ON THE SINGULARITIES OF COMPLETE HOLOMORPHIC VECTOR FIELDS IN DIMENSION TWO

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ABSTRACT. For a germ of singular holomorphic vector field on a complex manifold to be the local model of a complete one, it is necessary for its solutions to be univalent (and not multivalued). Rebelo formalized this local obstruction to completeness through the notion of *semicompleteness*. He started studying this some twenty-five years ago, and it has been investigated ever since by various authors. We here review this notion, from the foundational definitions to some local and global results, with a special emphasis on manifolds and analytic spaces of dimension two.

CONTENTS

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1. Introduction

For holomorphic vector fields on complex manifolds, completeness is not only a property to be understood at infinity, but, as it turns out, the singularities of the vector field play an important part in it. Some twenty-five years ago, Rebelo noticed and started exploiting the fact that multivaluedness constitutes a local obstruction for a holomorphic vector field to be complete. A systematic local study of complete holomorphic vector fields on complex manifolds ensued. For surfaces, a very comprehensive one was made possible by the thorough local understanding we have of holomorphic foliations in dimension two. While the existence of local obstructions for completeness was probably known, or, at least, not completely unexpected, the degree to which multivaluedness is a strong obstruction amid degenerate vector fields was certainly surprising.

The general problem was, of course, not new, as the quest for understanding the algebraic differential equations in the complex domain with single-valued solutions was an important one all throughout the second half of the 19th century, with results, to name some, of Briot and Bouquet, Poincaré, Picard and Painlevé (an account up to the turn of the XXth century is given in Painlevé's *Leçons de Stockholm* [Pai73]). While the local nature of Rebelo's point of view is, in principle, not very suitable to directly study transcendence issues (an important point behind Painlevé's motivation), it was a key element permitting its use in nonalgebraic contexts. For instance, an early application of the results obtained by Ghys and Rebelo was the completion of the classification of compact complex surfaces admitting vector fields due to Dloussky, Oeljeklaus and Toma [DOT00], [DOT01], which needed the understanding of vector fields on some nonalgebraic surfaces.

Rebelo's and Ghys and Rebelo's original efforts were followed by those of Reis and the author. Some of the ideas were successfully developed in other contexts: for other Lie groups, for meromorphic vector fields, in higher dimensions. Over the years we have gained a deeper understanding of some key notions, simplified proofs, realized not only that some ideas were already there (notably, Palais's theory on the integration of infinitesimal actions) but that some had been for a very long time (e.g. work by Briot and Bouquet, Kowalevski, Painlevé, . . .).

The aim of this survey is to present, from a personal point of view and with today's hindsight, some of the results concerning the singular points of complete holomorphic vector field in dimension two, both for manifolds and analytic spaces. (Many interesting parts have been regrettably left outside.)

We assume that the reader is acquainted with some basic facts around foliations on surfaces, like the material covered in the first few chapters of [Bru15] or [CCD13]. All manifolds will be complex; all maps and vector fields, holomorphic, and so on.

2. Semicompleteness

We begin by preparing for our core definition, that of *semicomplete* or *univalent* vector field (Definition 2.3). While the discussion at times is not too far from that in Palais's monograph [Pal57], some objects particular to holomorphic vector fields come into play, and intertwine with the rest of the theory.

Let M be a complex manifold, X a holomorphic vector field on M. We may think of X as an ordinary differential equation and consider *solutions* of X, functions $\phi : U \to M$, $U \subset \mathbf{C}$, such that $\phi'(t) = X|_{\phi(t)}$. The Existence and Uniqueness Theorem guarantees that for every $p \in M$ there exists an open connected subset $U \subset \mathbf{C}$, $0 \in U$, and a solution $\phi : U \to M$ of X such that $\phi(0) = p$; furthermore, two such solutions define the same germ at 0. We refer the reader to [Inc44], [Hil97] and [IY08] for basic facts around ordinary differential equations in the complex domain.

If for every p in M there is a solution $\phi_p : \mathbf{C} \to M$ of X with $\phi_p(0) = p$ we say that X is *complete*. In such a case we may glue together the solutions into a map $\Phi : \mathbf{C} \times M \to M$ defined by $\Phi(t, p) = \phi_p(t)$, that satisfies the conditions

- $\Phi(0, p) = p$,
- $\Phi(t, \Phi(s, p)) = \Phi(t+s, p)$ for all $s, t \in \mathbf{C}$, and for all $p \in M$

(the second one is a consequence of the fact that the ordinary differential equation associated to X is an autonomous one). The map Φ is a holomorphic *flow* or *action of* $\mathbf{C}$. Vector fields on compact manifolds are complete, but, in general, completeness is difficult to characterize.

Let $\mathrm{Sing}(X) = \{p \mid X(p) = 0\}$. On $M \setminus \mathrm{Sing}(X)$, when the images of two solutions intersect, their union defines an immersed curve. The maximal curves thus constructed are the *leaves* or *orbits* of X (or of the *foliation* induced by X) on $M \setminus \mathrm{Sing}(X)$. Each one of these curves is tangent to X at each one of its points, and inherits a special geometry.

A *translation structure* on a complex curve L is an atlas for its complex structure taking values in $\mathbf{C}$ whose changes of coordinates are given by translations, maps of the form $z \mapsto z + c$. On a curve, the following data are equivalent:

- a holomorphic vector field without zeros,
- a translation structure,
- a holomorphic one-form without zeros.

Indeed, let L be a curve. If X is a nowhere-zero holomorphic vector field on L, if two solutions $\phi_1 : U_1 \to L$ and $\phi_2 : U_2 \to L$ of X have overlapping images, by the uniqueness of solutions, they differ by a translation: there exists $c \in \mathbf{C}$ such that $\phi_1(t) = \phi_2(t+c)$ for every $t \in U_1 \cap (U_2 - c)$, and the local inverses of the solutions of X endow L with a translation structure, and we thus go from the vector field to a translation structure. The primitives of a nowhere-vanishing one-form on L are the charts of a translation structure, thus producing a translation structure out of a one-form. To obtain forms and vector fields from a translation structure, notice that since, in $\mathbf{C}$, translations preserve the vector field $\partial/\partial z$ as well as the one-form dz, these may both be pulled back to a curve endowed with a translation structure via its charts, producing a well-defined nowhere-zero vector field and a well-defined nowhere-zero one-form on the curve. Finally, given a nowhere-zero vector field X on L, there is a unique nowhere-zero one-form ω such that $\omega(X) \equiv 1$, the *time form* of X; the inverse procedure produces a vector field starting from a form.

Thus, each orbit of X on $M \setminus \mathrm{Sing}(M)$ is endowed with a translation structure.

Given a translation structure on the curve L with universal covering $\pi : \widetilde{L} \to L$, there is a *developing* map $\mathscr{D} : \widetilde{L} \to \mathbf{C}$, giving a global chart for the projective structure induced on $\widetilde{L}$, and a *monodromy* (or *period*) homomorphism $\mathrm{mon} : \pi_1(L) \to \mathbf{C}$ such that for every

$\alpha \in \pi_1(L)$,

$$\mathscr{D}(\alpha \cdot p) = \mathscr{D}(p) + \mathrm{mon}(\alpha) \tag{1}$$

(see [Thu97, Sect. 3.4]). If the translation structure is given by the nowhere-vanishing one-form ω, $\mathscr{D}$ is given by $x \mapsto \int^x \pi^*\omega$ ($\pi^*\omega$ is exact; $\mathscr{D}$ is one of its primitives), and the monodromy is induced by $\gamma \mapsto \int_\gamma \omega$.

A translation structure on a curve L is said to be *uniformizable* if L is isomorphic (as a curve with a translation structure) to the quotient of an open set $\Omega \subset \mathbf{C}$ under a group of translations acting properly discontinuously.

Proposition 2.1. *Let M be a complex manifold, X a holomorphic vector field on M, $p \in M \setminus \mathrm{Sing}(X)$, $L \subset M$ the orbit of X through p. The following are equivalent:*

1. *For every solution $\phi : U \to M$ ($U \subset \mathbf{C}$, $0 \in U$) of X with initial condition p and every pair of paths $\gamma_i : [0,1] \to \mathbf{C}$ with $\gamma_i(0) = p$, $\gamma_1(1) = \gamma_2(1)$, such that the germ of ϕ at 0 admits analytic continuations along γ_1 and γ_2, both analytic continuations define the same germ at $\gamma_i(1)$.*
2. *There exists $\Omega \subseteq \mathbf{C}$, $0 \in \Omega$ and $\phi : \Omega \to M$ a solution to X with initial condition p, such that the map $\phi \times i : \Omega \to M \times \mathbf{C}$ given by $t \mapsto (\phi(t), t)$ is proper.*
3. *For every path $\gamma : [0,1] \to L$, $\gamma(0) = p$, such that $\gamma(0) \neq \gamma(1)$, for the time form ω of X on L, $\int_\gamma \omega \neq 0$.*
4. *When considering L with its translation structure, for the covering map $\overline{\pi} : \overline{L} \to L$ associated to the kernel of the monodromy, the map $\overline{\mathscr{D}} : \overline{L} \to \mathbf{C}$ induced by $\mathscr{D}$ is one-to-one.*
5. *The translation structure induced by X on L is uniformizable.*

Proof. (1) $\Rightarrow$ (2). Let $\phi : U \to M$ a solution of X with initial condition p. Let $\Omega \subset \mathbf{C}$ be the set of all points t for which there exists a path $\gamma : [0,1] \to \mathbf{C}$, joining 0 to t, along which an analytic continuation of ϕ may be defined. By hypothesis, we have a well-defined map $\phi : \Omega \to M$, which is a solution to X (the analytic continuation of a solution is still a solution). It does not have any analytic continuation beyond Ω and is thus is maximal as a holomorphic function from a subset of $\mathbf{C}$ into M. Let $\{t_i\}$ be a sequence of points in Ω converging in $\mathbf{C}$ to a point t_∞ in $\partial\Omega$. Suppose that $q \in M$ is an accumulation point of $\{\phi(t_i)\}$. There exists a neighborhood V of q in M and $\varepsilon > 0$ such that, for every $x \in V$, a solution of X with initial condition x is defined in the disk of radius ε around 0 in $\mathbf{C}$. But the solution with initial condition $\phi(t_i)$ cannot be defined in a disk of radius greater than $|t_i - t_\infty|$. Thus, the sequence $\{\phi(t_i)\}$ does not have accumulation points. This proves that $\phi \times i$ is proper.

(2) $\Rightarrow$ (3). Let $\gamma : [0,1] \to L$ be a path such that $\gamma(0) \neq \gamma(1)$, $p = \gamma(0)$, $\phi : \Omega \to M$ be a solution with initial condition p such that $\phi \times i$ is proper. Observe that ϕ takes values in L and that it is a local biholomorphism. We will prove that it has the path-lifting property, that it is a covering map. Let $s_0 = \sup\{s \in [0,1] \mid \gamma|_{[0,s]} \text{ can be lifted to } \Omega\}$, and suppose that $s_0 < 1$. Let $\overline{\gamma} : [0, s_0) \to \Omega$ be a lift of the restriction of γ to $[0, s_0)$. We claim that $\overline{\gamma}$ is proper. Otherwise, if $\{\delta_i\}$ is a sequence of positive reals converging to 0 such that $\overline{\gamma}(s_0 - \delta_i)$ converges to $T \in \Omega$ then, since $\gamma(s_0 - \delta_i)$ converges to $\phi(T)$, $\overline{\gamma}$ may be extended to s_0 by considering the local biholomorphism between neighborhoods of T and $\phi(T)$

given by ϕ. Since $\phi\times i$ and $\overline{\gamma}$ are proper and since $\phi\circ\overline{\gamma}(s)=\gamma(s)$ for all $s<s_0$, the limit of $\phi\circ\overline{\gamma}(s)$ as s approaches s_0 exists, and thus $\overline{\gamma}\colon[0,s_0)\to\mathbf{C}$ is proper. This implies that $\lim_{s\to s_0^-}\int_{\overline{\gamma}|_{[0,s]}}\mathrm{d}t=\infty$, but this contradicts the fact that $\int_{\gamma|_{[0,s_0]}}\omega<\infty$ through the identity $\phi^*\omega=\mathrm{d}t$. We have thus proved that $s_0=1$, that γ can be lifted to a path $\overline{\gamma}\colon[0,1]\to\Omega$ (that ϕ is a covering map). Since $\gamma(0)\neq\gamma(1)$, $\overline{\gamma}(0)\neq\overline{\gamma}(1)$, and we have that $\int_\gamma\omega=\int_{\overline{\gamma}}\mathrm{d}t=\overline{\gamma}(1)-\overline{\gamma}(0)\neq0$.

(3) ⇒ (4). Let $\overline{\pi}:\overline{L}\to L$ be the covering associated to the kernel of the monodromy homomorphism; it is the smallest covering where the pull-back of the time form has no periods. From formula (1), there is a well-defined developing map $\overline{\mathscr{D}}:\overline{L}\to\mathbf{C}$, satisfying $\overline{\mathscr{D}}\equiv\mathscr{D}\circ\overline{\pi}$. Let p,q be such that $\overline{\mathscr{D}}(p)=\overline{\mathscr{D}}(q)$. Let $\overline{\gamma}\colon[0,1]\to\overline{L}$ be a path joining p and q, and let $\gamma=\overline{\pi}\circ\overline{\gamma}$. Since $\overline{\mathscr{D}}(x)=\int^x\overline{\pi}^*\omega$, $\int_{\overline{\gamma}}\overline{\pi}^*\omega=0$, and thus $\int_\gamma\omega=0$. By (3), γ is closed. Since $\int_\gamma\omega=0$ then, by the defining property of $\overline{L}$, the lift $\overline{\gamma}$ of γ is closed. Thus, $p=q$, and $\overline{\mathscr{D}}$ is one-to-one.

(4) ⇒ (5). Let $\overline{\pi}:\overline{L}\to L$ be the Galois covering associated to the kernel of the monodromy homomorphism and consider $\overline{\mathscr{D}}:\overline{L}\to\mathbf{C}$ the induced developing map, which is one-to-one by hypothesis. Let $\Omega\subset\mathbf{C}$ denote its image. We have an induced monodromy homomorphism $\overline{\mathrm{mon}}:\mathrm{Gal}(\overline{L}/L)\to\mathbf{C}$, which satisfies $\overline{\mathscr{D}}(\gamma\cdot x)=\overline{\mathscr{D}}(x)+\overline{\mathrm{mon}}(\gamma)$, and which, by the definition of $\overline{L}$, is also one-to-one. In this way, $\overline{\pi}\circ\overline{\mathscr{D}}^{-1}:\Omega\to L$ is a covering map whose group of deck transformations is given by translations, a covering map which uniformizes the translation structure on L.

(5) ⇒(1). Let $U\subseteq\mathbf{C}$, $0\in U$ and $\phi:U\to M$ a solution of X with initial condition p. By hypothesis, we have an open subset $\Omega\subseteq\mathbf{C}$, $0\in\Omega$, and a covering map $\overline{\mathscr{D}}:\Omega\to L$ with $\overline{\mathscr{D}}(0)=p$. Up to a translation, we may suppose that the germs of ϕ and $\mathscr{D}$ around 0 coincide. Let us consider the analytic continuation of the germ of ϕ around 0. Let $\gamma_1:[0,1]\to\mathbf{C}$, $\gamma_1(0)=0$. If the image of γ_1 is contained in Ω, the analytic continuation of the germ of ϕ at 0 along γ at $\gamma_1(1)$ is given by $\overline{\mathscr{D}}$. If $\gamma_1(t)\in\Omega$ for $t<t_0$ and $\gamma_1(t_0)\notin\Omega$, $\overline{\mathscr{D}}$ does not have an analytic continuation at $\gamma_1(1)$, for $\overline{\mathscr{D}}$, being a covering map, is proper. Thus, if the analytic continuation of ϕ may be defined along the paths $\gamma_i:[0,1]\to\mathbf{C}$, $\gamma_i(0)=0$, $i=1,2$, with $\gamma_1(1)=\gamma_2(1)$, these analytic continuations coincide at $\gamma_1(1)$, since they are given by $\mathscr{D}$. □

For a holomorphic vector field X on the complex manifold M, the *uniformizability locus of X* is the set of points of M where either X vanishes or where one (hence all) of the above conditions hold. We will see in Theorem 3.3 that this locus is closed.

Proposition 2.2. *Let X be a holomorphic vector field on the manifold M. The following are equivalent.*

1. *The uniformizability locus of X is all of M.*
2. *There exists an open subset $\Omega\subset\mathbf{C}\times M$, $\{0\}\times M\subset\Omega$, and $\Phi:\Omega\to M$, such that*
   - (a) *$\Phi(\cdot,p)$ is a solution of X,*
   - (b) *$\Phi(0,p)=p$,*
   - (c) *for every p in M, for $\Omega_p=\{t\in\mathbf{C}\mid(t,p)\in\Omega\}$, Ω_p is connected, and the map $j:\Omega_p\to M\times\mathbf{C}$, $t\mapsto(\Phi(t,p),p)$ is proper.*
3. *There exists an open subset $\Omega\subset\mathbf{C}\times M$, $\{0\}\times M\subset\Omega$, and $\Phi:\Omega\to M$, such that*
   - (a) *$\Phi(\cdot,p)$ is a solution of X.*

(b) $\Phi(0,p)=p$.
(c) *If* $(t,p)\in\Omega$ *and* $(s,\Phi(t,p))\in\Omega$ *then* $(t+s,p)\in\Omega$ *and* $\Phi(s,\Phi(t,p))=\Phi(t+s,p)$.

(4) *In the nonsingular foliation by curves on* $\mathbf{C}\times M$ *induced by* $-\partial/\partial t\oplus X$*, every leaf that intersects* $\{0\}\times M$ *does so at only one point.*

Proof. (1) $\Rightarrow$ (2). For $p\in M$, let $\phi_p:\Omega_p\to\mathbf{C}$ be the solution with initial condition p such that $\phi_p\times i$ is proper. Let $\Omega=\cup_{p\in M}(\Omega_p\times\{p\})$. It is an open subset of $\mathbf{C}\times M$ by the continuity of solutions with respect to initial conditions. Let $\Phi:\Omega\to M$ be given by $\Phi(t,p)=\phi_p(t)$. By definition, we have (2a) and (2b). Condition (2c) is simply a restatement of the properness condition.

(2) $\Rightarrow$ (3). Let $\Omega\subset\mathbf{C}\times M$ and $\Phi:\Omega\to M$ as in (2). They satisfy conditions (3a) and (3b). Let $t\in\Omega_p$, $s\in\Omega_{\Phi(t,p)}$. Let $\phi_1:\Omega_p\to M$ be given by $\phi_1(u)=\Phi(u,p)$, $\phi_2:(\Omega_{\Phi(t,p)}+t)\to M$ by $\phi_2(u)=\Phi(u-t,\Phi(t,p))$. Let W be the connected component of $\Omega_p\cap(\Omega_{\Phi(t,p)}+t)$ containing t. Since both ϕ_1 and ϕ_2 are solutions of X and $\phi_1(t)=\phi_2(t)$, ϕ_1 and ϕ_2 agree in restriction to W. Let $\{y_i\}$ be a sequence in W converging to a point $y_\infty\in\mathbf{C}$. If $y_\infty\in\partial\Omega_p\cap(\Omega_{\Phi(t,p)}+t)$ then, on the one hand, $\{\phi_1(y_i)\}$ escapes from every compact subset of M, and, on the other, since ϕ_2 is holomorphic in a neighborhood of y_∞, $\{\phi_2(y_i)\}$ converges. Thus, $(\Omega_{\Phi(t,p)}+t)$, which is connected by hypothesis, is contained in Ω_p. Since $s+t\in(\Omega_{\Phi(t,p)}+t)$, $s+t\in\Omega_p$, and $\phi_1(s+t)=\phi_2(s+t)$, this is, $\Phi(s+t,p)=\Phi(s,\Phi(t,p))$.

(3) $\Rightarrow$ (4). On $\mathbf{C}\times M$, consider the vector field such that, for every $t\in\mathbf{C}$, its restriction to $\{t\}\times M$ is X, and let us still denote it by X. Likewise, let $\partial/\partial t$ be the vector field on $\mathbf{C}\times M$ induced by the vector field $\partial/\partial t$ on the factor $\mathbf{C}$. Consider the vector field $-\partial/\partial t\oplus X$ on $\mathbf{C}\times M$ and let $\mathscr{G}$ be the nonsingular foliation by curves that it induces. Consider the action ρ of $\mathbf{C}$ on $\mathbf{C}\times M$ given by translations in the first factor, $\rho(s,(t,p))=(s+t,p)$. It preserves the vector field $-\partial/\partial t\oplus X$ and, a fortiori, the foliation $\mathscr{G}$.

For every $p\in M$, there is a disk $\Delta\subset\mathbf{C}$ centered at 0, where $s\mapsto(-s,\Phi(s,p))$ is defined; it parametrizes the leaf of $\mathscr{G}$ through $(0,p)$. Using the action ρ, the map $s\mapsto(t-s,\Phi(s,p))$ is defined in the same Δ and parametrizes the leaf of $\mathscr{G}$ through (t,p).

Let $\Omega'\subset\mathbf{C}\times M$ be the saturated of $\{0\}\times M$ by $\mathscr{G}$. We will begin by proving that $\Omega'\subseteq\Omega$. Let L be a leaf of $\mathscr{G}$ intersecting Ω. Let us prove that $L\subset\Omega$. Since Ω is open, it intersects L in an open subset. Let $(t,p)\in L$ be an accumulation point of points in $L\cap\Omega$ (considering L with its manifold topology). By the previous arguments, there exists $s_0\in\mathbf{C}$ such that $(t-s_0,\Phi(s_0,p))\in L\cap\Omega$. Since (s_0,p) and $(t-s_0,\Phi(s_0,p))$ are both in Ω, hypothesis (3) implies that $(t,p)\in\Omega$: Ω intersects L in a closed subset, and thus $L\subset\Omega$. Since $\{0\}\times M\subset\Omega$, $\Omega'\subset\Omega$, as stated.

Let $L\subset\Omega'$ be the leaf of $\mathscr{G}$ passing through $(0,q)$. If $(t,p)\in L$ and s is sufficiently small, $s\mapsto(t-s,\Phi(s,p))$ parametrizes a disk in L belonging to Ω and since, again by (3), $\Phi(t-s,\Phi(s,p))=\Phi(t,p)$, Φ is constant along L. Since $\Phi(0,q)=q$, Φ takes the value q at all points of L. In consequence, L intersects $\{0\}\times M$ at only one point.

(4) $\Rightarrow$ (1). Let us prove that (4) implies condition (3) from Proposition 2.1. Let L be a leaf of $\mathscr{F}$ (the foliation induced by X) in $M\setminus\operatorname{Sing}(X)$, ω its time form, $\gamma:[0,1]\to L$ a curve such that $\int_\gamma\omega=0$. We will prove that γ is closed. Let $p=\gamma(0)$. We keep the notations from the previous step. Let $t:\mathbf{C}\times M\to\mathbf{C}$ and $\pi:\mathbf{C}\times M\to M$ be, respectively, the projections onto the first and second factors. Let $\widehat{L}$ be the leaf of $\mathscr{G}$ passing through

$(0,p)$. The map $\pi|_{\widehat{L}}:\widehat{L}\to L$ is a covering one, and ω lifts via $\pi|_{\widehat{L}}$ to $-\mathrm{d}t|_{\widehat{L}}$. Thus, for the lift $\widehat{\gamma}\colon[0,1]\to\widehat{L}$ of γ based at $(0,p)$, $\widehat{\gamma}(1)=(-\int_\gamma\omega,\gamma(1))=(0,\gamma(1))$. Since, by hypothesis, $\widehat{L}$ may intersect $\{0\}\times M$ at no point other than $(0,p)$, $\gamma(1)=p$, and γ is closed. □

We arrive to our central definition.

Definition 2.3. Let M be a complex manifold, X be a holomorphic vector field on M. We say that X is *univalent* or *semicomplete* if any one (hence all) of the conditions of Proposition 2.2 is satisfied.

Notice that *complete vector fields are semicomplete*. Indeed, complete vector fields satisfy condition (3) in Proposition 2.2 with $\Omega=\mathbf{C}\times M$. Notice also that *the class of semicomplete vector fields is stable under restrictions*: if X is a semicomplete vector field on M (e.g., a complete vector field) and $U\subset M$ is an open subset, the restriction of X to U is still semicomplete. (This follows directly from condition (4) in Proposition 2.2.)

A consequence of the last condition is one of Rebelo's key observations: *it makes sense to speak about germs of semicomplete of vector fields*. If X is a vector field on M and $p\in\mathrm{Sing}(M)$, we say that *the germ of X at p is semicomplete* if there is a neighborhood of p in restriction to which X is semicomplete. In dire contrast with the real case, there exist germs of holomorphic vector fields that are not semicomplete! This opens the door to the local study of complete vector fields.

In the same way as an action can be associated to a complete vector field, a unique global object can be attached to a semicomplete vector field, from which it can be furthermore recovered.

Definition 2.4. Let M be a manifold. A *maximum* **C**-*transformation group* or *semiglobal flow* on M is a pair (Ω,Φ), with $\Omega\subset\mathbf{C}\times M$ an open subset containing $\{0\}\times M$, with $\Omega\cap(\mathbf{C}\times\{p\})$ connected for each p, and $\Phi:\Omega\to M$, a map such that

(1) $\Phi(0,p)=p$;
(2) if $(t,p)\in\Omega$ and $(s,\Phi(t,p))\in\Omega$ then $(t+s,p)\in\Omega$ and $\Phi(s,\Phi(t,p))=\Phi(t+s,p)$.

The second condition may be replaced by the tandem:

(2a) if (t,p), $(s,\Phi(t,p))$ and $(t+s,p)$ are all three in Ω, $\Phi(s,\Phi(t,p))=\Phi(t+s,p)$;
(2b) for every p in M, for $\Omega_p=\{t\in\mathbf{C}\mid(t,p)\in\Omega\}$, Ω_p, the map $j:\Omega_p\to M\times\mathbf{C}$, $t\mapsto(\Phi(t,p),p)$ is proper.

On its turn, the last condition may be rewritten as:

(2b') for every p in M, for every sequence $\{t_i\}$ such that $\{(p,t_i)\}$ belongs to Ω and converges to a point in $\partial\Omega$, the sequence $\{\Phi(t_i,p)\}$ escapes from every compact subset of M.

A standard flow (action of **C**) is naturally such an object. A semiglobal flow can be considered as a Lie groupoid $\Omega\rightrightarrows M$ over M having Φ for target map, and for source one the restriction to Ω of the projection $\mathbf{C}\times M\to M$. This generalizes the standard Lie groupoid associated to an action (see [Mac87, Ex. 1.1.9]).

These definitions of *maximum* **C**-*transformation group* and *univalence* are due to Palais, in the more general setting of infinitesimal Lie group actions on manifolds [Pal57, Ch. III, Defs. VI and VII]); they were rediscovered by Rebelo in the context of holomorphic vector fields under the names of *semiglobal flow* and *semicompleteness* [Reb96, Déf. 2.3]). The connection with condition (3) of Proposition 2.1 appears in [Reb96, Prop. 2.7] and [Reb00, Prop. 2.1]. The link with conditions (5) and (4) of the same proposition appears in [Gui06]. Some of these notions also make sense in the topological setting (some even in the set-theoretic one), and for this we refer the to the work of Abadie [Aba03], who studies the equivalent notion of *partial action*.

All the notions in this section and in the following ones can be naturally defined for analytic spaces.

3. Semicompleteness as a closed condition

Let us now present some general results, valid in all dimensions, related to the property of semicompleteness when vector fields are considered in families. The proofs of these have all the same underlying principle.

Recall that if L is a leaf of a foliation $\mathscr{F}$ and $p \in L$, if T is a transverse to $\mathscr{F}$ at p, by the process of lifting paths to nearby leaves we obtain a *holonomy* representation $\mathrm{hol}: \pi_1(L,p) \to \mathrm{Diff}(T,p)$ (see, for instance, [IY08, Ch. I, Sect. 2]). In the present setting we have natural choices for the submersions involved in its definition: given a transversal T though a point p on a leaf L, for a sufficiently small neighborhood $U \subset M$ of p, we may consider the unique submersion $\pi: U \to L$ that has T as one of its fibers and that is a translation in restriction to each leaf (with respect to each leaf's translation structure), and which, in particular, is the identity in restriction to L. Through such a construction, paths may be lifted *isometrically* to neighboring leaves.

3.1. Holonomy.

The holonomies of semicomplete vector fields are very special.

Theorem 3.1 (Fundamental Lemma)**.** *Let M be a complex manifold, X a univalent vector field on M. Let $p \in M$ be such that $X(p) \neq 0$, let L be the orbit of X through p, considered with the translation structure induced by X. There is a group homomorphism $\rho: \mathrm{mon}(\pi_1(L)) \to \mathrm{hol}(\pi_1(L))$ such that, for all $\gamma \in \pi_1(L)$,*

$$\mathrm{hol}(\gamma) = \rho \circ \mathrm{mon}(\gamma),$$

this is, the holonomy representation factors through the monodromy one. In particular, the holonomy is abelian.

Proof. We must define $\rho(\mathrm{mon}(\alpha))$ as $\mathrm{hol}(\alpha)$, and for this to be well-defined it is necessary and sufficient that the holonomy along a path of trivial monodromy is trivial. Let L be a leaf, ω its time form, $p \in L$. Let $\gamma: ([0,1],0) \to (L,p)$ be a smooth closed path representing a class in $\pi_1(L)$ with trivial monodromy, a path such that $\int_\gamma \omega = 0$. Let T be a transversal to L at p. For $q \in T$ sufficiently close to p, let L_q the orbit of X through q and ω_q the corresponding time form. Let $\gamma_q: ([0,1],0) \to (L_q,q)$ be such that $\omega_q(\gamma_q'(t)) = \omega(\gamma'(t))$ for every $t \in [0,1]$ (the isometric lift of γ through q). By construction, $\int_{\gamma_q} \omega_q = 0$, and, since X is univalent, γ_q is closed, and intersects T at q. The holonomy of γ is trivial. □

The following is one of the early incarnations of the Fundamental Lemma.

Corollary 3.2 ([GR97, Lemma 3.1])**.** *Let M be a manifold, X a univalent vector field on M inducing the foliation $\mathscr{F}$. Let L be an orbit of X with time form ω and $\gamma:[0,1]\to L$ a closed curve such that $\int_\gamma \omega = 0$. Then the holonomy of $\mathscr{F}$ along γ is trivial.*

3.2. **Univalence as a closed condition.** We now present two closely related results that present univalence as a closed condition. This was well-known in Painlevé's time, and was fundamental in his research on differential equations with uniform solutions [Pai00, §6] (see also [Inc44, §14.12]).

Theorem 3.3 ([GR12, Cor. 12])**.** *Let M be a complex manifold, X a vector field on M. The uniformizability locus of X is a closed subset of M (saturated by X). In particular, if the restriction of the vector field to an open and dense subset of M is semicomplete, so is the vector field.*

Proof. Let p be a point that is not in the uniformizability locus of X, L_p the orbit of X through p, ω_p the associated time form. Let $\gamma_p:[0,1]\to L_p$ be an open smooth path such that $\int_{\gamma_p}\omega_p = 0$. For $q\in M$ close to p, let L_q be the orbit of X through q and ω_q the corresponding time form. There exists a smooth path $\gamma_q:[0,1]\to L_q$, $\gamma_q(0)=q$ such that $\omega_q(\gamma_q'(t))=\omega_p(\gamma_p'(t))$ for every $t\in[0,1]$ (an isometric lift). Notice that $\int_{\gamma_q}\omega_q=0$ but that, since γ_q is close to γ_p, it is an open curve. □

As a prototypical application of this principle, we have:

Corollary 3.4 ([GR97, Cor. 2.6])**.** *Let X be a holomorphic vector field defined on $(\mathbf{C}^n,0)$. Let $X=X_k+X_{k+1}+\cdots$ be the Taylor series of X, where X_i is a homogeneous vector field of degree i and X_k is not identically zero. If the germ of X at 0 is semicomplete, X_k is semicomplete as a vector field on $\mathbf{C}^n$.*

Proof. Let $U\subset\mathbf{C}^n$, $0\in U$ be an open subset where X is semicomplete. For α in $\mathbf{C}^*$, let $h:\mathbf{C}\to\mathbf{C}$ be the homothety $h_\alpha(z)=\alpha^{-1}z$. The vector field

$$Z_\alpha=\frac{1}{\alpha^{k-1}}h_*X=X_k+\alpha X_{k+1}+\alpha^2X_{k+2}+\cdots,$$

defined in $\alpha^{-1}U$, is semicomplete. In the subset $A=(\cup_\alpha(\{\alpha\}\times\alpha^{-1}U))\cup(\{0\}\times\mathbf{C}^n)$ of $\mathbf{C}\times\mathbf{C}^n$, we have the holomorphic vector field Z given by Z_α in $\{\alpha\}\times\alpha^{-1}U$ and by X_k in $\{0\}\times\mathbf{C}^n$. Since it is semicomplete in $A\cap(\mathbf{C}^*\times\mathbf{C}^n)$, it is semicomplete in its closure A, and is thus semicomplete in restriction to $\{0\}\times\mathbf{C}^n$. □

Theorem 3.5 (Ghys-Rebelo [GR97, Sect. 2.2])**.** *Let M be a complex manifold. In the space of holomorphic vector fields on M, the univalent ones form a closed subset with respect to the topology of uniform convergence on compact sets.*

Proof. The comments at the beginning of this section remain valid for laminated structures other than foliations, and the proof we present will make use of this. Let $\{X_i\}$ a sequence of semicomplete holomorphic vector fields on M converging uniformly on compact subsets to the vector field X_∞ (this gives a continuous vector field in the laminated space $M\times(\mathbf{N}\cup\{\infty\})$—where the topology in $\mathbf{N}\cup\{\infty\}$ is that of $\{0\}\cup\{1/n\}_{n\in\mathbf{N}}$ in $\mathbf{R}$—that equals X_i in restriction to $M\times\{i\}$). Suppose that X_∞ is not semicomplete, that there exists a smooth open path $\gamma_\infty:[0,1]\to M$ tangent to an orbit L_∞ of M, with time form ω_∞, such that $\int_{\gamma_\infty}\omega_\infty = 0$. Let $p=\gamma_\infty(0)$, L_i the leaf of X_i through p, ω_i the time-form of X_i on L_i. If i is big enough, there exists a smooth path $\gamma_i:[0,1]\to L_i$, $\gamma_i(0)=p$, such that $\omega_i(\gamma_i'(t)) = \omega_\infty(\gamma_\infty'(t))$ for all $t\in[0,1]$ (an isometric lift in the laminated space). In particular, $\int_{\gamma_i}\omega_i = \int_{\gamma_\infty}\omega_\infty \neq 0$. Since the curves γ_i depend continuously on i, γ_i is open if i is big enough (for γ_∞ is), and X_i is not semicomplete. □

There exist sequences of complete holomorphic vector fields on manifolds that converge to vector fields which are not complete (see [For95, Sect. 3]).

The results of this section can be easily transported to Palais's more general context [Pal57].

4. Semicomplete vector fields in dimension one

For a vector field on a neighborhood of 0 in $\mathbf{C}$ of the form $(\lambda z+\cdots)\partial/\partial z$, $\lambda\neq 0$, its germ at 0 is linearizable (see [Lor21, Prop. 2.3], [IY08, Thm. 5.5]), and its germ is thus semicomplete. The vector field $z^2\partial/\partial z$ in $\mathbf{C}$ is semicomplete (and, in consequence, so is its germ at 0), for it extends as a holomorphic vector field when $\mathbf{C}$ is compactifed into $\mathbf{P}^1$: in the coordinate $w=-1/z$, it reads $\partial/\partial w$. These give all the germs of semicomplete vector fields in dimension one (see also [Reb96, Prop. 3.1]):

Proposition 4.1. *Let X be a germ of semicomplete vector field in $(\mathbf{C},0)$, having a singularity at 0. In suitable coordinates, X is given either by $z^2\partial/\partial z$ or by $\lambda z\partial/\partial z$ for some $\lambda\in\mathbf{C}^*$.*

Proof. Let Δ be a disk around 0 on which the restriction of X is semicomplete, and let $\Delta^*=\Delta\setminus\{0\}$. Let ω be the time form of X on Δ^*, $\gamma:[0,1]\to\Delta^*$ a closed curve winding simply and positively around 0.

If $\int_\gamma\omega = 0$, there is a well-defined developing map : $\Delta^*\to\mathbf{C}$ which maps X to $\partial/\partial w$. The semicompleteness hypothesis implies that it is one-to-one. By Picard's Little Theorem, it cannot have an essential singularity at 0, and it extends thus as a one-to-one map taking values in $\mathbf{P}^1$. It maps 0 to a singular point of the vector field, which can only be the point at infinity, leading to the first possibility in the statement.

If $\int_\gamma\omega\neq 0$, we may suppose, up to multiplying X by a constant, that $\int_\gamma\omega = 2i\pi$. Let $\mathscr{D}:\widetilde{\Delta^*}\to\mathbf{C}$ be the developing map, which is one-to-one by the semicompleteness hypothesis. The map $\exp\circ\mathscr{D}$ induces a well-defined map $f:\Delta^*\to\mathbf{C}^*$, mapping X to $z\partial/\partial z$. Let us prove that f is one-to one. Let p and q be such that $f(p)=f(q)$. There is a path $\rho:[0,1]\to\Delta^*$ joining p and q in Δ^* such that $\int_\rho\omega$ is a multiple of $2i\pi$, and by conveniently concatenating it with a closed path based at q, we may suppose that $\int_\rho\omega = 0$. Since X is semicomplete, ρ is closed and $p=q$: f is one-to-one. As before, f extends as a holomorphic map taking values in $\mathbf{P}^1$, and 0 must be mapped to a singular point of the vector field $z\partial/\partial z$, either 0 or ∞. But the second possibility cannot actually occur, for the period of $z\partial/\partial z$ around ∞ is $-2i\pi$ (in the coordinate $w=1/z$, $z\partial/\partial z$ reads $-w\partial/\partial w$). □

By the way, we have shown that, in dimension one, every semicomplete germ of vector field is the local model of a complete vector field (actually, of one defined on a compact curve).

Although seemingly naïf, the following corollary has far-reaching consequences.

Corollary 4.2. *Let X be a holomorphic vector field defined on a neighborhood of* 0 *in* $\mathbf{C}$*, and having there a zero of order r. If $r > 2$, X is not semicomplete.*

5. SEMICOMPLETE VECTOR FIELDS IN DIMENSION TWO

In dimension one, all nondegenerate singularities vector fields are semicomplete, while degenerate ones are very rare. This will also be the trend in dimension two. In the degenerate case, isolated singularities of semicomplete vector fields are so rare that they fit within small lists of orbital normal forms. We will discuss exclusively isolated singularities; for nonisolated ones, we refer the reader to [Reb99] as well as to Section 7. For the local theory of holomorphic foliations and vector fields in dimension two, the reader is referred to [Lor21, Ch. 6], [CS87], [Bru15, Ch. 1].

5.1. Nondegenerate vector fields.

Linear vector fields are complete, and, in consequence, all linearizable germs of vector fields are semicomplete as well. More generally, germs of vector fields having semicomplete normal forms are semicomplete. This allows to show, with barely any work (but relying on some very deep results) that many germs of vector fields are semicomplete.

Recall that a singular, nondegenerate vector field is said to belong to the *Poincaré domain* if the convex hull of the eigenvalues of its linear part at the singular point does not contain 0, and is said to belong to *Siegel's domain* otherwise. In the Poincaré domain, germs of vector fields admit a "Poincaré-Dulac" normal form (see [IY08, Thm 5.5]). As observed by Chaperon in [Cha86, Sect. 1.2], *Poincaré-Dulac normal forms are complete as vector fields on* $\mathbf{C}^n$, and their germs are thus semicomplete.

In the Siegel domain, most germs of vector fields are holomorphically linearizable [Brj71] (see also [IY08, Thm. 5.22]), and we can thus, a priori, affirm that they are semicomplete. Since the problem of semicompleteness concerns only the tangential geometry of the orbits, Chaperon's results on C^k-conjugacy imply that *weakly hyperbolic* germs of vector fields are semicomplete as well (see Theorem 2 and Corollary 8 in [Cha86]).

Beyond these results, in which semicompleteness comes as a collateral fact, it can be directly proved that, in all dimensions, germs of nondegenerate vector fields are semicomplete (see [Reb00, Thm. A] and [Rei06] for earlier results pointing in this direction):

Theorem 5.1 ([Gui20])**.** *Let X be a nondegenerate singular holomorphic vector field on* $(\mathbf{C}^n, 0)$*. Then the germ of X at* 0 *is semicomplete.*

Proof (in dimension two, in the Siegel domain). A vector field like the ones under consideration has two transverse separatrices, as it follows from the invariant manifold theorem [IY08, Thm. 7.1] (see also [MM80, Appendix II]). They can be redressed onto the coordinate axis, and thus the vector field may be written, up to a constant factor, as

$$X = zf(z,w)\frac{\partial}{\partial z} - \lambda w g(z,w)\frac{\partial}{\partial w},$$

with $\lambda \in \mathbf{R}$, $\lambda > 0$, $f(0) = 1$ and $g(0) = 1$. Let $\Re(X)$ be the real vector field given by the real part of X. We have

$$\Re(X) \cdot |z|^2 = 2|z|^2\Re(f),\ \Re(X) \cdot |w|^2 = -2\lambda|w|^2\Re(g). \tag{2}$$

By conveniently scaling the variables we may suppose that the vector field is defined in a neighborhood of the closure of the polydisk Δ^2 of radius 1, that $\Re(X)\cdot|z|^2$ is strictly positive away from $\{z=0\}$, and that $\Re(X)\cdot|w|^2$ is strictly negative away from $\{w=0\}$.

Let $\gamma\colon[0,1]\to\Delta^2$ be a path taking values in the leaf L (different from the separatrices), such that $\int_\gamma\omega=0$, where ω is the time form of X on L. We will prove that γ is closed. Let B be the boundary component of Δ^2 given by $|z|=1$. If an orbit of $\Re(X)$ starts within Δ^2, it will intersect B in finite time. In fact, by (2), $|z|^2$ will increase along the orbit while $|w|^2$ decreases, and thus, if the orbit exits Δ^2, it ought to do so through B. If an orbit of $\Re(X)$ had an ω-limit set within Δ^2, this set would have to be contained in a level set of $|z|^2$, but since, by (2), such level sets are transverse to $\Re(X)$, they cannot contain forward-invariant sets.

By following positively the orbits of $\Re(X)$, we can deform the path γ, through a homotopy with fixed endpoints remaining within L, into one that is the concatenation of three smooth paths:

- a first one, ρ_1, going from $\gamma(0)$ to B through an orbit of $\Re(X)$,
- a second one, τ contained in $L\cap B$, and
- a third one, ρ_2, going back to $\gamma(1)$ by following negatively an orbit of $\Re(X)$.

Notice that $\omega(\Re(X))\in\mathbf{R}$ and thus, for $i=1,2$, $\int_{\rho_i}\omega\in\mathbf{R}$. Up to a homotopy with fixed endpoints, we can suppose that τ is either constant or, by (2), always transverse to $\Re(X)$. In the second case, $\Im(\omega(\tau'(s)))$ does not vanish (has constant sign), and thus $\Im(\int_\tau\omega)\neq 0$; however, $\Im(\int_\tau\omega)$ vanishes, for it is equal to $\Im(\int_\gamma\omega)$. Thus, τ is constant, and ρ_1 and ρ_2 go along the same orbit of $\Re(X)$. Since $\int_{\rho_1*\rho_2}\omega=0$, and $\rho_1*\rho_2$ is actually closed. This proves that γ is a closed path. By condition (3) in Proposition 2.1, X is univalent in restriction to Δ^2. □

5.2. **Saddle-nodes.** The least degenerate among degenerate vector fields are *saddle-nodes*, singular points which have one vanishing and one nonvanishing eigenvalue. From the orbital point of view, germs of saddle-nodes form an infinite-dimensional family of equivalence classes. In it, only countably many of these come from semicomplete vector fields.

Theorem 5.2 (Rebelo [Reb00, Thm. 4.1]). *Let X be a holomorphic vector field defined on a neighborhood of 0 in $\mathbf{C}^2$, whose germ at 0 is semicomplete. Suppose that 0 is an isolated equilibrium point of X and that the linear part of X at 0 has one vanishing and one nonvanishing eigenvalue. Then, in suitable coordinates and up to multiplication by a nonvanishing holomorphic function, X may be written as*

$$x^2\frac{\partial}{\partial y}+y(1+\lambda x)\frac{\partial}{\partial y} \tag{3}$$

for some $\lambda\in\mathbf{Z}$.

Observe that the germs of the vector fields (3) are semicomplete since, as vector fields on $\mathbf{C}^2$ their local flow in time t is given by

$$(x,y)\mapsto\left(\frac{x}{1-tx},\frac{ye^t}{(1-tx)^\lambda}\right).$$

Thanks to the works Dulac [Dul05], Hukuahara-Kimura-Matuda [HKM61] and Martinet-Ramis [MR82] (among others), we have a very complete description of the saddle-node singularities of foliations. We will present some fact around saddle-nodes following Loray's account [Lor21, Sect. 6.4] to some extent before proceeding to the proof of the theorem.

Dulac proved that there are coordinates where the foliation induced by X is the one induced by a one-form of the form $x^{p+1}\,\mathrm{d}y - A(x,y)\,\mathrm{d}x$, with $\partial A/\partial y|_0 \neq 0$. In particular, there is a unique invariant analytic curve, given in these coordinates by $x=0$, tangent to the eigenspace associated to the nonvanishing eigenvalue and a local fibration, $(x,y)\mapsto x$, which is transverse to the foliation $\mathscr{F}$ induced by X away from this curve. For such a form, there is a unique $p\in\mathbf{N}$, $p\geq 1$ and a unique $\lambda\in\mathbf{C}$ such that a formal fibered transformation, $(x,y)\mapsto(x,\phi(x,y))$, redresses the foliation into the one induced by

$$\omega_{p,\lambda} = x^{p+1}\,\mathrm{d}y - y(1+\lambda x^p)\,\mathrm{d}x, \tag{4}$$

having the first integral

$$yx^{-\lambda}\mathrm{e}^{1/px^p}. \tag{5}$$

The number $p+1$ is the *multiplicity* of the saddle node, λ its *formal invariant*. Thus, at the formal level (but, in general, not at the holomorphic one) there is a second invariant curve, given by $y=0$ in the above expression, the *weak* or *central separatrix*.

For $j\in\mathbf{Z}/2p\mathbf{Z}$, consider the sector

$$U_j = \left\{ \frac{2j+1}{2p}\pi - \frac{\pi}{p} + \varepsilon < \arg(x) < \frac{2j+1}{2p}\pi + \frac{\pi}{p} - \varepsilon \right\},$$

of angle barely smaller than $2\pi/p$. Hukuahara, Kimura and Matuda have shown that, above each one these sectors, the previous formal transformation represents a holomorphic one, that there exists a holomorphic map ψ_j such that, for $x\in U_j$, the map $(x,y)\mapsto(x,\phi_j(y))$ redresses the restriction of $\mathscr{F}$ to $\pi^{-1}(U_j)$ to its formal model [HKM61, Ch. III, Sect. II, §6]. This reduces the problem of understanding $\mathscr{F}$ to the problem of understanding first the formal model (4) in each sector, and then the transition maps between neighboring sectors. For $i\in\mathbf{Z}/p\mathbf{Z}$, let $V_i^+ = U_{2i}\cap U_{2i+1}$. For $x\in V_i^+$, $\Re(x^p)>0$. Above V_i^+, the behavior is of node type: all leaves converge to the origin and any determination of the first integral takes, in any neighborhood of the origin, all complex values. The transition map $\phi_i^+ = \psi_{2i+2}\circ\psi_{2i+1}^{-1}$ preserves the foliation induced by the form (4) above V_i^+, and is a globally defined affine map. Similarly, let $V_i^- = U_{2i+1}\cap U_{2i+2}$; for $x\in V_i^-$, $\Re(x^p)<0$. Above V_i^-, the behavior of $\mathscr{F}$ is of saddle type: the separatrix given by $y=0$ is the only leaf that converges to the origin, and all the others escape away from it; a determination of the first integral (5) takes values only in a neighborhood of 0. The map $\phi_i^- = \psi_{2i+1}\circ\psi_{2i+2}^{-1}$ preserves the foliation induced by the form (4) above V_i^-, and is a local biholomorphism of $\mathbf{C}$ fixing 0. The data $(\phi_0^+,\phi_0^-,\ldots,\phi_{p-1}^+,\phi_{p-1}^-)$ determines the foliation. The weak separatrix will be convergent if and only if all the affine maps ϕ_i^+ are linear (fixing 0 and allowing the determination of the weak separatrix in one sector to be continued to the neighboring one). In this last case the holonomy of the weak separatrix is $\phi_{p-1}^-\circ\phi_{p-1}^+\circ\phi_{p-2}^-\circ\cdots\circ\phi_1^-\circ\phi_0^-\circ\phi_0^+$.

For the foliations given by the formal normal forms (4) with $p=1$ the holonomy is $y\mapsto \mathrm{e}^{2i\pi\lambda}y$. The result of Martinet and Ramis implies that *if a saddle-node of multiplicity* *two ($p=1$) has a convergent weak separatrix and if the holonomy around this separatrix is trivial, it is holomorphically equivalent to the foliation given by (4), with $\lambda \in \mathbf{Z}$.* In fact, under the hypothesis, $\phi_0^- = (\phi_0^+)^{-1}$ is linear, and this reads, within the framework of Martinet and Ramis's theorem, as the triviality of the "cohomological part" of their invariant, which implies that the foliation is holomorphically conjugated to its formal normal form.

Proof of Theorem 5.2. We follow Rebelo's original proof. Let X be a germ of vector field like in the statement of the theorem. By Dulac's theorem, X may be written as $f(x,y)[x^{p+1}\partial/\partial x + A(x,y)\partial/\partial y]$ for some holomorphic, nonvanishing function f. Above the union of sectors $\Delta = U_1 \cup U_2$, we may consider the union of the weak separatrices, in order to get an invariant curve C over a sector of angle slightly smaller than $3\pi/p$, and which projects injectively under π if $p>1$. The image of this curve under the developing map (inverse of a solution) $\mathscr{D}$ is a sector at infinity of angle slightly smaller than 3π, as a sector of angle α gets mapped to a sector of angle $p\alpha$ at infinity. In particular, if $p>1$ there are two different points in the sector that have the same image under $\mathscr{D}$. This shows that the restriction of X to Δ is not semicomplete if $p>1$. If $p=1$ these two points must be the same one, and this is only possible if the weak separatrix is convergent. In a suitable coordinate, by Proposition 4.1, the restriction of X to it is conjugate to $z^2\partial/\partial z$. By Corollary 3.2, its holonomy is trivial. The theorem is now a direct consequence of the theorem of Martinet and Ramis. □

Question 5.3. What can be said about the factor in the statement of Theorem 5.2?

In the hierarchy of degenerate vector fields, those with nilpotent (nonzero) linear part come next. The following result illustrates how incredibly restrictive the semicompleteness condition becomes as the degenerateness of the vector field increases.

Theorem 5.4 (Ghys-Rebelo [GR97, Prop. 3.16])**.** *Let X be a holomorphic vector field defined on a neighborhood of 0 in $\mathbf{C}^2$, whose germ at 0 is semicomplete. Suppose that 0 is an isolated equilibrium point of X and that the linear part of X at 0 is nilpotent but not trivial. Then, in suitable coordinates, and up to multiplication by a nonvanishing holomorphic function, X is one of the following:*

1. $(2y-x^2)\partial/\partial x + 2xy\partial/\partial y$,
2. $(3y-x^2)\partial/\partial x + 4xy\partial/\partial y$,
3. $(y-2x^2)\partial/\partial x - 2xy\partial/\partial y$,
4. $2y\partial/\partial x - 3x^2\partial/\partial y$.

5.3. **More degenerate singular points.** The isolated singularities of semicomplete vector fields cannot be too degenerate:

Theorem 5.5 (Rebelo [Reb96])**.** *Let X be a holomorphic vector field defined on a neighborhood of 0 in $\mathbf{C}^2$. Suppose that 0 is an isolated equilibrium point of X. Then the second jet of X at 0 is nontrivial.*

Proof. By the theorem of Camacho and Sad [CS82], there exists a *separatrix* of X, a curve $\gamma : (\Delta,0) \to (\mathbf{C}^2,0)$ whose image is tangent to the orbits of X. Let $\gamma_i(t) = c_i t^{s_i} + \cdots$ and suppose that $s_1 \leq s_2$ and that $c_1 \neq 0$. If $X = f\partial/\partial x + g\partial/\partial y$, its restriction to this separatrix is

$$\frac{f(\gamma_1(t),\gamma_2(t))}{\gamma_1'(t)}\frac{\partial}{\partial t}.$$

Suppose that X has order d, and let $f=\sum_{i+j\geq d} a_{ij}x^iy^j$. The first term of the contribution of $a_{ij}x^iy^j$ to the above vector field is $t^{is_1+js_2-(s_1-1)}$. Since $s_1\geq 1$,

$$is_1+js_2-(s_1-1)\geq (i+j-1)s_1+1\geq (d-1)s_1+1\geq d. \tag{6}$$

The order of X in restriction to the separatrix is at least d. Since according to Proposition 4.1 this number is at most 2 for some (i,j), we must have $d\leq 2$. Furthermore, if $d=2$ then $s_1=1$ (the separatrix is smooth). □

Thus, germs of semicomplete vector fields cannot have a vanishing second jet around an isolated singularity (Theorem 5.5). We understand incredibly well those that have a trivial first jet:

Theorem 5.6 (Ghys-Rebelo, [GR97, Thm. A])**.** *Let X be a holomorphic vector field defined on a neighborhood of 0 in $\mathbf{C}^2$, whose germ at 0 is semicomplete. Suppose that 0 is an isolated equilibrium point of X and that the first jet of X at 0 vanishes. Then, in suitable coordinates and up to multiplication by a nonvanishing holomorphic function, X is one of the following:*

1. $x^2\partial/\partial x-y[(m-1)x-my]\partial/\partial y$, $m\in\mathbf{Z}$, $m\geq 1$,
2. $x(x-2y)\partial/\partial x+y(y-2x)\partial/\partial y$,
3. $x(x-3y)\partial/\partial x+y(y-3x)\partial/\partial y$,
4. $x(2x-5y)\partial/\partial x+y(y-4x)\partial/\partial y$.

In particular, the vector fields in the above list are all the semicomplete quadratic homogeneous vector fields on $\mathbf{C}^2$ with an isolated singularity at the origin. The vector field in the first item is semicomplete, for its solution with initial condition (x,y) is given by

$$t\mapsto\left(\frac{x}{1-tx},\frac{xy(1-tx)^{m-1}}{x-y+y(1-tx)^m}\right);$$

the others have polynomial first integrals with elliptic level curves and can be integrated by elliptic functions.

Ghys and Rebelo's strategy to prove the above result starts, following Corollary 3.4, by determining the few semicomplete quadratic homogeneous vector fields on $\mathbf{C}^2$ (a result that can be traced back to Briot and Bouquet [BB55]), and then, for each one of them, by studying the vector fields that have these as 2-jets. We will mention an approach to the first part in Remark 6.9 and Section 9.2.1, and briefly discuss the second part in Remark 6.10 and Section 8.1.

With some of the results described so far we can prove the following:

Theorem 5.7 (Rebelo, [Reb00, Lemme 6.1])**.** *Let S be a Stein surface, X a complete holomorphic vector field on S. Let $p\in S$ be an isolated singularity of X. Then p is a nondegenerate one.*

Proof. Let $\gamma\colon(\Delta,0)\to(S,p)$ be a separatrix. Let Y be the vector field on Δ induced by X via γ. According to Proposition 4.1, Y is either equivalent to $\lambda w\partial/\partial w$ or to $w^2\partial/\partial w$. In the last case, the orbit to which γ belongs is a rational curve inside S. However, S is Stein, and has no complete rational curves. Thus, Y is equivalent to $\lambda w\partial/\partial w$ at 0. From (6), X has a nontrivial linear part at 0, and the inequalities in (6) become equalities, this is, $s_1 = 1$ and $s_2 = 1$. But this means that γ is smooth at 0, and thus λ is an eigenvalue of X at 0: the linear part of X is non-nilpotent. If X were a saddle-node, by Theorem 5.2, it would have a second separatrix where the restriction of X would be conjugate to $w^2\partial/\partial w$, but, again, this cannot occur in a Stein surface. □

A more general statement appears in [Reb00, Thm. 6.3]; an extension to singular analytic Stein spaces of dimension two will be discussed in Theorem 8.3.

6. Affine structures and nonisolated singularities

Some obstructions for semicompleteness are localized along the locus of zeros of a vector field. In order to exploit this this, translation structures must make place for affine ones. This will enable a better understanding of semicompleteness in the presence of nonisolated singularities.

The foliation induced by X can, in some cases, be extended to some points of $\mathrm{Sing}(X)$. Locally, X may be written as fX_0, where f is a holomorphic function and X_0 is a holomorphic vector field with singular set of codimension at least two, both defined up to units. This decomposition defines a foliation $\mathscr{F}$ on M, locally defined by X_0, with *singular set* $\mathrm{Sing}(\mathscr{F})$, locally given by the zeros of X_0 (of codimension at least two), and the *divisor of zeros* of X, given by f. A leaf of $\mathscr{F}$ may intersect the divisor of zeros of X, and $\mathrm{Sing}(\mathscr{F}) \subset \mathrm{Sing}(X)$.

6.1. **Affine structures.** In the study of semicomplete vector fields, affine structures on curves have been around ever since (to witness, Briot and Bouquet's 1855 note [BB55]). Within the thread of ideas here discussed, they first appeared in [Reb99] through the notion of *renormalized time forms*, but were not truly recognized as affine structures until [GR12].

Consider the group $\mathrm{Aff}(\mathbf{C}) = \{z \mapsto az+b\}$ of affine transformations of $\mathbf{C}$. It contains the group of translations as well as the group of orientation-preserving Euclidean isometries (corresponding to $|a| = 1$). An *affine structure* on a curve L is an atlas for its complex structure taking values in $\mathbf{C}$ with changes of coordinates in $\mathrm{Aff}(\mathbf{C})$. Translations structures are examples of affine ones. An affine structure comes with a developing map $\mathscr{D} : \widetilde{L} \to \mathbf{C}$ (a global chart for the affine structure induced on $\widetilde{L}$), and a monodromy homomorphism $\mathrm{mon} : \pi_1(L) \to \mathrm{Aff}(\mathbf{C})$ such that for every $\alpha \in \pi_1(L)$, $\mathscr{D}(\alpha \cdot p) = \mathrm{mon}(\alpha) \cdot \mathscr{D}(p)$; see [Thu97, Sect. 3.4] for details.

We will distinguish a particular class of affine structures.

Proposition 6.1. *Let L be a curve endowed with an affine structure. The following are equivalent.*

(1) *There exists $\Omega \subset \mathbf{C}$ and a Galois covering $\rho : \Omega \to L$ that is a map between curves with affine structures (whose deck transformations are given by restrictions of affine maps).*
(2) *For every open path $\gamma : [0,1] \to L$, if $\widetilde{\gamma} : [0,1] \to \widetilde{L}$ denotes one of its lifts to $\widetilde{L} \to L$, $\mathscr{D} \circ \widetilde{\gamma} : [0,1] \to \mathbf{C}$ is an open path.*

(3) *For the covering $\overline{L} \to L$ associated to the kernel of the monodromy, the induced developing map $\overline{\mathscr{D}} : \overline{L} \to \mathbf{C}$ is one-to-one.*

Proof. (1) ⇒ (2). Let $\pi : \widetilde{L} \to L$ be the universal covering. Since Ω covers L, we have an intermediate covering $\mathscr{D} : \widetilde{L} \to \Omega$, which is also a developing map for the affine structure. Notice that $\pi = \rho \circ \mathscr{D}$. If $\mathscr{D} \circ \widetilde{\gamma}$ were closed, so would $\rho \circ \mathscr{D} \circ \widetilde{\gamma} = \pi \circ \widetilde{\gamma} = \gamma$, but this curve is, by hypothesis, an open one.

(2) ⇒ (3). Let $\pi : \widetilde{L} \to L$ be the universal covering, $\mathscr{D} : \widetilde{L} \to \mathbf{C}$ the developing map; let $\overline{\pi} : \overline{L} \to L$ be the intermediate covering associated to the kernel of the monodromy, and $\overline{\mathscr{D}} : \overline{L} \to \mathbf{C}$ the map induced by $\mathscr{D}$. Let p and q be points in $\overline{L}$ having the same image under $\overline{\mathscr{D}}$ and let $\overline{\gamma} : [0,1] \to \overline{L}$ be a curve joining them, so that $\overline{\mathscr{D}} \circ \overline{\gamma}$ is a closed path. Let $\widetilde{\gamma}$ be the lift of $\overline{\gamma}$ to $\widetilde{L}$, and let $\gamma = \overline{\pi} \circ \overline{\gamma}$. Since $\mathscr{D} \circ \widetilde{\gamma}$ equals $\overline{\mathscr{D}} \circ \overline{\gamma}$, it is a closed path, and, by (2), γ is a closed path in L. This extends to neighborhoods of p and q: a point close to q identified with a point close to p via $\overline{\mathscr{D}}$ is also its image under the deck transformation in $\overline{L}$ associated to γ. This means that the monodromy of γ is trivial. Thus, by the definition of $\overline{L}$, $\overline{\gamma}$ is a closed path: p and q agree.

(3) ⇒ (1). The map $\overline{\mathscr{D}}$ identifies $\overline{L}$ with its image, $\Omega \subset \mathbf{C}$. The map $\rho : \Omega \to L$ induced by $\overline{\pi}$ is the sought one. □

An affine structure on a curve satisfying any one of the previous conditions is said to be *uniformizable*. Observe that a translation structure is uniformizable if and only if it is uniformizable as an affine one.

Example 6.2. Let $T \subset \mathbf{R}^2$ be a plane triangle, T' its image under a reflection. By identifying T and T' along the edges and removing the vertices, we get a Euclidean (hence affine) structure on the thrice-punctured sphere. This affine structure is uniformizable if and only if the triangle tesselates the plane through the group generated by the reflections on its sides. This happens if and only all of the internal angles of the triangle are aliquot parts of π, of the form π/p, π/q and π/r with p, q and r integers. The only possibilities for (p,q,r) are $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$.

6.1.1. *Connections.* Affine structures on curves may also be defined in terms of connections. Since the standard affine connection ∇ on $\mathbf{C}$, the one for which

$$\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} \equiv 0, \tag{7}$$

is invariant under the action of the affine group, the connection may be pulled back to a curve endowed with an affine structure. Reciprocally, if ∇ is an affine connection on a curve, it is necessarily torsion-free and flat, and there exists a coordinate z for which (7) holds. Such a coordinate is a chart of a well-defined affine structure.

On a curve, in the presence of a connection, the *Christoffel symbol* of a vector field Z is the function $\Gamma(Z)$ such that $\nabla_Z Z = \Gamma(Z)Z$. For a function f,

$$\Gamma(fZ) = Z(f) + f\nabla(Z). \tag{8}$$

A vector field Z is said to be *parallel* if $\Gamma(Z) = 0$. A nonconstant function will be a chart of the affine structure if its second derivative along a parallel vector field vanishes. It follows that on a curve endowed with an affine structure, given a (locally defined) vector field Z, a function ψ will be a chart of the affine structure if and only if it is a nonconstant solution of the second-order linear equation

$$Z^2\psi - \Gamma(Z)Z\psi = 0. \tag{9}$$

6.2. **Foliated connections.** A vector field on a manifold induces a foliation away from its zeros, and, along its leaves, a translation (hence affine) structure varying holomorphically in the transverse direction: a *foliated affine structure*. By gluing the associated leafwise connections we obtain a *foliated connection*, a map that associates to a vector field Z tangent to $\mathscr{F}$ a holomorphic function $\Gamma(Z)$, its (foliated) *Christoffel symbol*, satisfying (8). This can be used as an alternative definition of a foliated affine structure: a map $\Gamma : T_{\mathscr{F}} \to \mathscr{O}_M$ (associating holomorphic functions to vector fields tangent to the foliation) satisfying this relation. We refer to [DG23] for details.

The advantage provided by leafwise affine structures over translation ones resides in the following two propositions. They allow to localize obstructions for semicompleteness of a vector field along leaves of the foliation contained in the sets of zeros of the vector field, where the latter appears to give no information at all.

Proposition 6.3 ([GR12, Prop. 8])**.** *Let X be a holomorphic vector field on the complex manifold M, C a component of the divisor of zeros of X that is invariant by $\mathscr{F}$, the foliation induced by X. At every point of C which is both a regular point of $\mathscr{F}$ and a smooth point of the divisor of zeros of X, the affine structure induced by X on the leaves of $\mathscr{F}$ where it does not vanish extends holomorphically to those in C.*

Proof. Let f be a local equation defining C in a neighborhood of p, Z a vector field that does not vanish at p generating the foliation induced by X. Since C is invariant by Z, there exists a function h such that $Zf = hf$, and X is thus of the form $X = gf^nZ$ for $n \in \mathbf{Z}$ and a nonvanishing holomorphic function g. Away from C, $\Gamma(X) \equiv 0$, for the affine structure is the translation one induced by X. Hence, for $u = gf^n$,

$$\Gamma(Z) = \Gamma\left(\frac{1}{u}(uZ)\right) = uZ\left(\frac{1}{u}\right) + \frac{1}{u}\Gamma(uZ) = -\frac{1}{u}Z(u) + \frac{1}{u}\Gamma(X) =$$
$$= -\frac{n}{f}Z(f) - \frac{1}{g}Z(g) = -nh - \frac{1}{g}Z(g),$$

which is holomorphic: the connection (and thus the affine structure) extends to C in a unique way. □

Proposition 6.4 ([GR12, Prop. 9])**.** *Let X be a vector field in the manifold M, C an irreducible component of the divisor of zeros of X that is invariant by $\mathscr{F}$. Let $L \subset C$ be a leaf of $\mathscr{F}$ within the locus of zeros of X. If X is semicomplete, the affine structure on L is uniformizable.*

First sketch of proof (after [GR12]*).* Suppose that the affine structure on L fails to be uniformizable, and let $\gamma : [0,1] \to L$ a smooth open path that develops onto a closed one. For a sufficiently close leaf L', where X is holomorphic and nonzero, the curve γ may be lifted isometrically (with respect to the affine structures along the leaves), yielding an open path in L' that develops into a closed one. But this implies that the affine structure on L' is not uniformizable, that X is not semicomplete. □

Second sketch of proof. Suppose that all the one-dimensional components of the divisor of zeros of X are invariant by the foliation $\mathscr{F}$ induced by X. Let $D \subset M$ be the support of the divisor of zeros of X. Let $T_{\mathscr{F}} \to X$ be the tangent (line) bundle of the foliation induced by X, let $\sigma : M \to T_{\mathscr{F}}$ be the tautological lift of X. Let Y be the vector field on $T_{\mathscr{F}}$ generating the homotheties. Let $\widetilde{X}$ be the unique vector field on $\pi^{-1}(M \setminus D)$ whose restriction to the image of σ is the lift of X and that satisfies the Lie-algebraic relation $[Y,\widetilde{X}] = \widetilde{X}$. (We refer the reader to [DG23, Sect. 4.1] for details.) Above $M \setminus D$, $\widetilde{X}$ is semicomplete, as it is given by multiples of $\sigma_* X$ on homothetic images of $\sigma(M)$. The vector field $\widetilde{X}$ extends as a non-identically-zero holomorphic vector field to $\pi^{-1}(D)$—this is essentially a consequence of Proposition 6.3—, which is semicomplete by Theorem 3.3. Let $\overline{L}$ be an integral curve of $\widetilde{X}$ projecting onto L. It is endowed with the uniformizable translation structure induced by $\widetilde{X}$. The map $\pi|_{\overline{L}} : \overline{L} \to L$ is a covering one, and its group of deck transformations acts affinely with respect to the translation structure. This uniformizes the affine structure on L. (Details for this last part may be found in [Gui06, Sect. 2.2].) ☐

With the same order of ideas, we have the following extension of Lemma 3.1 (see [GR12, Sect. 4.2] for a proof).

Lemma 6.5 (Fundamental Lemma, affine version)**.** *Let M be a complex manifold, $\mathscr{F}$ be a nonsingular foliation by curves such that each curve is endowed with an affine structure that varies holomorphically and that is, in restriction to each leaf, uniformizable. Let L be a leaf of $\mathscr{F}$. There is a group homomorphism $\rho : \mathrm{mon}(\pi_1(L)) \to \mathrm{hol}(\pi_1(L))$ such that for all $\gamma \in \pi_1(L)$, $\mathrm{hol}(\gamma) = \rho \circ \mathrm{mon}(\gamma)$, this is, the holonomy representation factors through the monodromy one.*

Together, these results imply that germs of semicomplete vector fields have solvable *projective holonomy groups* (see [Lor21, Sect. 7.3]), and thus very limited possible behaviors (see [Lor21, Sect. 4.1]). This contributes to the scarcity of orbital models in Theorems 5.4 and 5.6.

6.3. **Singularities of affine structures.** On a curve C, an affine structure defined on the open subset $C_0 \subset C$ is said to have a *singularity* at $p \in C \setminus C_0$ if the affine structure does not extend to p (equivalently, if for a vector field on C that does not vanish at p, its Christoffel symbol does not admit a holomorphic extension to p). A singular point of an affine structure p is said to be *Fuchsian* if for a holomorphic vector field Z that does not vanish at p, $\Gamma(Z)$ has a simple pole at p (by (8), this is well-defined; in this case, the second-order linear equation (9) satisfied by the charts of the affine structure belongs to the *Fuchsian class* [Inc44, Ch. XV]). For a Fuchsian singularity p, the *ramification index* of the structure at p is the element $\nu \in \mathbf{C}^* \cup \{\infty\}$ such that, for a coordinate z around p, $\Gamma(\partial/\partial z) = (\frac{1}{\nu} - 1)z^{-1} + \cdots$ (this is, again, well-defined).

Proposition 6.6. *A uniformizable isolated singularity of an affine structure is Fuchsian, and its ramification index ν belongs to $\mathbf{Z}^* \cup \{\infty\}$. In a suitable coordinate z, the developing map is given by $z \mapsto z^{1/\nu}$ if $\nu \in \mathbf{Z}$, and by $z \mapsto \log(z)$ if $\nu = \infty$.*

Proof. Let us first prove that the monodromy cannot be hyperbolic. If it is of the form $w \mapsto \alpha w$ with $|\alpha| \neq 1$, for the elliptic curve $E = \mathbf{C}^*/\alpha$, the developing map induces a well-defined one-to-one map $f : \Delta^* \to E$, which induces a nontrivial homomorphism at the level of fundamental groups. This map cannot be onto and it thus takes values in a punctured elliptic curve $E\setminus\{q\}$. But since Δ^* and $E\setminus\{q\}$ are both hyperbolic Riemann surfaces, in order for the hyperbolic distance to decrease under f, we must have that $\lim_{z\to 0} f(z)=q$, which is absurd. Since the monodromy is discrete, it is either parabolic or finite. From Proposition 4.1 and its proof, in a suitable coordinate z, the affine structure is either the one induced by $z\partial/\partial z$ or, for some $\mu\in\mathbf{N}$, for $w=z^{1/\mu}$, by either $\partial/\partial w$ or $w^2\partial/\partial w$. Thus, in a suitable coordinate, the affine structure is the one induced by the determinations of a multivalued vector field of the form $z^{1-1/\nu}\partial/\partial z$ with $\nu\in\mathbf{Z}^*\cup\{\infty\}$, for which $\Gamma(\partial/\partial z)=(1/\nu-1)z^{-1}$. □

6.3.1. *On compact curves.* The Poincaré-Hopf index theorem takes the following form for affine structures with Fuchsian singularities:

Proposition 6.7 ([GR12, Prop. 5])**.** *Let C be a compact curve with Euler characteristic $\chi(C)$ endowed with an affine structure with Fuchsian singularities having ramification indices $\nu_1,\ldots,\nu_k$. Then $\sum_i(1-1/\nu_i)=\chi(C)$.*

On its turn, this greatly limits the possibilities for uniformizable affine structures with singularities on curves:

Corollary 6.8. *Elliptic curves are the only ones supporting nonsingular uniformizable affine structures. Rational curves are the only ones admitting uniformizable affine structures with non trivial singular set, with sets of ramification indices (-1), $(n,-n)$ ($n\in\mathbf{Z}^*$), (∞,∞), $(2,3,6)$, $(2,4,4)$, $(3,3,3)$, $(2,2,\infty)$ and $(2,2,2,2)$.*

Remark 6.9. Consider a quadratic homogeneous vector field X on $\mathbf{C}^2$ such that $(x\partial/\partial x+y\partial/\partial y)\wedge X=\ell_1\ell_2\ell_3\,\partial/\partial x\wedge\partial/\partial y$ for three different homogeneous linear forms ℓ_i. After blowing up the origin, the exceptional divisor D is invariant by the foliation. The transformed vector field has a zero of order one along D, and the affine structure on D has Fuchsian singularities at each one of the three singularities of the foliation corresponding to each of the ℓ_i. We may have singularities of ramification index one, corresponding to points that are singular for the foliation but not for the affine structure. The vector fields in Theorem 5.6 are the semicomplete ones for which the origin is an isolated zero, and correspond to the triples of ramification indices $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ and $(1,m,-m)$ for $m\in\mathbf{Z}^*$.

Remark 6.10. Let discuss the lines of part of the proof of Theorem 5.6 in [GR97, Sect. 3.1]. Let Z be a semicomplete vector field in a neighborhood U of the origin in $\mathbf{C}^2$ such that its second jet is given by the vector field Z_0 in item (2) in the statement of Theorem 5.6, the one associated to the triple of ramification indices $(3,3,3)$. After blowing up the origin we find a divisor D invariant by the foliation, along which the affine structure is the one induced by Z_0. By Lemma 6.5, the holonomy of the foliation along D (the *hidden holonomy* of the original foliation) is a representation into the group $\mathrm{Diff}(\mathbf{C},0)$ of the monodromy of the affine structure on D, the triangle group $T_{3,3,3}=\langle u,v,w\mid uvw=u^3=v^3=w^3=1\rangle$. These representations are very well understood. Other than the linear representation into $\mathbf{C}^*$—the one associated to the homogeneous case—, we have the tautological representation of the monodromy into $\mathrm{Aff}(\mathbf{C})$ viewed from the point at infinity, plus the ones obtained from it by ramifications (see [CM88], [LM94]). Except for the linear one, the holonomy representations have orbits accumulating to 0 with nontrivial stabilizer. The integral curve L of the vector field associated to such an orbit of the holonomy is stabilized by a nontrivial $\tau \in \mathbf{C}$ under the semilocal flow Φ associated to Z (in the sense that $\Phi(\tau, x) = x$ for every x in L). But this contradicts the fact that the set stabilized by τ is a proper analytic subset. We conclude that the holonomy is linearizable, conjugated to that of the homogeneous case. The equivalence of the holonomies may then be promoted, with some work, to an orbital equivalence of the vector fields.

7. The birational point of view

One cannot overstress the importance of Seidenberg's theorem in the theory of foliations on surfaces. It constitutes, together with the comprehensive description of simple singularities associated to the names of Martinet and Ramis and the Camacho-Sad index theorem, the starting point of many developments. For the study of semicomplete vector fields, we will define a notion on top of Seidenberg's one concerning the position of the curve of zeros of a vector field with respect to its foliation (compare with [Can04, Def. 13] and [CC92, Def. 2.2]).

Recall that a foliation on a surface is said to be *reduced* in Seidenberg's sense if at every singularity the linear part of a tangent vector field with isolated singularities is non-nilpotent with nonresonant eigenvalues. Seidenberg's theorem [Sei68] affirms that any foliation can be transformed to a reduced one through a locally finite number of blowups (see also [Bru15, Ch. 1, Sect. 2], [CCD13, Ch. 4]).

Definition 7.1. Let S be a surface, X a holomorphic vector field on S, C its curve of zeros, and $\mathscr{F}$ the foliation it induces. The vector field X is said to be *reduced* if

- $\mathscr{F}$ is reduced in Seidenberg's sense, and
- in the neighborhood of every point p of S, the union of C and the curves through p that are invariant by $\mathscr{F}$ has normal crossings.

When blowing up a point that is a zero of a holomorphic vector field, the transformed of the vector field is still holomorphic. It is not difficult to see that *every vector field can be transformed into a reduced one by a locally finite number of blowups (of zeros)*, and that *reduced vector fields are stable under blowups of singular points*. By Theorem 3.3, the original vector field will be semicomplete if and only if its transformed is, and thus *every semicomplete vector field can be transformed into a reduced semicomplete one by a locally finite number of blowups*.

We have the following result for germs of reduced semicomplete vector fields, giving orbital normal forms in many cases.

Theorem 7.2 ([GR12, Prop. 17])**.** *Let M be a surface, X a reduced semicomplete vector field on M. Let p be a point where $X(p) = 0$. Either p is an isolated nondegenerate singularity of X or, in suitable coordinates around p, and up to multiplication by a nonvanishing holomorphic function, X is one of the following:*

(1) $x(1+\lambda y)\partial/\partial x + y^2\partial/\partial y$, $\lambda \in \mathbf{Z}$,
(2) $x^p y^q(mx\partial/\partial x - ny\partial/\partial y)$, $pm - qn = 1$ ($m,n \in \mathbf{Z}$, $m,n \geq 0$, $p,q \geq 0$),

(3) $x^p y^q [x(q+\cdots)\partial/\partial x - y(p+\cdots)\partial/\partial y]$, $p,q \geq 1$,
(4) $x^r y^q \partial/\partial x$ *with* $r \in \{0,1,2\}$, $q \geq 0$.

Let us give a proof in the case where X is a reduced vector field with a curve of zeros tangent to a foliation having a reduced singularity with nondegenerate linear part, a case leading to items (2) and (3) in the above statement. In this case the foliation has two transverse separatrices, and since X is reduced, the curve of zeros is contained in their union. The vector field may thus be written as

$$x^p y^q \left(x f(x,y) \frac{\partial}{\partial x} + y g(x,y) \frac{\partial}{\partial y} \right),$$

with p and q non-negative integers (at least one of them nonzero), and f and g holomorphic functions that do not vanish at the origin, where they take, respectively, the values f_0 and g_0. The vector field induces a foliated affine structure. With respect to this structure, for $Z = x^{-p-1} y^{-1} f^{-1} X$,

$$\Gamma(Z) = -\frac{1}{x} \left(1 + p + q\frac{g}{f} + x\frac{f_x}{f} + y\frac{f_y}{f}\frac{g}{f} \right).$$

The restriction of Z to the invariant curve $y = 0$ is $\partial/\partial x$, and by restricting the above Christoffel symbol, we have that the affine structure on $y = 0$ has a Fuchsian singularity at 0 with ramification index $\nu = -f_0/(pf_0 + qg_0)$. By the same reasoning, the ramification index of the singularity at 0 of the restriction of the foliated affine structure to the curve $x = 0$ is $\mu = -g_0/(pf_0 + qg_0)$. Either $pf_0 + qg_0 = 0$, leading to case (3) in Theorem 7.2, or both ramification indices are integers (Proposition 6.6), satisfying moreover the relation $p\nu + q\mu = -1$ (observe that, since p and q are non-negative, μ and ν have different signs). Since $f_0/g_0 = \nu/\mu \in \mathbf{Q}^-$, the foliation is a *resonant saddle*. By Lemma 6.5, the monodromy of the foliation along each one of the separatrices has finite order, and by the theorem of Martinet and Ramis [MR83], the foliation is linearizable. Thus, the vector field falls within case (2) in the statement of Theorem 7.2.

Remark 7.3. The multiplicative factor in item (2) of Theorem 7.2 can be actually taken to be constant (see [GR12, Prop. 18]).

8. Some applications

For a reduced semicomplete vector field on a surface we have the following situation. Each compact irreducible curve invariant by the foliation has an affine structure with singularities, belonging to the very restrictive list of Corollary 6.8 (notice that only rational and elliptic curves may appear). Singular points for the affine structures have ramification indices (belonging to $\mathbf{Z}^* \cup \{\infty\}$), and are singular points of the foliation. The vector field has an order (an integer) along each curve, and each curve has a self-intersection (another integer), which may be described through local data by the Camacho-Sad index theorem. Curves intersect at points described by Theorem 7.2, where, for instance, the arithmetic relation in condition (2) is a relation between orders of vector fields and ramification indices of the affine structures. The combinatorics associated to all this can be successfully worked out in some settings, while taking into account that some curves may be blown down while keeping the vector field reduced.

If a singular analytic space is endowed with a holomorphic vector field, its transformed in the resolution will still be holomorphic [Sei66], and the above can be thought to take place after such a resolution.

All this allows for an enhanced understanding of semicompleteness of vector fields without isolated singularities and vector fields on singular analytic spaces.

8.1. **Some univalent vector fields without linear part.** As a first illustration of the principle and methods, let us go back to Theorem 5.6. Let $m \in \mathbf{Z}$, and let Z_0 be the quadratic homogeneous vector field in item (1) in its statement. We will prove that *a vector field Z satisfying the hypothesis of the theorem and whose two-jet at 0 is given by Z_0 may be locally redressed onto Z_0*. This is, along with this particular instance of Theorem 5.6, we will prove that, for it, the factor appearing in its statement can be chosen to be constant—a fact already observed in [FRR18, Prop. 4.1].

Let thus Z be such a vector field. Upon blowing up the origin we find an exceptional divisor C_m, along which the transformed vector field has a zero of order one. The induced foliation is tangent to this divisor, and has three singular points on it:

- a point a, where Z has the form $v[u(1+\cdots)\partial/\partial u - v(1+\cdots)\partial/\partial v]$, with the exceptional divisor given by $v=0$. Along the divisor, the affine structure has a singularity with ramification index 1 (an *apparent* singularity);
- a point q, where Z has the form $v[u(m+\cdots)\partial/\partial u - v(1+\cdots)\partial/\partial v]$;
- a point p, where the foliation is in the Poincaré domain and may be put in Poincaré-Dulac normal form, so that the vector field reads

$$f(u,v)v\left[(mu+\varepsilon v^n)\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right], \tag{10}$$

  with $\varepsilon \in \{0,1\}$ and f a holomorphic and nonvanishing function.

According to Theorem 7.2 (taking Remark 7.3 into account), Z may be brought, in a neighborhood of a, into the normal form $v(u\partial/\partial u - v\partial/\partial v)$. In restriction to a bidisk of radius δ, the orbits of Z other than the separatrices are, as translation surfaces, identified with annular subsets of $\mathbf{C}$, with the inner boundary component corresponding to $|u| = \delta$, and the exterior one to $|v| = \delta$. On $\mathbf{C}^2$, with coordinates (U,V), consider the vector field $\partial/\partial V$, and restrict it to $\Delta \times \mathbf{C}$. Graft the latter with a neighborhood of a deprived of its intersection with the exceptional divisor through the identification $(U,V) = (uv, 1/v)$. This operation fills the bounded part of the complement of each annuli, and, through it, the separatrix through a that is not contained in the exceptional divisor extends to a rational curve E. By the Camacho-Sad formula, $E^2 = -1$. Collapse this divisor. The resulting point is a nonsingular one for the foliation. We now have that $C_m^2 = 0$, and on C_m, we still have the singular points p and q.

When resolving p, if $\varepsilon = 1$, after n blowups, the foliation has a saddle node with a convergent weak separatrix, tangent to the exceptional divisor, with the vector field having zeros both along this divisor and along the strong separatrix. But by Theorem 7.2, such a vector field is not semicomplete (in a semicomplete vector field, the saddle-nodes of the foliation are isolated singularities of the vector field), and hence $\varepsilon = 0$.[1] Through the

[1] One may also proceed as in the proof of Theorem 7.2. When restricting to $v = 0$, for the projective structure induced by (10), $\Gamma(\partial/\partial u) = (-1-1/m - uf_u/f)/u$, the point 0 is a Fuchsian singularity with ramification

resolution of p, we obtain a chain of embedded rational curves C_m, C_{m-1}, , …, C_1, C_0 ($C_i \cdot C_{i+1} = 1$, $C_i \cdot C_j = 0$ if $|i-j| \geq 2$) such that $C_m^2 = -1$, $C_i^2 = -2$ for $1 \leq i \leq n-1$, and $C_0^2 = -1$. For $i \geq 1$, C_i is invariant by the foliation and the vector field has a zero of order one along it. The component C_0 is everywhere transverse to the foliation (a dicritical component), along which Z has a zero of order two. We may now successively collapse C_n, then C_{n-1}, and so on, until finally collapsing C_1. We end with a curve C_0 of vanishing self-intersection, everywhere transverse to the foliation, with a vector field tangent to the foliation having a zero of order two at C_0, locally conjugated, on each leaf, to the germ at 0 of the vector field $x^2\partial/\partial x$. By Savel'ev theorem [Sav82], a neighborhood of C_0 may be redressed to a neighborhood of $y=0$ in $\Delta \times \mathbf{P}^1$. This can be done by a map that redresses the vector field onto $x^2\partial/\partial x$ and that maps the image of the last blown-down divisor to $(0,0)$.

Consequently, the original vector field Z may be constructed by reversing this process. The only choice involved is the one of the point blown-up to produce, in the last divisor, the point a, but this choice is rendered irrelevant by the action of the vector field $y\partial/\partial y$. This shows that, up to local biholomorphisms, there is only one vector field like the ones under consideration, that they are all conjugate to the germ of the homogeneous model Z_0 at the origin. This description makes also patent that the vector field may be extended to a compact ambient (rational) surface.

8.2. **Kato surfaces and vector fields without separatrices.** Theorem 5.5 greatly limits the possible isolated singularities of complete vector fields on smooth surfaces, and also those, in singular surfaces, that have separatrices. Foliations by curves on singular spaces need not have separatrices, and semicomplete vector fields associated to these are not subject to such limitations. Nevertheless, germs of semicomplete vector fields on singular spaces that do not have separatrices are remarkably special:

Theorem 8.1 ([Gui14, Thm. A])**.** *Let S be a reduced normal, connected, two-dimensional analytic space endowed with a complete holomorphic vector field X. Let $p \in S$ be a singular point such that there is no curve through p invariant by X. Then the minimal resolution of S at p is an intermediate Kato surface.*

Kato surfaces are non-Kähler compact complex surfaces of vanishing algebraic dimension, belonging to the class VII_0 in the Enriques-Kodaira classification [BHPV04] introduced by Kato in 1977 [Kat77] (see also [Dlo84]). For Kato surfaces in the *intermediate* class, the union of the compact curves in the surface is a connected divisor that contains a cycle, and the intersection matrix of its irreducible components is negative definite (its contraction produces an analytic space). Some of these admit holomorphic vector fields which produces, after contraction of the maximal divisor, a singular analytic space endowed with a complete vector field without separatrices at the singular point.

We stress the fact that a consequence of the above theorem is the compactness of the ambient space. Let us say some words about its proof. After resolving first the singular point of the space and then the vector field, we find a divisor in a smooth surface, in the neighborhood of which we have a semicomplete vector field. The combinatorial data of both the divisor and the vector field is so special that it produces a germ of intermediate

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index $-m$. By Lemma 6.5, the holonomy of $v=0$ around 0 should be trivial, but this only happens when the foliation is linearizable, when $\varepsilon = 0$.

Kato surface around its divisor (the *Kato data* needed to build the surface can be extracted from it). A Kato surface can then be "grown" from this germ through the fact that, for vector fields on intermediate Kato surfaces, the maximal divisor is an attractor.

In this proof, Kato surfaces arrive directly and naturally, and not through a characterization or by narrowing down a classification. If Kato surfaces (at least those with vector fields) were not already there, they could have been discovered through the theorem!

Question 8.2. When collapsing the curves of an intermediate Kato surface with a vector field, the resulting germ of analytic space (S,p) can be, for some n, embedded in $(\mathbf{C}^n,0)$, and the vector field extends to a vector field on the ambient space. What can be said about the order of this vector field at 0? (See also Question 9.1.)

8.3. **Singular two-dimensional Stein spaces.** We have seen in Theorem 5.7 that the isolated singularities of complete vector fields on Stein surfaces are very special. This is still the case in the singular setting:

Theorem 8.3 ([Gui14, Thm. B])**.** *Let V be a two-dimensional normal reduced Stein space, X a complete holomorphic vector field on V, $p \in V$ a singular point of V where the zeros of X do not accumulate. Then either*

- *p is a weighted homogeneous singularity, and X generates the weighted homotheties; or*
- *p is a cyclic quotient (Hirzebruch-Jung) singularity, and X has either one or two separatrices through p.*

As an example of the first situation, we have the restriction of the vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$ to the quadratic cone $x^2+y^2+z^2=0$. For the second one, recall that the *Hirzebruch-Jung* or *cyclic quotient* surface singularity $A_{n,m}$ is the quotient of $(\mathbf{C}^2,0)$ under the linear action of $\mathbf{Z}/n\mathbf{Z}$ given, for a primitive n-th root of unity ω, by $(z,w) \mapsto (\omega z, \omega^m w)$. Some complete vector fields on $\mathbf{C}^2$, like $z\partial/\partial z + (mw+z^m)\partial/\partial w$ or $\lambda z\partial/\partial z + \mu w\partial/\partial w$, are preserved by this symmetry, and give univalent vector fields on $A_{n,m}$. The resolution of $A_{n,m}$ is completely determined by its combinatorics [BHPV04, Ch. III, §5, Thm. 5.1].

For the proof, Theorem 8.1 can be used as an "odd separatrix theorem." Since Stein surfaces are never compact, Theorem 8.1 implies that there is a separatrix through p. Since V is Stein, by the argument in Theorem 5.7, the restriction of X to each separatrix has a simple zero. If there is a dicritical component in the divisor of the resolution, the action factors through an action of $\mathbf{C}^*$, and, if not, the divisor turns out to have the combinatorics of the resolution of some $A_{n,m}$, and is thus biholomorphically equivalent to it.

8.4. **Vector fields on compact complex surfaces.** As of 1990, we almost had a classification of the minimal compact complex surfaces supporting holomorphic vector fields [GH90]. There remained the problem of classifying vector fields with zeros on minimal compact complex surfaces of the class VII_0 with strictly positive second Betti number and vanishing algebraic dimension, and, by the results in [Hau95], one needed only consider the case in which the induced action of $\mathbf{C}$ is effective. A major difficulty in classifying the missing ones was that, for all we knew, they could be defined on surfaces that remained yet to be discovered, dwelling in the nebulous territory of VII_0 surfaces within the Enriques-Kodaira classification [BHPV04, Ch. VI, Sect. 1]. At the turn of the century, Dloussky, Oeljekalus and Toma completed the classification in [DOT00] and [DOT01]. The work of Ghys and Rebelo discussed in Section 5 is a key ingredient in their work. The birational point of view here explained allows to give an alternative simple proof of the last piece of the classification, by profiting from the natural way in which Kato surfaces appear.

Theorem 8.4 ([Gui14, Thm. C])**.** *Let S be a minimal compact complex surface, X a holomorphic vector field on S with zeros and without a first integral. Either:*

- *S is rational or ruled,*
- *the flow of X factors through an action of $\mathbf{C}^*$,*
- *X is an intermediate Kato surface, or*
- *there is a divisor D of vanishing self-intersection, invariant by X.*

In order to use this statement to go from the one in [GH90] and [Hau95] to a complete classification, it suffices to recall that surfaces of the class VII_0 with strictly positive second Betti number supporting a divisor of vanishing self-intersection were characterized by Enoki [Eno81]: they constitute a particular class of Kato surfaces nowadays called *Enoki surfaces*. The vector fields completing the classification are, in consequence, those defined on Enoki and intermediate Kato surfaces. Vector fields on the first were classified in [DO99] (the proof of Theorem 8.1 also gives a complete description); those on Enoki surfaces were classified in [DK98] (see also [Gui17]).

Let us sketch the broad lines of the proof of Theorem 8.4. In the absence of a first integral, by a result of Jouanoulou (extended by Ghys [Ghy00]), there are only finitely many invariant algebraic curves. The union of the invariant algebraic curves is *isolated* in the sense that any germ of invariant curve intersecting this union is contained in it (if there is a germ of curve along which the vector field has a double zero, it extends as a complete rational curve; if it has a simple zero, all the points in it have the same nontrivial element in their stabilizer, a condition that defines a proper analytic subset if the flow does not factor through an action of $\mathbf{C}^*$). In particular, by the Camacho-Sad separatrix theorem, the divisor of curves is nonempty as soon as there are zeros of X. If there are rational curves whose self-intersection is non-negative, the surface is rational or ruled. Leaving this case aside, for each connected component of the union of the invariant curves, the combinatorial analysis implies that, if the connected components of the union of the curves does not have vanishing self-intersection, it has the combinatorics of the curves in an intermediate Kato surface, and, as we discussed in Section 8.2, a Kato surface can then be recovered from these.

9. Other directions

The problem of understanding complete vector fields is an ample and many-faceted one, and we will not touch upon many of its aspects. Let us briefly discuss some of these. A notable omission will be the "Nevanlinna" approach, which lays the basis for Brunella's landmark work on complete polynomial vector fields on $\mathbf{C}^2$ [Bru04]. Many global problems remain barely addressed, like that of understanding complete holomorphic vector fields on $\mathbf{C}^2$ that are not polynomial (or even of giving significant examples).

9.1. **Higher dimensions.** It would be interesting to study the extent to which results like those stated in Section 5 apply to higher dimensions. In all dimensions, germs of nondegenerate vector field are semicomplete [Gui20]. The least degenerate vector fields among degenerate ones are saddle-node singularities having exactly one vanishing eigenvalue; for these, Theorem 5.2 has been generalized by Reis to all dimensions [Rei08]. For more degenerate germs of vector fields we know almost nothing. It would be interesting to say something about germs of semicomplete vector fields in dimension three having only one nonvanishing eigenvalue.

One of the main difficulties in the study of higher-dimensional vector fields is that, for $n \geq 3$, there exist vector fields on $(\mathbf{C}^n, 0)$ that do not have separatrices: Gómez-Mont and Luengo have given explicit families in dimension three [GL92]; higher dimensional analogues appear in [LO00].

Question 9.1. If X is a germ of semicomplete vector field on $(\mathbf{C}^3, 0)$ with an isolated singularity at 0, is its second jet at 0 non trivial? (Attributed to Ghys.) More generally, is there a way in which X is not too degenerate at 0? (See also Question 8.2.)

9.1.1. *The birational point of view.* In dimension three, we have the relatively recent theorems on the resolutions of singularities of holomorphic foliations by curves due to McQuillan-Panazzolo [MP13] and to Cano-Roche-Spivakovsky [CRS14]. Based on these, Rebelo and Reis have proved that, for semicomplete vector fields, with the exception of some special ones with nilpotent (but nonzero) linear part, they can all be resolved by standard blowups [RR21, Thm. B]. This result urges us to understand semicompleteness among these special vector fields, for which relatively simple prenormal forms are available [RR21, Thm. 3]. We refer the reader to the examples and the discussion in Rebelo and Reis's article.

9.2. **Homogeneous systems.** As it may be hinted by Corollary 3.4, the study of semicompleteness among homogeneous (and quasihomogeneous) systems of differential equations is unavoidable in the study of semicompleteness of germs of vector fields. For instance, the orbital normal forms of Theorem 5.6 are homogeneous, and that those of Theorem 5.4 are quasihomogeneous ones.

This places us fully within the context of algebraic differential equations. If semicomplete homogeneous vector fields in dimension two may be integrated by rational or elliptic functions, the situation in higher dimensions is more complicated, where interesting phenomena, like the existence of essential boundaries for the solutions, appear.

9.2.1. *Kowalevski exponents.* Homogeneous systems have some special orbits to which some numbers can be associated and where an obstruction for semicompleteness manifests as a Diophantine condition. These numbers have appeared in many contexts (notably, in Kowalevski's study of the top [Kow89, §1]), and have received many names. We refer to [Gor00] for a detailed study of their history, definitions and properties. Let us here restrict to quadratic homogeneous systems. Let $X = \sum_i P_i \partial/\partial z_i$ be a polynomial vector field on $\mathbf{C}^n$, with P_i quadratic homogeneous. Let ℓ be a *radial orbit*, a line through the origin of $\mathbf{C}^n$ containing a nonstationary solution of X. The *equation of variations* along this solution gives the linear system $tv_i' = -\sum_j (\partial P_i/\partial z_j)|_p v_j$. The *resonance numbers*, *Fuchs indices* or *Kowalevski exponents* of (X,ℓ) are, for $M=(\partial P_i/\partial z_j)|_p$, the $n-1$ eigenvalues of $\mathbf{I}-M$ other than -1 (which is always the eigenvalue in the direction of the orbit). For a constant matrix A with an eigenvector w associated to the eigenvalue λ, the equation $tv'=Av$ has the solution $t\mapsto t^{\lambda}w$, which is multivalued unless $\lambda\in\mathbf{Z}$. Naturally, if X has only single-valued solutions, so will the equation of variations, and thus, *if X is semicomplete, the Kowalevski exponents of all radial orbits are integers* (see also the argument in [Gui06, Cor. 2.6]).

The Kowalevski exponents give thus some Diophantine obstructions for semicompleteness, and it is tempting to try to approach the study of semicompleteness among homogeneous systems through their study. In $\mathbf{C}^2$, quadratic homogeneous systems have three Kowalevski exponents ν_i, from which the vector field can be recovered, satisfying moreover the relation $\sum_{i=1}^{3}1/\nu_i=1$, directly related to Proposition 6.7. The vector fields where these are integers are exactly those appearing in the statement of Theorem 5.6. A similar analysis can also be achieved for quasihomogeneous vector fields associated to some particular weights in higher dimensions (like in the setting of Chazy's study [Cha11], or in Cosgrove's analysis of fourth-order equations [Cos00, Sect. 2.1]), but becomes very complex very soon.

As for quadratic homogeneous vector fields on $\mathbf{C}^3$, a generic one has seven radial orbits; the associated Kowalevski exponents $\nu_1,\mu_1,\dots,\nu_7,\mu_7$ are bound by the relations

$$\sum_{i=1}^{7}\frac{1}{\nu_i\mu_i}=1,\ \sum_{i=1}^{7}\frac{\nu_i+\mu_1}{\nu_i\mu_i}=-4,\ \sum_{i=1}^{7}\frac{(\nu_i+\mu_1)^2}{\nu_i\mu_i}=16; \tag{11}$$

see [Gui04, Cor. 12]. We can go a long way with these equations—and the help of the computer! (The partial classification attained in [Gui18] exhibits fifteen infinite families with up to four parameters each, plus thirteen "sporadic" vector fields.) There are other algebraic relations beyond those in (11), of which we know in general very little, but for which some things can be said in particular cases, like the one discussed in [GR19]. It would be interesting to have a better understanding of the Kowalevksi exponents as functions in the space of vector fields.

9.3. **Meromorphic vector fields.** The tools we have described so far, specially those in Section 8, can be adapted to deal with meromorphic vector fields. This allows, as noticed early on (see [LM00]), to study semicompleteness for polynomial vector fields on affine surfaces by localizing some obstructions at infinity. By following a scheme not too far from the one described in Section 8, we have, in collaboration with Rebelo, been able to give a complete description of *semicomplete meromorphic* vector fields (semicomplete in restriction to the the subset where the vector field is holomorphic) on compact surfaces:

Theorem 9.2 ([GR12, Thm. A], [Gui17])**.** *Let X be a semicomplete meromorphic vector field on the compact complex surface S. Then, up to a birational transformation, either X is holomorphic, or X preserves a fibration on S.*

The study of semicomplete polynomial vector fields on $\mathbf{C}^3$ may be, in principle, approached through similar methods. It would be interesting, for instance, to understand the semicompleteness of saddle-nodes in the presence of a nontrivial divisor of poles. (The Lorenz system has such a singularity at infinity, and there are parameters giving semicomplete vector fields [Kud15].)

9.4. **Other Lie groups.** In principle, analogous results to the ones here discussed hold for germs of Lie algebras of holomorphic vector fields: a good understanding of their germs will allow for that of the actions of the corresponding Lie group. This has been done only for a few groups, and only in small dimensions. We have a complete understanding of the actions of $\mathrm{SL}(2,\mathbf{C})$ on compact complex threefolds [Gui07], where the very strong conditions imposed by univalence and the perfectness of the Lie algebra impose very restrictive conditions on the local models. Another Lie group where this has been studied is $\mathbf{C}^2$; the poorer structure of the Lie algebra comes with enhanced difficulties, even in the "codimension zero" case. We refer the reader to [RR16] and [FRR18] for details.

9.5. **Beyond univalence.** For a semicomplete vector field X on a manifold M, Palais constructed a *universal globalization*, a (not necessarily Hausdorff) manifold $\overline{M}$ endowed with a complete vector field $\overline{X}$ where M embeds equivariantly. The construction is natural and enjoys a universality property [Pal57, Ch. III, Thm. IX]). The space is actually Hausdorff in many situations (see the discussion around Theorem A in [FRR19]), and we have criteria for its Hausdorffness [Aba03, Prop. 1.2]. However, this concerns vector fields on manifolds, and not germs, which demand a subtler approach. In order to apply Palais's construction to a germ of (semicomplete) vector field, one must choose a representative, and the Hausdorfness of the globalization may depend on this choice (see [FRR19, Remark 2.11 and Prop. 2.12]).

Univalence is a necessary condition for a germ of vector field to be the local model of a complete one. We ignore if it is also a sufficient one: are there germs of semicomplete vector fields which are not the local models of complete ones?

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References

[Aba03] F. Abadie, *Enveloping actions and Takai duality for partial actions*, J. Funct. Anal. **197** (2003), no. 1, 14–67, doi:10.1016/S0022-1236(02)00032-0.

[BHPV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 4, Springer-Verlag, Berlin, 2004. doi:10.1007/978-3-642-57739-0.

[BB55] C. Briot and J.-C. Bouquet, *Recherches fur les fonctions doublement périodiques*, C. R. Hebdomadaires des Scéances de l'Académie des Sciences **40** (1855), no. 7, 342–344.

[Brj71] A. D. Brjuno, *Analytic form of differential equations. I, II*, Trudy Moskov. Mat. Obšč. **25** (1971), 119–262; ibid. 26 (1972), 199–239.

[Bru04] M. Brunella, *Complete polynomial vector fields on the complex plane*, Topology **43** (2004), no. 2, 433–445, doi:10.1016/S0040-9383(03)00050-8.

[Bru15] M. Brunella, *Birational geometry of foliations*, IMPA Monographs, vol. 1, Springer, Cham, 2015, doi:10.1007/978-3-319-14310-1.

[CS82] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. (2) **115** (1982), no. 3, 579–595, doi:10.2307/2007013.

[CS87] C. Camacho and P. Sad, *Pontos singulares de equações diferenciais analíticas*, 16º Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1987.

[Can04] F. Cano, *Reduction of the singularities of codimension one singular foliations in dimension three*, Ann. of Math. (2) **160** (2004), no. 3, 907–1011, doi:10.4007/annals.2004.160.907.

[CC92] F. Cano and D. Cerveau, *Desingularization of nondicritical holomorphic foliations and existence of separatrices*, Acta Math. **169** (1992), no. 1-2, 1–103, doi:10.1007/BF02392757.

[CCD13] F. Cano, D. Cerveau, and J. Déserti, *Théorie élémentaire des feuilletages holomorphes singuliers*, Echelles, Belin, 2013.

[CRS14] F. Cano, C. Roche, and M. Spivakovsky, *Reduction of singularities of three-dimensional line foliations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **108** (2014), no. 1, 221–258, doi:10.1007/s13398-013-0117-7.

[CM88] D. Cerveau and R. Moussu, *Groupes d'automorphismes de* $(\mathbf{C},0)$ *et équations différentielles* $ydy+\cdots=0$, Bull. Soc. Math. France **116** (1988), no. 4, 459–488 (1989), doi:10.24033/bsmf.2108.

[Cha86] M. Chaperon, C^k*-conjugacy of holomorphic flows near a singularity*, Inst. Hautes Études Sci. Publ. Math. (1986), no. 64, 143–183, URL: http://www.numdam.org/item?id=PMIHES_1986__64__143_0.

[Cha11] J. Chazy, *Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, Acta Math. **34** (1911), no. 1, 317–385, doi:10.1007/BF02393131.

[Cos00] C. M. Cosgrove, *Higher-order Painlevé equations in the polynomial class. I. Bureau symbol* P2, Stud. Appl. Math. **104** (2000), no. 1, 1–65, doi:10.1111/1467-9590.00130.

[DG23] B. Deroin and A. Guillot, *Foliated affine and projective structures*, Compos. Math. **159** (2023), no. 6, 1153–1187, doi:10.1112/S0010437X2300711X.

[Dlo84] G. Dloussky, *Structure des surfaces de Kato*, Mém. Soc. Math. France (N.S.) (1984), no. 14, ii+120, doi:10.24033/msmf.297.

[DK98] G. Dloussky and F. Kohler. Classification of singular germs of mappings and deformations of compact surfaces of class VII_0. *Ann. Polon. Math.* **70** (1998), 49–83. Proceedings of the conference "Complex analysis and applications" (Warsaw, 1997), doi:10.4064/ap-70-1-49-83.

[DO99] G. Dloussky and K. Oeljeklaus, *Vector fields and foliations on compact surfaces of class* VII_0, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 5, 1503–1545, doi:10.5802/aif.1728.

[DOT00] G. Dloussky, K. Oeljeklaus, and M. Toma, *Surfaces de la classe* VII_0 *admettant un champ de vecteurs*, Comment. Math. Helv. **75** (2000), no. 2, 255–270, doi:10.1007/PL00000374.

[DOT01] G. Dloussky, K. Oeljeklaus, and M. Toma, *Surfaces de la classe* VII_0 *admettant un champ de vecteurs. II*, Comment. Math. Helv. **76** (2001), no. 4, 640–664, doi:10.1007/s00014-001-8323-x.

[Dul05] H. Dulac, *Recherches sur les points singuliers des équations différentielles*, J. École Polytech. (1905), 1–125, 2è s. (C. no. 9).

[Eno81] I. Enoki, *Surfaces of class* VII_0 *with curves*, Tohoku Math. J. (2) **33** (1981), no. 4, 453–492, doi:10.2748/tmj/1178229349.

[FRR18] A. C. Ferreira, J. C. Rebelo, and H. Reis, *Complete commuting vector fields and their singular points in dimension 2*, preprint arXiv:1809.09255v1, 2018, doi:10.48550/arXiv.1809.09255.

[FRR19] A. C. Ferreira, J. C. Rebelo, and H. Reis, *Palais leaf-space manifolds and surfaces carrying holomorphic flows*, Mosc. Math. J. **19** (2019), no. 2, 275–305, doi:10.17323/1609-4514-2019-19-2-275-305.

[For95] F. Forstnerič, *Limits of complete holomorphic vector fields*, Math. Res. Lett. **2** (1995), no. 4, 401–414, doi:10.4310/MRL.1995.v2.n4.a3.

[GH90] C. Gellhaus and P. Heinzner, *Komplexe Flächen mit holomorphen Vektorfeldern*, Abh. Math. Sem. Univ. Hamburg **60** (1990), 37–46, doi:10.1007/BF02941046.

[Ghy00] É. Ghys, *À propos d'un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes*, Rend. Circ. Mat. Palermo (2) **49** (2000), no. 1, 175–180, doi:10.1007/BF02904228.

[GR97] É. Ghys and J.-C. Rebelo, *Singularités des flots holomorphes. II*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 4, 1117–1174, doi:10.5802/aif.1783.

[GL92] X. Gómez-Mont and I. Luengo, *Germs of holomorphic vector fields in* $\mathbf{C}^3$ *without a separatrix*, Invent. Math. **109** (1992), no. 2, 211–219, doi:10.1007/BF01232024.

[Gor00] A. Goriely, *A brief history of Kovalevskaya exponents and modern developments*, Regul. Chaotic Dyn. **5** (2000), no. 1, 3–15, doi:10.1070/rd2000v005n01ABEH000120.

[Gui04] A. Guillot, *Un théorème de point fixe pour les endomorphismes de l'espace projectif avec des applications aux feuilletages algébriques*, Bull. Braz. Math. Soc. (N.S.) **35** (2004), no. 3, 345–362, doi:10.1007/s00574-004-0018-7.

[Gui06] A. Guillot, *Semicompleteness of homogeneous quadratic vector fields*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 5, 1583–1615, doi:10.5802/aif.2221.

[Gui07] A. Guillot, *Sur les équations d'Halphen et les actions de* $\mathrm{SL}_2(\mathbf{C})$, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 221–294, doi:10.1007/s10240-007-0008-6.

[Gui14] A. Guillot, *Vector fields, separatrices and Kato surfaces*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 3, 1331–1361, doi:10.5802/aif.2882.

[Gui17] A. Guillot, *Semicomplete vector fields on non-Kähler surfaces*, Proc. Japan Acad. Ser. A Math. Sci. **93** (2017), no. 8, 73–76, doi:10.3792/pjaa.93.73.

[Gui18] A. Guillot, *Quadratic differential equations in three variables without multivalued solutions: Part I*, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2018), Paper No. 122, 46, doi:10.3842/SIGMA.2018.122.

[Gui20] A. Guillot, *On the local univalence of nondegenerate holomorphic vector fields*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 7, 877–880, doi:10.5802/crmath.100.

[GR19] A. Guillot and V. Ramírez, *On the multipliers at fixed points of quadratic self-maps of the projective plane with an invariant line*, Comput. Methods Funct. Theory **19** (2019), no. 4, 687–716, doi:10.1007/s40315-019-00293-w.

[GR12] A. Guillot and J. Rebelo, *Semicomplete meromorphic vector fields on complex surfaces*, J. Reine Angew. Math. **667** (2012), 27–65, doi:10.1515/CRELLE.2011.127.

[Hau95] J. Hausen, *Zur Klassifikation glatter kompakter* $\mathbf{C}^*$*-Flächen*, Math. Ann. **301** (1995), no. 4, 763–769, doi:10.1007/BF01446658.

[Hil97] E. Hille, *Ordinary differential equations in the complex domain*, Dover Publications, Inc., Mineola, NY, 1997, Reprint of the 1976 original.

[HKM61] M. Hukuhara, T. Kimura, and T. Matuda, *Equations différentielles ordinaires du premier ordre dans le champ complexe*, Publications of the Mathematical Society of Japan, 7. The Mathematical Society of Japan, Tokyo, 1961.

[IY08] Y. Ilyashenko and S. Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008.

[Inc44] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944.

[Kat77] M. Kato, *Compact complex manifolds containing "global" spherical shells*, Proc. Japan Acad. **53** (1977), no. 1, 15–16, URL: http://projecteuclid.org/euclid.pja/1195517920.

[Kow89] S. Kowalevski, *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Math. **12** (1889), no. 1, 177–232, doi:10.1007/BF02391879.

[Kud15] N. A. Kudryashov, *Analytical solutions of the Lorenz system*, Regul. Chaotic Dyn. **20** (2015), no. 2, 123–133, doi:10.1134/S1560354715020021.

[LM00] J. L. López and J. Muciño Raymundo, *On the problem of deciding whether a holomorphic vector field is complete*, Complex analysis and related topics (Cuernavaca, 1996), Oper. Theory Adv. Appl., vol. 114, Birkhäuser, Basel, 2000, pp. 171–195, doi:10.1007/978-3-0348-8698-7_13.

[Lor21] F. Loray, *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*, Ensaios Matemáticos, vol. 36, Sociedade Brasileira de Matemática, Rio de Janeiro, 2021, Pages 53–274.

[LM94] F. Loray and R. Meziani, *Classification de certains feuilletages associés à un cusp*, Bol. Soc. Brasil. Mat. (N.S.) **25** (1994), no. 1, 93–106, doi:10.1007/BF01232937.

[LO00] I. Luengo and J. Olivares, *Germs of holomorphic vector fields in* $\mathbf{C}^m$ *without a separatrix*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5511–5524, doi:10.1090/S0002-9947-00-02677-5.

[Mac87] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, 1987, doi:10.1017/CBO9780511661839.

[MR82] J. Martinet and J.-P. Ramis, *Problèmes de modules pour des équations différentielles non linéaires du premier ordre*, Inst. Hautes Études Sci. Publ. Math. (1982), no. 55, 63–164, URL: http://www.numdam.org/item?id=PMIHES_1982__55__63_0.

[MR83] J. Martinet and J.-P. Ramis, *Classification analytique des équations différentielles non linéaires résonnantes du premier ordre*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 4, 571–621 (1984), doi:10.24033/asens.1462.

[MM80] J.-F. Mattei and R. Moussu, *Holonomie et intégrales premières*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 469–523, doi:10.24033/asens.1393.

[MP13] M. McQuillan and D. Panazzolo, *Almost étale resolution of foliations*, J. Differential Geom. **95** (2013), no. 2, 279–319, doi:10.4310/jdg/1376053448.

[Pai00] P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bulletin de la Société Mathématique de France **28** (1900), 201–261, doi:10.24033/bsmf.633.

[Pai73] P. Painlevé, *Oeuvres de Paul Painlevé. Tome I*, Éditions du Centre National de la Recherche Scientifique, Paris, 1973, preface by R. Garnier, compiled by R. Gérard, G. Reeb and A. Sec.

[Pal57] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. **22** (1957), iii+123.

[Reb96] J. C. Rebelo, *Singularités des flots holomorphes*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 2, 411–428, doi:10.5802/aif.1519.

[Reb99] J. C. Rebelo, *Champs complets avec singularités non isolées sur les surfaces complexes*, Bol. Soc. Mat. Mexicana (3) **5** (1999), no. 2, 359–395.

[Reb00] J. C. Rebelo, *Réalisation de germes de feuilletages holomorphes par des champs semi-complets en dimension 2*, Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), no. 4, 735–763, URL: http://www.numdam.org/item?id=AFST_2000_6_9_4_735_0.

[RR16] J. C. Rebelo and H. Reis, *2-dimensional Lie algebras and separatrices for vector fields on* $(\mathbf{C}^3,0)$, J. Math. Pures Appl. (9) **105** (2016), no. 2, 248–264, doi:10.1016/j.matpur.2015.10.005.

[RR21] J. C. Rebelo and H. Reis, *On resolution of 1-dimensional foliations on 3-manifolds*, Uspekhi Mat. Nauk **76** (2021), no. 2(458), 103–176, doi:10.4213/rm9993.

[Rei06] H. Reis, *Equivalence and semi-completude of foliations*, Nonlinear Anal. **64** (2006), no. 8, 1654–1665, doi:10.1016/j.na.2005.07.013.

[Rei08] H. Reis, *Semi-complete vector fields of saddle-node type in* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **360** (2008), no. 12, 6611–6630, doi:10.1090/S0002-9947-08-04516-9.

[Sav82] V. I. Savel'ev, *Zero-type embeddings of the sphere into complex surfaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh, 1982, no. 4, 28–32.

[Sei66] A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167–173, URL: http://projecteuclid.org/euclid.pjm/1102995095.

[Sei68] A. Seidenberg, *Reduction of singularities of the differential equation* $A\,dy = B\,dx$, Amer. J. Math. **90** (1968), 248–269, doi:10.2307/2373435.

[Thu97] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997.

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