RATIONALITY OF CYCLES AND SPECIAL CORRESPONDENCES

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In this note, we are interested in [2, Theorem SC.1] due to Nikita Karpenko and Alexander Merkurjev. We deal with the case of equality of this theorem. We refer to [2, Appendix SC. Special correspondences] for notation and vocabulary. The proof of the following proposition is widely inspired by the proof of [2, Theorem SC.1]. Note that, in a way, the following proposition is a generalization of [1, Proposition 2.1] (which deal with the case $p = 2$) to any prime integer p.

Proposition 1. *Let X be an A-trivial F-variety for $\mathbb{F}_p$ equivalent to an A-trivial F-variety of dimension $p^n - 1$ possessing a special correspondence. Then for any smooth irreducible F-variety Y, any $m \in \mathbb{Z}$, and any $y \in Ch^m(Y_{F(X)})$, there exists a polynomial P of degree $\leq p-1$ with rational coefficients in $Ch(Y_{F(X)})$ such that the element $S^s(y) + P(y) \in Ch^{m+s}(Y_{F(X)})$, with $s = (m-b)(p-1)$, is rational up to the class modulo p of an exponent p element of $CH^{m+s}(Y_{F(X)})$.*

Proof. We make the assumption that $\dim(Y) > 0$ (otherwise, the conclusion is immediate). We use the same notation as those introduced during the proof of [2, Theorem SC.1]. According to the proof of [2, Theorem SC.1], one can find an element $x \in Ch^m(X \times Y)$ such that x decomposes over $F(X)$ as

$$x_{F(X)} = 1 \times y + H \times x_1 + \cdots + H^{p-1} \times x_{p-1} \tag{1}$$

with some $x_1, ..., x_{p-1} \in Ch(Y_{F(X)})$. By the same reasoning as those done in the beginning of [2, Proposition SC.12] and since $S^{d+s}(x) = x^p$, we get that p divides the element $A + pr_*(\tilde{x}^p)$ in $CH(Y)$, where

$$A = \sum_{\substack{i+j+k+l_1+\cdots+l_{p-1}=d+s \\ i>0;\ j,k,l_1,\ldots,l_{p-1}\geq 0}} pr_*(b_j \cdot (pr_*(b_i \cdot S_\sigma^{l_1} \cdot \ldots \cdot S_\sigma^{l_{p-1}})) \cdot S_x^k),$$

and where $\tilde{x} \in CH^m(X \times Y)$ is an integral representative of $x \in Ch^m(X \times Y)$.

Furthermore, according to the proof of [2, Proposition SC.12], there exist a cycle $\beta \in CH(Y_{F(X)})$, a cycle $\gamma \in CH(Y)$, and a prime to p integer q such that

$$A_{F(X)} = p^2\beta + p\gamma_{F(X)} + q\deg(b_d)S_y^s.$$

Therefore, modifying β and γ, one can write

$$q\deg(b_d)S_y^s + pr_*(\tilde{x}^p_{F(X)}) = p^2\beta + p\gamma_{F(X)}. \tag{2}$$

Moreover, according to the decompositon (1), there exists a cycle $\alpha \in CH^m(X \times Y)_{F(X)}$ such that

$$\tilde{x}_{F(X)} = 1 \times \tilde{y} + H \times \tilde{x}_1 + \cdots + H^{p-1} \times \tilde{x}_{p-1} + p\alpha,$$

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where the cycles $\tilde{y}, \tilde{x_1}, ..., \tilde{x}_{p-1}$ are some integral representatives of the cycles $y, x_1, ..., x_{p-1}$. Thus, we have

$$\tilde{x}^p_{F(X)} = \sum_{k=0}^{p} \binom{p}{k} (p\alpha)^{p-k} \cdot (1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \tilde{x}_{p-1})^k.$$

In the lattest expression, each summand, except the one corresponding to $k = p$, is divisible by p^2. Thus, modifying β, we deduce from the equation (2) the following identity

$$q\mathrm{deg}(b_d) S^s_y + pr_*((1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \tilde{x}_{p-1})^p) = p^2\beta + p\gamma_{F(X)}. \tag{3}$$

Furthermore, by the multinomial Theorem, the cycle $pr_*((1 \times \tilde{y} + H \times \tilde{x_1} + \cdots + H^{p-1} \times \ \tilde{x}_{p-1})^p)$ is equal to

$$\sum_{\substack{k_0+k_1+\cdots+k_{p-1}=p \\ k_1+2k_2+\cdots+(p-1)k_p-1=p-1}} \binom{p}{k_0, k_1, ..., k_{p-1}} \tilde{y}^{k_0} \cdot \tilde{x_1}^{k_1} \cdots \tilde{x}^{k_{p-1}}_{p-1}.$$

Since for any $i = 0, ..., p-1$, one has $k_i < p$, each multinomial coefficient appearing in the previous sum is a multiple of p. Thus, the sum can be rewritten as

$$p \sum_{k=1}^{p-1} a_k \cdot \tilde{y}^k,$$

where

$$a_k := \sum_{\substack{k_1+k_2+\cdots+k_{p-1}=p-k \\ k_1+2k_2+\cdots+(p-1)k_p-1=p-1}} \binom{p-1}{k, k_1, ..., k_{p-1}} \tilde{x_1}^{k_1} \cdot \tilde{x_2}^{k_2} \cdots \tilde{x}^{k_{p-1}}_{p-1} \in CH(Y_{F(X)}).$$

Therefore, since the integer $\mathrm{deg}(b_d)$ is divisible by p but not by p^2 (see proof of [2, Proposition SC.12]), we deduce from the equation (3) that the element

$$S^s(y) + \sum_{k=1}^{p-1} a_k \cdot y^k \tag{4}$$

is rational up to the class modulo p of an exponent p element of $CH^{m+s}(Y_{F(X)})$ (for $k = 1, ..., p-1$, we still write a_k for the class in $Ch(Y_{F(X)})$, and we replace the coefficient in $\mathbb{F}^\star_p$ near $S^s(y)$ by 1). From now on, we work with Chow groups modulo p. For any $k = 1, ..., p-1$, one has (we use the Projection Formula to get the last identity)

$$\begin{aligned}
a_k &= \binom{p-1}{k-1} k^{-1} pr_*((H \times x_1 + H^2 \times x_2 + ... + H^{p-1} \times x_{p-1})^{p-k}) \\
&= (-1)^{k-1} k^{-1} pr_*((x_{F(X)} - 1 \times y)^{p-k}) \\
&= (-1)^{k-1} k^{-1} \sum_{i=0}^{k} \binom{p-k}{i} (-1)^i pr_*({x_{F(X)}}^{p-k-i} \cdot (1 \times y^i)) \\
&= (-1)^{k-1} k^{-1} \sum_{i=0}^{k} \binom{p-k}{i} (-1)^i y^i \cdot pr_*({x_{F(X)}}^{p-k-i}).
\end{aligned}$$

Therefore, setting for every $j = 1, ..., p-1$,

$$e_j := \left(\sum_{l=1}^{j} l^{-1}\binom{p-l}{j-l}\right)(-1)^{j-1}pr_*(x^{p-j}) \in Ch(Y),$$

we get that

$$\sum_{k=1}^{p-1} a_k \cdot y^k = P(y), \tag{5}$$

where P is the polynomial in variable Z with coefficient in $Ch(Y_{F(X)})$ such that $P(Z) = \sum_{j=1}^{p-1} {e_j}_{F(X)} \cdot Z^j$ (there is no coefficient e_p because $pr_*(1) = 0$). We get the desired result by combining (4) and (5). $\square$

References

[1] Fino, R. Around rationality of cycles. *Linear Algebraic Groups and Related Structures (preprint serveur)*, 450 (2011, Nov 16), 11 pages.

[2] Karpenko, N., and Merkurjev, A. On standard norm varieties. *Linear Algebraic Groups and Related Structures (preprint serveur)*, 456 (2012, Jan 4), 37 pages.

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