lemma 5.5.** *For any subsets  $I, J \subset [1, 2n + 1]_{\text{od}}$ , with  $1 \notin I \cup J$  if  $\dim(h) = 2n + 1$ , one has in  $\text{Ch}_K(\bar{X})$ ,*

$$(x_J)_*(e_I) = \begin{cases} e_{I \cap J} & \text{if } I \cup J = [1, 2n + 1]_{\text{od}}, \\ & \text{or } I \cup J = [3, 2n + 1]_{\text{od}} \text{ for odd-dimensional } h; \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Assume that  $\dim(h) = 2n + 2$ . Since

$$x_J = \sum_{J_1 \subset J} e_{J_1} \times e_{J \setminus J_1},$$

one has

$$(x_J)_*(e_I) = \sum_{J_1 \subset J} \deg(e_I \cdot e_{J_1}) e_{J \setminus J_1},$$

in  $\text{Ch}_K(\bar{X})$ , where  $\deg : \text{Ch}_K(\bar{X}) \rightarrow \text{Ch}_K(\text{Spec } F(X)) = \mathbb{Z}/2\mathbb{Z}$  is the degree homomorphism associated with the push-forward of the structure morphism. Therefore,



it suffices to show that for any subsets  $I, J_1 \subset [1, 2n+1]_{\text{od}}$ , one has

$$(5.6) \quad \deg(e_I \cdot e_{J_1}) = \begin{cases} 1 \bmod 2 & \text{if } J_1 = [1, 2n+1]_{\text{od}} \setminus I; \\ 0 \bmod 2 & \text{otherwise,} \end{cases}$$

to get the conclusion. For  $J_1 = [1, 2n+1]_{\text{od}} \setminus I$ , this follows from [Corollary 4.7](#). For  $J_1 \neq [1, 2n+1]_{\text{od}} \setminus I$ , using [Corollary 4.14](#) (or [Theorem 4.9](#)), one gets that  $e_I \cdot e_{J_1}$  is either zero or the monomial  $e_L$  for some  $L$  different from  $[1, 2n+1]_{\text{od}}$ , thus  $\deg(e_I \cdot e_{J_1}) = 0 \bmod 2$ . The proof in the odd case is similar.  $\square$

We are now able to prove the  $K/F$ -hermitian analogue of [\[Vishik 2005, Main Theorem 5.8\]](#).

**Theorem 5.7.** *Let  $K/F$  be a quadratic separable field extension and  $h$  a nondegenerate  $K/F$ -hermitian form of dimension  $2n+2$  or  $2n+1$ . Let  $X$  be the variety of maximal totally isotropic subspaces in  $h$ . Then the ring  $\overline{\text{Ch}}_K(X)$  is generated by all  $e_k$ ,  $k \in [3, 2n+1]_{\text{od}}$  and also  $k = 1$  for  $h$  of even dimension, such that  $e_k \in \overline{\text{Ch}}_K(X)$ .*

*Proof.* Assume that  $\dim(h) = 2n+2$ . By [Theorem 4.9](#), one has to show that if an element

$$\alpha = \sum a_I e_I \in \text{Ch}_K(\bar{X})$$

(with  $I \subset [1, 2n+1]_{\text{od}}$  and  $a_I \in \mathbb{Z}/2\mathbb{Z}$ ) is rational, then for every  $I$  satisfying  $a_I = 1$  and any  $k \in I$ , the element  $e_k \in \text{Ch}_K(\bar{X})$  is rational.

One may assume that  $\alpha$  is homogeneous. We induct on the number of nonzero coefficients of  $\alpha$ .

Let  $I$  be a subset with largest  $|I|$  such that  $a_I = 1$ . Let  $k \in I$  and set

$$J = ([1, 2n+1]_{\text{od}} \setminus I) \cup \{k\}.$$

We claim that  $(x_J)_*(\alpha) = e_k$ . By [Lemma 5.5](#), it suffices to prove that if  $I'$  is a subset such that  $a_{I'} = 1$  and  $I' \cup J = [1, 2n+1]_{\text{od}}$  then  $I' = I$ . Since  $I' \cup J = [1, 2n+1]_{\text{od}}$ , one has

$$I' = ([1, 2n+1]_{\text{od}} \setminus J) \cup (J \cap I').$$

By maximality of  $|I|$ , the subset  $J \cap I'$  is either empty or consists of a single element. Hence, it follows from the homogeneity of  $\alpha$  that  $J \cap I' = \{k\}$ , that is  $I' = I$ . The claim is proven.

Thus, since  $x_J$  is rational, the cycle  $e_k$  is also rational, for all  $k \in I$ . Consequently, the elements  $e_I$  and  $\alpha - e_I$  are rational. By the induction hypothesis, every element  $e_k$  appearing in the decomposition of  $\alpha - e_I$  is rational and it is therefore so for  $\alpha$ . This concludes the even case and the odd case can be treated similarly.  $\square$

## 6. The invariant $J(h)$

In this section, we define a new invariant of nondegenerate  $K/F$ -hermitian forms on the model of the  $J$ -invariant for nondegenerate quadratic forms defined by Vishik [2005] (although this section follows the thread of [Elman et al. 2008, §88], where the latter is defined in the “opposite way”).

Let  $h$  be a nondegenerate  $K/F$ -hermitian form of dimension  $2n + 2$  or  $2n + 1$  and  $X$  the variety of maximal totally isotropic subspaces in  $h$ . We use notation introduced in the previous sections. We still denote by  $e_k$  the generators of  $\text{Ch}_K(\bar{X})$ . The discrete  $J$ -invariant  $J(h)$  is defined as follows:

$$J(h) = \begin{cases} \{k \in [1, 2n + 1]_{\text{od}} \text{ with } e_k \in \overline{\text{Ch}}_K(X)\} & \text{if } \dim(h) = 2n + 2; \\ \{k \in [3, 2n + 1]_{\text{od}} \text{ with } e_k \in \overline{\text{Ch}}_K(X)\} & \text{if } \dim(h) = 2n + 1. \end{cases}$$

For a subset  $I \subset [1, 2n + 1]$  let us denote by  $\|I\|$  the sum of all  $k \in I$ .

**Proposition 6.1.** *The biggest codimension  $i$  such that  $\overline{\text{Ch}}_K^i(X) \neq 0$  is equal to  $\|J(h)\|$ .*

*Proof.* The element  $\prod_{k \in J(h)} e_k \in \overline{\text{Ch}}_K(X)$  is nontrivial by Theorem 4.9 and has the biggest codimension among the nontrivial elements of  $\overline{\text{Ch}}_K(X)$  by Theorem 5.7.  $\square$

**Proposition 6.2.** *A nondegenerate  $K/F$ -hermitian form  $h$  is split if and only if  $J(h)$  is maximal.*

*Proof.* If  $h$  is split then the fact that  $J(h)$  is maximal follows from the definition. If  $J(h)$  is maximal then, by Corollary 4.7, the class of a rational point of  $\bar{X}$  belongs to  $\overline{\text{Ch}}_K(X)$ . Consequently, the variety  $X$  admits a closed point  $x$  of odd degree (recall that the degree map is well defined on  $\text{Ch}_K$ ). Combining the fact that the residue field  $F(x)$  is a splitting field of  $h$  with Springer’s theorem for quadrics, one gets the identities

$$\lfloor \dim(h)/2 \rfloor = i_0(h_{F(x)}) = i_0(q_{F(x)})/2 = i_0(q)/2 = i_0(h).$$

Therefore  $h$  is split.  $\square$

**Lemma 6.3.** *Let  $h = \tilde{h} \perp \mathbb{H}$ . Then  $J(h) = J(\tilde{h}) \cup \{2n + 1\}$ .*

*Proof.* Since  $e_{2n+1} = [\tilde{X}]$  (see Lemma 4.4), one has  $2n + 1 \in J(h)$ . Let  $i \leq 2n - 1$ . From decomposition (3.3) (where  $d = 2n + 1$ ), one gets  $\text{Ch}_K^i(X) \simeq \text{Ch}_K^i(\tilde{X})$  with  $e_i$  corresponding to  $\tilde{e}_i$  by Lemma 4.5(i). The conclusion follows.  $\square$

**Corollary 6.4.** *Let  $h$  and  $h'$  be Witt-equivalent  $K/F$ -hermitian forms with  $h \simeq h' \perp j\mathbb{H}$ . Then  $J(h) = J(h') \cup \{2n + 1, 2n - 1, \dots, 2n + 1 - 2(j - 1)\}$ .*

The following statement is the  $\text{Ch}_K$ -version of [Elman et al. 2008, Lemma 88.5] of Karpenko and Merkurjev for classical Chow groups (see Remark 2.1).

**Proposition 6.5.** *Let  $Z$  be a smooth  $F$ -variety and  $Y$  an equidimensional  $F$ -variety. Given an integer  $m$  such that for any nonnegative integer  $i$  and any point  $y \in Y$  of codimension  $i$  the change of field homomorphism*

$$\mathrm{Ch}_K^{m-i}(Z) \longrightarrow \mathrm{Ch}_K^{m-i}(Z_{F(y)})$$

*is surjective, the change of field homomorphism*

$$\mathrm{Ch}_K^m(Y) \longrightarrow \mathrm{Ch}_K^m(Y_{F(Z)})$$

*is also surjective.*

For any integer  $k$ , let us denote by  $X_k$  the  $F$ -variety of  $k$ -dimensional totally isotropic subspaces in  $h$  (so  $X_{\lfloor \dim(h)/2 \rfloor} = X$  and  $X_k = \emptyset$  for  $k \notin [0, \lfloor \dim(h)/2 \rfloor]$ ). By the result [Karpenko 2012, Corollary 7.3] (which is also a consequence of [Karpenko 2000, Theorem 15.8]), for any integer  $k$ , there is a Chow motivic decomposition with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients

$$(6.6) \quad M(X_k) \simeq M_k \oplus M,$$

where  $M$  is a sum of shifts of  $M(\mathrm{Spec}(K))$  and  $M_k$  splits as a sum of Tate motives over any splitting field of  $h$ . The motive  $M_k$  is defined by  $X_k$  uniquely up to an isomorphism and is called the *essential* motive of  $X_k$ .

**Lemma 6.7.** *The change of field homomorphism  $\mathrm{Ch}_K^i(X) \rightarrow \mathrm{Ch}_K^i(X_{F(X_1)})$  is surjective for any  $i \leq 2n$  if  $\dim(h) = 2n + 2$  and for any  $i \leq 2n - 1$  if  $\dim(h) = 2n + 1$ .*

*Proof.* By Proposition 6.5, it is sufficient to prove that for any  $x \in X$  the change of field homomorphism  $\mathrm{Ch}_K^i(X_1) \rightarrow \mathrm{Ch}_K^i(X_{1_{F(x)}})$  is surjective, for any  $i \leq 2n$  if  $\dim(h) = 2n + 2$  and for any  $i \leq 2n - 1$  if  $\dim(h) = 2n + 1$ , to get the conclusion.

It follows from decomposition (6.6) that

$$(6.8) \quad \mathrm{Ch}_K(X_1) \simeq \mathrm{Ch}_K(M_1).$$

Furthermore, by [Karpenko 2012, Corollary 9.6], one has the following Chow motivic decomposition with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients

$$(6.9) \quad M(Q) \simeq M_1 \oplus M_1\{1\}$$

(where  $Q$  is the smooth projective quadric associated with the nondegenerate quadratic form  $q : V \rightarrow F$ ,  $v \mapsto h(v, v)$ ).

Combining (6.8) with (6.9), we see that it suffices to show that for any  $x \in X$  the change of field homomorphism

$$(6.10) \quad \mathrm{Ch}_K^i(Q) \rightarrow \mathrm{Ch}_K^i(Q_{F(x)})$$

is surjective, for any  $i \leq 2n$  if  $\dim(h) = 2n + 2$  and for any  $i \leq 2n - 1$  if  $\dim(h) = 2n + 1$ , to get the conclusion. In fact, (6.10) is already surjective at the level

of integral Chow groups. Indeed, since  $F(x)$  is a splitting field of the hermitian form  $h$ , one has  $i_0(q_{F(x)})$  is equal to  $2n + 2$  or  $2n$  depending on whether  $\dim(h)$  is respectively even or odd. Therefore, the group  $\mathrm{CH}^i(Q_{F(x)})$  is generated by  $h^i$  (always rational) for  $i \leq 2n$  or  $i \leq 2n - 1$  depending on whether  $\dim(h)$  is respectively even or odd (see [Fino 2013, §1] for example).  $\square$

**Corollary 6.11.**  $J(h) \subset J(h_{F(X_1)}) \subset J(h) \cup \{2n + 1\}$ .

The following proposition relates the set  $J(h)$  and the absolute Witt indices of  $h$ . It follows from Corollaries 6.4 and 6.11.

**Proposition 6.12.** *Let  $h$  be a nondegenerate  $K/F$ -hermitian form of dimension  $2n + 2$  or  $2n + 1$  with height  $\mathfrak{h}(h)$ . Then  $J(h)$  contains the complementary of the set*

$$\{2n + 1 - 2j_0(h), 2n + 1 - 2j_1(h), \dots, 2n + 1 - 2j_{\mathfrak{h}(h)-1}(h)\}$$

*in  $[1, 2n + 1]_{\mathrm{od}}$ , excluding 1 for  $h$  of odd dimension. In particular,  $|J(h)| \geq n - \mathfrak{h}(h)$  and the inequality is strict for  $h$  of even dimension.*

## 7. Steenrod operations

Let  $h$  be a nondegenerate  $K/F$ -hermitian form on  $V$  of dimension  $2n + 2$  or  $2n + 1$  and let  $X$  be the variety of maximal totally isotropic subspaces in  $h$ . This section is the hermitian replica of [Elman et al. 2008, §89], where we compute the Steenrod operations on  $\mathrm{Ch}(\bar{X})$ .

We use notation introduced in the previous sections and we write  $\pi_X$  and  $\pi_Q$  for the respective compositions  $p_X \circ \mathrm{in}$  and  $p_Q \circ \mathrm{in}$ . Let  $\mathcal{L}$  be the canonical line bundle over  $\mathbb{P}(E)$  and  $\mathcal{T}$  the relative tangent bundle of  $\pi_X$ . By [Elman et al. 2008, Example 104.20], there is an exact sequence of vector bundles over  $\mathbb{P}(E)$ :

$$0 \rightarrow \mathbb{1} \rightarrow \mathcal{L} \otimes \pi_X^*(E) \rightarrow \mathcal{T} \rightarrow 0.$$

Hence, since  $c_i(E) = 0$  in  $\mathrm{Ch}(\bar{X})$  for all  $i > 1$  and  $c_1(E) = 0$  in  $\mathrm{Ch}(\bar{X})$  for even-dimensional  $h$  and  $c_1(E) = 0$  in  $\mathrm{Ch}_K(\bar{X})$  for odd-dimensional  $h$  (follows from Proposition 4.10 and the fact that  $\mathrm{Ch}_K^1(\bar{X})$  is trivial in the odd case), one deduces from the Whitney sum formula and [Fulton 1984, Remark 3.2.3 (b)] that

$$c(\mathcal{T}) = c(\mathcal{L} \otimes \pi_X^*(E)) = c(\mathcal{L} \otimes \mathbb{1}^{2r}) = c(\mathcal{L})^{2r}$$

in  $\mathrm{Ch}(\bar{X})$  if  $\dim(h)$  is even and in  $\mathrm{Ch}_K(\bar{X})$  otherwise, with  $r = \lfloor \dim(h)/2 \rfloor$  (recall that  $E$  has rank  $2r$ ). Furthermore, since  $\mathcal{L}$  coincides with the pull-back with respect to  $\pi_Q$  of the canonical line bundle over  $Q$ , one has  $c(\mathcal{L}) = 1 + \pi_Q^*(h^1)$  in  $\mathrm{CH}(Q)$ . Consequently, one has

$$(7.1) \quad c(\mathcal{T}) = (1 + \pi_Q^*(h^1))^{2r}$$

in  $\mathrm{Ch}(\bar{X})$  if  $\dim(h)$  is even, and in  $\mathrm{Ch}_K(\bar{X})$  otherwise.

The following statement is the  $K/F$ -hermitian analogue of the result [Vishik 2005, Theorem 4.1] for quadratic forms.

**Theorem 7.2.** *Assume  $\text{char}(F) \neq 2$ . Let  $h$  be a nondegenerate  $K/F$ -hermitian form of dimension  $2n + 2$  or  $2n + 1$  and  $X$  the variety of maximal totally isotropic subspaces in  $h$ . Let  $S_{\bar{X}} : \text{Ch}(\bar{X}) \rightarrow \text{Ch}(\bar{X})$  denote the Steenrod operation of cohomological type on  $\bar{X}$ . Then one has*

$$S_{\bar{X}}^i(e_k) = \binom{k}{i} e_{k+i}$$

*in  $\text{Ch}(\bar{X})$  if  $\dim(h)$  is even, and in  $\text{Ch}_K(\bar{X})$  otherwise, for all  $i$  and  $k \in [2, 2n + 1]$  and also for  $i = k = 1$  if  $\dim(h)$  is even.*

*Proof.* By [Elman et al. 2008, Corollary 78.5], one has

$$S_Q(l_{2n+1-k}) = (1 + h^1)^{2r+k} \cdot l_{2n+1-k}.$$

It follows from (4.3), (7.1) and [Elman et al. 2008, Theorem 61.9 and Proposition 61.10] that one has

$$\begin{aligned} S_{\bar{X}}(e_k) &= S_X \circ \pi_{X*} \circ \pi_Q^*(l_{2n+1-k}) \\ &= \pi_{X*} \circ c(-\mathcal{T}) \circ S_{\mathbb{P}(E)} \circ \pi_Q^*(l_{2n+1-k}) \\ &= \pi_{X*}((1 + \pi_Q^*(h^1))^{-2r} \cdot \pi_Q^* \circ S_Q(l_{2n+1-k})) \\ &= \pi_{X*} \circ \pi_Q^*((1 + h^1)^{-2r} \cdot (1 + h^1)^{2r+k} \cdot l_{2n+1-k}) \\ &= \pi_{X*} \circ \pi_Q^*((1 + h^1)^k \cdot l_{2n+1-k}) \\ &= \sum_{i \geq 0} \binom{k}{i} \pi_{X*} \circ \pi_Q^*(l_{2n+1-k-i}) = \sum_{i \geq 0} \binom{k}{i} e_{k+i} \end{aligned}$$

in  $\text{Ch}(\bar{X})$  if  $\dim(h)$  is even, and in  $\text{Ch}_K(\bar{X})$  otherwise. □

Note that Theorem 7.2 is also a direct consequence of the quadratic case [Vishik 2005, Theorem 4.1]; see Remark 4.12.

## 8. Canonical dimension

In this section, we compute the canonical 2-dimension  $\text{cdim}_2(X)$  of the maximal unitary grassmannian  $X$  associated with a nondegenerate  $K/F$ -hermitian form  $h$  in terms of the  $J$ -invariant  $J(h)$ .

We recall the definition of the canonical 2-dimension of a variety (see [Karpenko 2010] for an introduction on canonical dimension and for a more geometric definition).

Let  $X$  be an  $F$ -variety. An isotropy field  $L$  of  $X$  is an extension  $L/F$  such that  $X(L) \neq \emptyset$  (note that if  $X$  is a maximal unitary grassmannian then this is the same thing as a splitting field of the corresponding hermitian form  $h$ ).

An isotropy field  $E$  is called *2-generic* if for any isotropy field  $L$  there is an  $F$ -place  $E \rightarrow L'$  for some finite extension  $L'/L$  of odd degree (see [Elman et al. 2008, §103] for an introduction to  $F$ -places). For example, the function field  $F(X)$  is 2-generic (because it is generic).

The *canonical 2-dimension*  $\text{cdim}_2(X)$  of  $X$  is the minimum of the transcendence degree  $\text{tr.deg}_F(E)$  over all 2-generic field extensions  $E/F$ . If  $X$  is smooth then  $\text{cdim}_2(X) \leq \dim(X)$ . One says that  $X$  is *2-incompressible* if  $\text{cdim}_2(X) = \dim(X)$ .

The proof of the theorem below is a very slight modification of the corresponding one for quadratic forms; see [Elman et al. 2008, Theorem 90.3]. Namely, at some point, one just has to consider the Chow rings  $\text{Ch}_K$ . Nevertheless, we write down this modified version.

The two following facts about  $F$ -places are used in the proof (contained in [Elman et al. 2008, §103]).

One can compose  $F$ -places. In particular, any  $F$ -place  $E \rightarrow L$  can be restricted to a subfield  $E'$  of  $E$  containing  $F$  (since field extensions over  $F$  are  $F$ -places).

For any proper  $F$ -variety  $Y$  equipped with an  $F$ -place  $\pi : F(Y) \rightarrow L$ , there is a morphism  $\text{Spec}(L) \rightarrow Y$ .

The following  $\text{Ch}_K$ -version of the result [Elman et al. 2008, Proposition 58.18] for classical Chow groups will also be needed (see Remark 2.1).

**Proposition 8.1.** *Let  $Z$  be a smooth  $F$ -scheme and  $Y$  a  $Z$ -scheme. Suppose there is a flat morphism of  $F$ -schemes  $f : Y \rightarrow Y'$ . If for every  $y' \in Y'$ , the pull-back homomorphism  $\text{Ch}_K(Z) \rightarrow \text{Ch}_K(Y \times_{Y'} \text{Spec}(F(y')))$  is surjective then the homomorphism*

$$\text{Ch}_K(Y') \otimes \text{Ch}_K(Z) \rightarrow \text{Ch}_K(Y), \quad \alpha \otimes \beta \mapsto (f)_K^*(\alpha) \cdot \beta$$

*is surjective.*

**Theorem 8.2.** *Let  $h$  be a nondegenerate  $K/F$ -hermitian form and  $X$  the associated maximal unitary grassmannian. Then*

$$\text{cdim}_2(X) = \dim(X) - J(h).$$

*Proof.* Let  $E$  be a 2-generic isotropy field of  $X$  with minimum transcendence degree  $\text{cdim}_2(X)$  and  $Y$  be the closure of the  $F$ -morphism  $\text{Spec}(E) \rightarrow X$ . Since  $F(Y)$  is a subfield of  $E$ , one has

$$(8.3) \quad \text{tr.deg}_F(E) \geq \dim(Y).$$

Moreover, since  $E$  is 2-generic, there is an  $F$ -place  $E \rightarrow L$ , with  $L$  a field extension of  $F(X)$  of odd degree. Restricting  $E \rightarrow L$  to  $F(Y)$ , one gets an  $F$ -place

$\pi : F(Y) \rightarrow L$ , and thus the existence of a morphism  $f : \operatorname{Spec}(L) \rightarrow Y$ . Let  $g : \operatorname{Spec}(L) \rightarrow X$  be the morphism induced by the extension  $L/F(X)$  and let  $Z$  be the closure of the image of  $(f, g) : \operatorname{Spec}(L) \rightarrow Y \times X$ . Then  $[F(Z) : F(X)]$  is odd since it divides  $[L : F(X)]$ . Thus, the image of  $[Z]$  under the composition

$$\operatorname{Ch}(Y \times X) \xrightarrow{(i \times 1)_*} \operatorname{Ch}(X^2) \xrightarrow{p_2} {}_* \operatorname{Ch}(X),$$

where  $i : Y \rightarrow X$  is the closed embedding and  $p_2$  is the second projection, is equal to  $[X]$ . It follows that  $(i \times 1)_*([Z])_{F(X)} \neq 0$  in  $\overline{\operatorname{Ch}}_K(X^2) \subset \operatorname{Ch}_K(\bar{X}^2)$ .

We claim that the homomorphism

$$(8.4) \quad \operatorname{Ch}_K(Y) \otimes \operatorname{Ch}_K(X^2) \rightarrow \operatorname{Ch}_K(Y \times X), \quad \alpha \otimes \beta \mapsto (p_Y)_K^*(\alpha) \cdot (i \times 1)_K^*(\beta),$$

where  $p_Y$  is the projection  $Y \times X \rightarrow Y$ , is surjective. By [Proposition 8.1](#), it suffices to show that for any  $y \in Y$  the homomorphism  $\operatorname{Ch}_K(X^2) \rightarrow \operatorname{Ch}_K(X_{F(y)})$  associated with the pull-back of the induced morphism  $\operatorname{Spec}(F(y)) \times X \rightarrow X^2$  (where the second factor is the identity) is surjective to prove the claim. The pull-back homomorphism  $\operatorname{Ch}(X^2) \rightarrow \operatorname{Ch}(X_{F(y)})$  sends an element in the fiber of the rational cycle  $x_I \in \operatorname{Ch}(\bar{X}^2)$  (introduced in [\(5.4\)](#)) under restriction to  $e_I \in \operatorname{Ch}(X_{F(y)}) \simeq \operatorname{Ch}(\bar{X})$ . Therefore, since the classes  $e_I$  generate  $\operatorname{Ch}_K(\bar{X})$  ([Theorem 4.9](#)), one deduces that the composition

$$\operatorname{Ch}(X^2) \rightarrow \operatorname{Ch}(X_{F(y)}) \rightarrow \operatorname{Ch}_K(X_{F(y)})$$

is surjective. Therefore the homomorphism  $\operatorname{Ch}_K(X^2) \rightarrow \operatorname{Ch}_K(X_{F(y)})$  is also surjective and the claim is proven.

As a consequence of [\[Elman et al. 2008, Proposition 49.20 and 58.17\]](#), one gets that the diagram

$$\begin{array}{ccc} \operatorname{Ch}_K(Y) \otimes \operatorname{Ch}_K(X^2) & \longrightarrow & \operatorname{Ch}_K(Y \times X) \\ \downarrow i_* \otimes 1 & & \downarrow (i \times 1)_* \\ \operatorname{Ch}_K(X) \otimes \operatorname{Ch}_K(X^2) & \longrightarrow & \operatorname{Ch}_K(X^2) \\ \downarrow & & \downarrow \\ \operatorname{Ch}_K(\bar{X}) \otimes \operatorname{Ch}_K(X^2) & \longrightarrow & \operatorname{Ch}_K(\bar{X}^2) \end{array}$$

where the horizontal arrows are defined as in [\(8.4\)](#), is commutative. Using the fact that  $(i \times 1)_*([Z])_{F(X)} \neq 0$  in  $\overline{\operatorname{Ch}}_K(X^2)$  and the claim, one obtains that the composition

$$\operatorname{Ch}_K(Y) \rightarrow \operatorname{Ch}_K(X) \rightarrow \operatorname{Ch}_K(\bar{X})$$

is nontrivial. Therefore, by [Proposition 6.1](#) one has  $\dim(Y) \geq \dim(X) - J(h)$ . Thus,

combining with inequality (8.3), one gets

$$\mathrm{cdim}_2(X) \geq \dim(X) - J(h).$$

By Proposition 6.1, there is a closed subvariety  $Y \subset X$  of dimension  $\dim(X) - J(h)$  such that  $[Y] \neq 0$  in  $\overline{\mathrm{Ch}}_K(X)$ . In particular, one has  $[Y] \neq 0$  in  $\overline{\mathrm{Ch}}(X)$ . From this moment on, the remaining of the proof of the second inequality is strictly the same as the one in the orthogonal case; see [Elman et al. 2008, Theorem 90.3].  $\square$

We recall that if  $\dim(h) = 2n + 1$  then

$$\dim(X) = n(n + 2) = 3 + 5 + \cdots + (2n - 1) + (2n + 1)$$

and if  $\dim(h) = 2n + 2$  then

$$\dim(X) = (n + 1)^2 = 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1).$$

**Corollary 8.5.** *The variety  $X$  is 2-incompressible if and only if  $J(h)$  is empty.*

**Remark 8.6.** We call the *generic hermitian form* of dimension  $d$  (for the fixed separable extension  $K/F$ ) the diagonal  $K(t_1, \dots, t_d)/F(t_1, \dots, t_d)$ -hermitian form  $\langle t_1, \dots, t_d \rangle$ , where  $t_1, \dots, t_d$  are variables. Since any hermitian form can be diagonalized (see [Scharlau 1985, Theorem 6.3 of Chapter 7]), any  $d$ -dimensional  $K/F$ -hermitian form is a specialization of the generic hermitian form of dimension  $d$ . In the case of a generic hermitian form, the maximal unitary grassmannian is 2-incompressible by [Karpenko 2012, Theorem 8.1]. Therefore, by Corollary 8.5, the  $J$ -invariant of a generic hermitian form is empty.

## 9. Motivic decomposition

In this section, we determine the complete motivic decomposition of the  $\mathrm{Ch}_K$ -motive  $M^K(X)$  (see Section 2) of the maximal unitary grassmannian  $X$  associated with a nondegenerate  $K/F$ -hermitian form  $h$  in terms of the  $J$ -invariant  $J(h)$  (Theorem 9.4).

For  $I \subset [1, 2n + 1]_{\mathrm{od}}$  or  $[3, 2n + 1]_{\mathrm{od}}$  depending on whether  $\dim(h)$  is, respectively, equal to  $2n + 2$  or  $2n + 1$ , let  $\bar{I}$  denote the complementary set. We set  $J = J(h)$ . One always has  $\|J\| + \|\bar{J}\| = \dim(X)$ . We also use notation introduced in the previous sections. By the very definition of the  $J$ -invariant and Proposition 5.1, for any  $S \subset \bar{J}$  and  $L, L' \subset J$ , the cycle

$$\theta_{S,L,L'} := x_S \cdot (e_L \times e_{J \setminus L'}),$$

where  $x_S$  is defined in (5.4), belongs to  $\overline{\mathrm{Ch}}_K(X^2)$ . Note that  $\theta_{S,L,L'}$  can be rewritten as

$$\sum_{M \subset S} e_{M \sqcup L} \times e_{(S \setminus M) \sqcup (J \setminus L')},$$

where  $\sqcup$  is the disjoint union of sets.



We write  $\theta_L$  for  $\theta_{\bar{J}, L, L}$  and  $\theta_{L, L'}$  for  $\theta_{\bar{J}, L, L'}$ . Let  $\Delta_{\bar{X}}$  denote the class of the diagonal in  $\mathrm{Ch}_K(\bar{X}^2)$ .

**Lemma 9.1.** (i) *The set  $\mathcal{B} = \{\theta_{S, L, L'} \mid S \subset \bar{J} ; L, L' \subset J\}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -basis of  $\overline{\mathrm{Ch}}_K(X^2)$ .*

(ii) *One has*

$$\theta_{S_2, L_2, L'_2} \circ \theta_{S_1, L_1, L'_1} = \delta_{L_2, L'_1} \cdot \delta_{S_1 \cup S_2, \bar{J}} \cdot \theta_{S_1 \cap S_2, L_1, L'_2}.$$

(iii) *As correspondences of degree 0, the elements of  $\{\theta_L \mid L \subset J\}$  are pairwise orthogonal projectors such that  $\sum_{L \subset J} \theta_L = \Delta_{\bar{X}}$ .*

*Proof.* The set  $\mathcal{B}$  is a free family. Indeed, assume that  $\sum_{S, L, L'} \alpha_{S, L, L'} \cdot \theta_{S, L, L'} = 0$  for some  $\alpha_{S, L, L'}$  in  $\mathbb{Z}/2\mathbb{Z}$ . Choose  $L_0 \subset J$ . By multiplying the latter equation by  $e_{\bar{J} \sqcup (J \setminus L_0)} \times 1$ , one gets

$$\sum_{S, L, L'} \alpha_{S, L, L'} \cdot \left( \sum_{M \subset S} e_{\bar{J} \sqcup (J \setminus L_0)} \cdot e_{M \sqcup L} \times e_{(S \setminus M) \sqcup (J \setminus L')} \right) = 0.$$

By using (5.6), taking the image of the previous equation under the homomorphism  $(p_{2*})_K$  associated with the push-forward of the second projection  $\bar{X} \times \bar{X} \rightarrow \bar{X}$ , one gets

$$\sum_{S, L'} \alpha_{S, L_0, L'} \cdot e_{S \sqcup (J \setminus L')} = 0.$$

Hence, since the family  $\{e_I\}$  is free (it is even a basis of  $\mathrm{Ch}_K(\bar{X})$ ; see Theorem 4.9), one obtains that  $\alpha_{S, L_0, L'} = 0$  for any  $S \subset \bar{J}$  and any  $L' \subset J$ . Therefore, the family  $\mathcal{B}$  is free.

Moreover, since the  $\mathrm{Ch}_K$ -motive  $M^K(\bar{X})$  is split, by the analogue of [Elman et al. 2008, Proposition 64.3] for  $\mathrm{Ch}_K$ -motives, the map

$$\mathrm{Ch}_K(\bar{X}) \otimes \mathrm{Ch}_K(\bar{X}) \rightarrow \mathrm{Ch}_K(\bar{X} \times \bar{X})$$

(given by the external product) is an isomorphism. Thus, by Theorem 4.9, any element of  $\mathrm{Ch}_K(\bar{X} \times \bar{X})$  can be written as  $\sum_{I_1, I_2} \alpha_{I_1, I_2} \cdot (e_{I_1} \times e_{I_2})$  where the sum runs over all subsets  $I_1, I_2 \subset [1, 2n+1]_{\mathrm{od}}$  or  $[3, 2n+1]_{\mathrm{od}}$  depending on whether  $\dim(h)$  is, respectively, equal to  $2n+2$  or  $2n+1$ , and  $\alpha_{I_1, I_2} \in \mathbb{Z}/2\mathbb{Z}$ . Let  $a = \sum_{I_1, I_2} \alpha_{I_1, I_2} \cdot (e_{I_1} \times e_{I_2}) \in \overline{\mathrm{Ch}}_K(X^2)$ . Then, for any subset  $I_2$ , the element

$$p_{1*}(x_{\bar{I}_2} \cdot a) = \sum_{I_1, M; M \subset \bar{I}_1 \cap \bar{I}_2} \alpha_{I_1, I_2 \sqcup M} \cdot e_{I_1 \sqcup M}$$

is rational. Hence, by Theorem 5.7, for any subset  $I \not\subset J$ , one has

$$\sum_{M \subset \bar{I}_2 \cap I} \alpha_{I \setminus M, I_2 \sqcup M} = 0.$$

Furthermore, the above relations are linearly independent because, for any subset  $I_2$  and  $I \not\subset J$ , the sum  $r_{I_2, I} := \sum_{M \subset \bar{I}_2 \cap I} \alpha_{I \setminus M, I_2 \sqcup M}$  contains  $\alpha_{I, I_2}$  and for any  $(I'_2, I') \neq (I_2, I)$  such that  $\alpha_{I, I_2} \in r_{I'_2, I'}$ , one has  $|I'_2| < |I_2|$ . It follows that

$$\text{rank}_{\mathbb{Z}/2\mathbb{Z}} \overline{\text{Ch}}_K(X^2) \leq 2^r \cdot 2^r - 2^r \cdot (2^r - 2^{|J|}) = 2^{r+|J|} = |\mathcal{B}|$$

(recall that  $\text{rank}_{\mathbb{Z}/2\mathbb{Z}} \text{Ch}_K(\bar{X}) = 2^r$ , with  $r = \lfloor \dim(h)/2 \rfloor$ ). Consequently, the family  $\mathcal{B}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -basis of  $\overline{\text{Ch}}_K(X^2)$ .

For any subsets  $S_1, S_2 \subset \bar{J}$  and  $L_1, L'_1, L_2, L'_2 \subset J$ , the composition of correspondences  $\theta_{S_2, L_2, L'_2} \circ \theta_{S_1, L_1, L'_1}$  is equal to

$$\sum_{M_1, M_2 \subset \bar{J}} (p_{13*})_K (e_{M_1 \sqcup L_1} \times (e_{((S_1 \setminus M_1) \sqcup (J \setminus L'_1))} \cdot e_{M_2 \sqcup L_2}) \times e_{(S_2 \setminus M_2) \sqcup (J \setminus L'_2)}).$$

Hence, the assertion (ii) follows from (5.6).

By (ii), the elements of  $\{\theta_L \mid L \subset J\}$  are pairwise orthogonal projectors. Furthermore, one has the identity

$$\sum_{L \subset J} \theta_L = \sum_{I \subset [1, 2n+1]_{\text{od}}} e_I \times e_{\bar{I}}.$$

Therefore, for any  $I \subset [1, 2n+1]_{\text{od}}$  (or  $[3, 2n+1]_{\text{od}}$  for odd-dimensional  $h$ ), one has  $(\sum_{L \subset J} \theta_L)^*(e_I) = e_I$ . Since the elements  $e_I$  generate the ring  $\text{Ch}_K(\bar{X})$ , one deduces that  $\sum_{L \subset J} \theta_L = \Delta_{\bar{X}}$  and (iii) is proven.  $\square$

The next proposition is the  $\text{Ch}_K$ -version of the Rost lemma (see [Rost 1998, Proposition 1] or [Elman et al. 2008, Theorem 67.1] and Remark 2.1).

**Proposition 9.2.** *Let  $Y$  and  $Z$  be smooth proper  $F$ -varieties. If a correspondence  $\alpha \in \text{Ch}_K(Y \times Y)$  is such that  $\alpha \circ \text{Ch}_K(Y_{F(z)}) = 0$  for every  $z \in Z$  then*

$$\alpha^{\dim(Z)+1} \circ \text{Ch}_K(Z \times Y) = 0.$$

For any smooth proper variety  $Y$ , by the very definition of the category of  $\text{Ch}_K$ -motives, one has  $\text{End}^j(M^K(Y)) = \text{Ch}_K^{\dim(Y)+j}(Y \times Y)$  for any  $j$ , with the composition of endomorphisms given by the composition of correspondences. If  $h$  is a nondegenerate  $K/F$ -hermitian form and  $x$  is a point of the associated maximal unitary grassmannian  $X$  then  $h$  splits over  $F(x)$ , so  $\text{Ch}_K(X_{F(x)}) = \text{Ch}_K(\bar{X})$ . Hence, Proposition 9.2 (applied with  $Y = Z = X$ ) implies the statement below, which says that Rost nilpotence holds for  $X$  at the level of  $K$ -Chow rings.

**Corollary 9.3.** *Let  $X$  be the maximal unitary grassmannian associated with a non-degenerate  $K/F$ -hermitian form. The kernel of the restriction ring homomorphism*

$$\text{End}^*(M^K(X)) \rightarrow \text{End}^*(M^K(\bar{X}))$$

*consists of nilpotent elements.*

We are now able to prove the following  $\mathrm{Ch}_K$ -motivic decomposition (in the spirit of [Petrov et al. 2008, Theorem 5.13]).

**Theorem 9.4.** *Let  $h$  be a nondegenerate  $K/F$ -hermitian form and  $X$  the associated maximal unitary grassmannian. Then the  $\mathrm{Ch}_K$ -motive of  $X$  decomposes as*

$$M^K(X) \simeq \bigoplus_{L \subset J(h)} \mathcal{R}(h)\{\|L\|\},$$

where  $\mathcal{R}(h)$  is an indecomposable motive.

Moreover, over a splitting field of  $h$ , the  $\mathrm{Ch}_K$ -motive  $\mathcal{R}(h)$  decomposes as a sum of shifts of the Tate motive. More precisely, one has

$$\overline{\mathcal{R}(h)} \simeq \bigoplus_{M \subset \overline{J(h)}} \mathbb{Z}/2\mathbb{Z}\{\|M\|\}.$$

*Proof.* By Lemma 9.1(ii), one has

$$\theta_{L,N} \circ \theta_{L',N'} = \delta_{L,N'} \cdot \theta_{L',N}.$$

In particular, this implies that for any  $L, L' \subset J$ , the projectors  $\theta_L$  and  $\theta_{L'}$  are isomorphic (in the sense of [Petrov et al. 2008, §2.1]); the isomorphism being given by  $\theta_{L,L'}$  and  $\theta_{L',L}$ .

We claim that the projectors  $\theta_L$  are indecomposable. For any integers  $0 \leq k \leq \|\bar{J}\|$  and  $0 \leq l \leq \dim(X)$ , we write

$$\overline{\mathrm{Ch}}_K^{k,l}(X^2)$$

for the subspace of  $\overline{\mathrm{Ch}}_K(X^2)$  spanned by the elements  $\theta_{S,L,L'}$  with  $\|S\| \leq k$  and codimension  $\leq l$ . For any correspondences  $\alpha, \alpha' \in \mathrm{Ch}_K(\bar{X}^2)$  one has

$$\mathrm{codim}(\alpha \circ \alpha') = \mathrm{codim}(\alpha) + \mathrm{codim}(\alpha') - \dim(X).$$

Thus, by Lemma 9.1(i) (or Lemma 9.1(ii)), the space  $\overline{\mathrm{Ch}}_K^{\|\bar{J}\|, \dim(X)}(X^2)$  equipped with the composition of correspondences is a ring. Furthermore, using the same codimensional considerations and the formula of Lemma 9.1(ii), one sees that, for any  $0 \leq k \leq \|\bar{J}\|$  and  $0 \leq l \leq \dim(X)$ , the subspace  $\overline{\mathrm{Ch}}_K^{k,l}(X^2)$  is an ideal of

$$\overline{\mathrm{Ch}}_K^{\|\bar{J}\|, \dim(X)}(X^2).$$

We denote by  $\mathcal{A}$  the factor ring of  $\overline{\mathrm{Ch}}_K^{\|\bar{J}\|, \dim(X)}(X^2)$  by the sum of ideals  $\overline{\mathrm{Ch}}_K^{k,l}(X^2)$  for all  $(k, l) \in ([0, \|\bar{J}\|] \times [0, \dim(X)]) \setminus \{(\|\bar{J}\|, \dim(X))\}$ . By Lemma 9.1(i), a  $\mathbb{Z}/2\mathbb{Z}$ -basis of  $\mathcal{A}$  is given by the classes of the elements  $\theta_{L,L'}$  with  $\|L\| = \|L'\|$ . Hence, since  $\theta_{L,L'}^*(e_N) = \delta_{N,L'} \cdot e_L$  for any  $N \subset J$  (by (5.6)), it follows from

**Lemma 9.1(ii)** that the assignment  $\theta_{L,L'} \mapsto \theta_{L,L'}^*$  for all  $\theta_{L,L'}$  with  $\|L\| = \|L'\|$  gives rise to an anti-isomorphism between  $\mathcal{A}$  and the product of matrix rings

$$\prod_{k=0}^{\dim(X)} \text{End}(\mathcal{E}_k),$$

where  $\mathcal{E}_k$  is the subspace of  $\overline{\text{Ch}}_K(X)$  spanned by the elements  $e_N$  with  $\|N\| = k$ . Under this identification, for any  $L \subset J$ , the class of  $\theta_L$  in  $\mathcal{A}$  corresponds to an idempotent element of rank 1 and therefore is indecomposable. Moreover, by **Lemma 9.1(ii)** and codimensional reasons (for any correspondence  $\alpha \in \text{Ch}_K(\bar{X}^2)$  of codimension  $< \dim(X)$ , one has  $\text{codim}(\alpha^{\circ i}) > \text{codim}(\alpha^{\circ(i+1)})$ ), the kernel of the projection

$$\pi : \overline{\text{Ch}}_K^{\|\bar{J}\|, \dim(X)}(X^2) \rightarrow \mathcal{A}$$

is nilpotent. Thus, the projectors  $\theta_L$  are indecomposable in  $\overline{\text{Ch}}_K(X^2)$ . The claim is proven.

Consequently, combining this result with **Lemma 9.1(iii)** and Rost nilpotence (**Corollary 9.3**), one gets that there exists a family  $\{\psi_L \mid L \subset J\}$  of pairwise orthogonal projectors in  $\text{Ch}_K^{\dim(X)}(X \times X)$ , satisfying  $\overline{\psi}_L = \theta_L$ , all isomorphic to  $\psi_J$  (with respect to the correspondences  $\theta_{L,J}$ ), and such that  $\sum_{L \subset J} \psi_L = \Delta_X$  (see [Petrov et al. 2008, Proposition 2.6]). Note that Rost nilpotence implies that the projectors  $\psi_L$  are also indecomposable. Let  $\mathcal{R}(h)$  denote the indecomposable  $\text{Ch}_K$ -motive  $(X, \psi_\emptyset)^K$ . Thus, for any  $L \subset J$ , one has  $(X, \psi_L)^K = \mathcal{R}(h)\{\|L\|\}$  (since  $\text{codim}(\theta_{L,J}) = \dim(X) - \|J \setminus L\|$ ). In other words, one has the desired  $\text{Ch}_K$ -motivic decomposition of  $M^K(X)$ .

We prove now the last assertion of the theorem. Over a splitting field, one has  $\overline{\mathcal{R}}(\bar{h}) = (\bar{X}, \theta_\emptyset)^K$ . Moreover,  $\theta_\emptyset = \sum_{M \subset \bar{J}} e_M \times e_{(\bar{J} \setminus M) \sqcup J}$  is a decomposition as a sum of pairwise orthogonal projectors in  $\text{Ch}_K(\bar{X} \times \bar{X})$ . These projectors are all isomorphic to  $1 \times e_{\bar{J} \sqcup J}$  with  $(\bar{X}, e_M \times e_{(\bar{J} \setminus M) \sqcup J})^K \simeq (\bar{X}, 1 \times e_{\bar{J} \sqcup J})^K \{\|M\|\}$  (recall that  $e_{\bar{J} \sqcup J}$  is the class of a rational point). Furthermore, one easily checks that the  $\text{Ch}_K$ -motive  $(\bar{X}, 1 \times e_{\bar{J} \sqcup J})^K$  is isomorphic to the Tate motive  $\mathbb{Z}/2\mathbb{Z}$  in the category of  $\text{Ch}_K$ -motives.  $\square$

The following statement can be viewed as the  $\text{Ch}_K$ -version of the Krull–Schmidt principle for the  $F$ -variety  $X$ .

**Corollary 9.5.** *Any direct summand of  $M^K(X)$  is isomorphic to a sum of shifts of the motive  $\mathcal{R}(h)$ .*

*Proof.* We use notation and material introduced in the proof of **Theorem 9.4**. Any idempotent in the ring  $\mathcal{A}$  is isomorphic (in the sense of [Petrov et al. 2008, §2.1]) to a sum of classes of  $\theta_L$  for some  $L \subset J$ . Moreover, since  $\text{Ker}(\pi)$  is nilpotent, the projection  $\pi$  lifts isomorphisms; see [Petrov et al. 2008, Proposition 2.6].  $\square$

**Remark 9.6.** It follows from [Theorem 9.4](#) that  $M^K(X)$  is indecomposable if and only if  $J(h)$  is empty, i.e., if and only if  $X$  is 2-incompressible. In particular, this applies to generic hermitian forms.

**Remark 9.7.** It follows from [\(6.6\)](#) that there is a group isomorphism

$$\mathrm{Ch}(M_e \otimes M_e) \rightarrow \mathrm{Ch}_K(X \times X),$$

with  $M_e$  the essential Chow motive of  $X$ . Moreover, let  $\varphi \in \mathrm{Ch}(X \times X)$  be the projector giving  $M_e$ . Then the group  $\mathrm{Ch}(M_e \otimes M_e) = \varphi \circ \mathrm{Ch}(X \times X) \circ \varphi$  is equipped with the ring structure given by the composition of correspondences (the neutral element being  $\varphi$ ). Since the above isomorphism is the restriction to  $\varphi \circ \mathrm{Ch}(X \times X) \circ \varphi$  of the projection  $\mathrm{Ch}(X \times X) \rightarrow \mathrm{Ch}_K(X \times X)$ , it is a ring isomorphism (with  $\mathrm{Ch}_K(X \times X)$  equipped with the composition of  $\mathrm{Ch}_K$ -correspondences). Hence, the essential motive  $M_e$  is decomposable if and only if the  $\mathrm{Ch}_K$ -motive  $M^K(X)$  is decomposable.

## 10. Comparison with quadratic forms

Let  $h$  be a nondegenerate  $K/F$ -hermitian form and  $q$  the associated nondegenerate  $F$ -quadratic form. In this section, we compare the  $J$ -invariant  $J(h)$  with the  $J$ -invariant  $J(q)$  as defined by Vishik [\[2005\]](#). We recall that if  $\dim(h) = 2n + 2$  then  $J(q)$  is a subset of  $[0, 2n + 1]$ , otherwise, if  $\dim(h) = 2n + 1$  then it is a subset of  $[0, 2n]$  and in both cases  $J(h)$  is a subset of  $[1, 2n + 1]_{\mathrm{od}}$ .

Let  $X$  be the maximal hermitian grassmannian of  $h$  and  $Y$  the maximal orthogonal grassmannian of  $q$ . For even-dimensional  $h$ , let  $im$  denote the natural closed embedding  $X \hookrightarrow Y$  (see [Remark 4.12](#)).

**Proposition 10.1.** *One has*

$$J(q) = \begin{cases} J(h) \cup [0, 2n]_{\mathrm{ev}} & \text{if } \dim(h) = 2n + 2; \\ [1, 2n] & \text{if } \dim(h) = 2n + 1, \end{cases}$$

where  $[0, 2n]_{\mathrm{ev}}$  stands for the even part of the set  $[0, 2n]$ .

*Proof.* Assume that  $\dim(h) = 2n + 2$ . We use notation and material introduced in [Remark 4.12](#). Since  $im^*(z_k) = e_k$  in  $\mathrm{Ch}(\bar{X})$  for any  $1 \leq k \leq 2n + 1$  (see [Remark 4.12](#)) and the generators  $z_k$  of the ring  $\mathrm{Ch}(\bar{Y})$  define  $J(q)$  the same way the elements  $e_k \in \mathrm{Ch}_K(\bar{X})$ , for  $k$  odd, define  $J(h)$  (see [\[Vishik 2005\]](#) or [\[Elman et al. 2008\]](#)), one has  $J(q)_{\mathrm{od}} \subset J(h)$ . Moreover, since the quadratic form  $q$  is obtained from a  $K/F$ -hermitian form, the absolute Witt indices of  $q$  are even. Hence, by [\[Elman et al. 2008, Proposition 88.8\]](#) (this is the quadratic equivalent of [Proposition 6.12](#), with the  $J$ -invariant defined in the opposite way), the set  $[0, 2n]_{\mathrm{ev}}$  is contained in  $J(q)$ . Furthermore, by [\[Karpenko 2012, Corollary 9.3\]](#), the varieties  $X$  and  $Y$

have the same canonical 2-dimension. Thus, by [Elman et al. 2008, Theorem 90.3] and Theorem 8.2, one has

$$\|J(q)\| - \|J(h)\| = \dim(Y) - \dim(X) = 0 + 2 + \cdots + (2n - 2) + 2n.$$

Consequently, one has  $J(q) = J(h) \cup [0, 2n]_{\text{ev}}$ .

Assume that  $\dim(h) = 2n + 1$ . In this case, one has  $\text{cdim}_2(Y) = 0$  (see [Karpenko 2012, Corollary 9.3]). Therefore, by [Elman et al. 2008, Theorem 90.3], one has

$$\|J(q)\| = \dim(Y) = \frac{2n(2n + 1)}{2}.$$

Moreover  $0 \notin J(q)$  because the discriminant of  $q$  is not trivial. Consequently, one has  $J(q) = [1, 2n]$ .  $\square$

**Remark 10.2.** Proposition 10.1 allows one to recover the smallest value of the  $J$ -invariant of a nondegenerate quadratic form  $q$  associated with a hermitian form  $h$  of even dimension  $2n + 2$  over a quadratic separable field extension of the base field, i.e.,  $q$  is given by the tensor product of a  $(2n+2)$ -dimensional bilinear form by a binary quadratic form. Namely, this value is  $[0, 2n]_{\text{ev}}$  and it is obtained for  $h$  generic. This was originally proven by Karpenko; see [Karpenko 2012, Corollary 9.4].

**Remark 10.3.** For even-dimensional nondegenerate  $K/F$ -hermitian forms, combined with Theorem 5.7, Proposition 10.1 and its proof provide another argument for the surjectivity of the homomorphism

$$\overline{\text{Ch}}(Y) \rightarrow \overline{\text{Ch}}_K(X),$$

associated with the pull-back  $im^*$ . This was originally observed by Maksim Zhykhovich; see [Karpenko 2012, Lemma 9.8].

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