HIPERBOLICITY AND EXPONENTIAL CONVERGENCE OF THE LAX-OLEINIK SEMIGROUP

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Abstract. For a convex superlinear Lagrangian $L : TM \to \mathbb{R}$ on a compact manifold M it is known that there is a unique number c such that the Lax Oleinik semigroup $\mathcal{L}_t : C(M, \mathbb{R}) \to C(M, \mathbb{R})$ has a fixed point. Moreover for any $v \in C(M, \mathbb{R})$ the uniform limit $\tilde{u} = \lim_{t\to\infty} \mathcal{L}_t u$ exists.

In this paper we assume that the Aubry set consists in a finite number of periodic orbits or critical points and study the relation of the hyperbolicity of the Aubry set to the exponential rate of convergence of the Lax Oleinik semigroup.

Keywords: Hamilton-Jacobi equation, viscosity solutions, Aubry set.

1. Introduction

Consider a convex superlinear Lagrangian $L : TM \to \mathbb{R}$ on a d-dimensional compact manifold M. For $t \geq 0$ define the (backward) Lax Oleinik semigroup $\mathcal{L}_t : C(M, \mathbb{R}) \to C(M, \mathbb{R})$ by

$$\mathcal{L}_t u(x) = \inf\{u(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma}) : \gamma : [0, t] \to M \text{ is piecewise } C^1, \gamma(t) = x\}$$

The function $S : M \times \mathbb{R}^+ \to \mathbb{R}$ given by $S(x, t) = \mathcal{L}_t u(x)$ is a viscosity solution of the Hamilton Jacobi initial value equation

$$S_t + H(x, S_x) = 0, \quad S(x, 0) = u(x) \tag{1}$$

It was shown in [1], [2] that there is a unique number $c = c(L)$ such that $\mathcal{L}_t + ct$ has a fixed point for any $t > 0$. Any fixed point u is a backward viscosity solution of

$$H(x, Du(x)) = c. \tag{2}$$

Moreover for any $u \in C(M, \mathbb{R})$ the uniform limit

$$\tilde{u} = \lim_{t\to\infty} \mathcal{L}_t u + ct$$

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exists. One can also define the forward Lax Oleinik semigroup $\mathcal{L}_t^*$ by

$$\mathcal{L}_t^*u(x) = \sup\{u(\gamma(t)) - \int_0^t L(\gamma,\dot\gamma) : \gamma : [0,t] \to M \text{ is piecewise } C^1, \gamma(0) = x\}.$$

Again $\mathcal{L}_t^* - ct$ has a fixed point for any $t > 0$ and any such fixed point u is a forward viscosity solution of (2). The semigroup $\mathcal{L}_t^*$ gives the solution to a Hamilton Jacobi final value problem.

Our goal in this paper is to stablish a relation of the hyperbolicity of the Aubry set to the exponential rate of convergenge of the semigroup $\mathcal{L}_t + ct$.

Theorem 1. *Assume that the Aubry set consists in a finite number of hyperbolic periodic orbits or critical points of the Euler-Lagrange flow. Then, there is $\mu > 0$ such that for any $u \in C(M,\mathbb{R})$ there is $K > 0$ such that*

$$\|\mathcal{L}_t u + ct - \bar u\|_0 \le K e^{-\mu t} \quad \forall t \ge 0. \tag{3}$$

Theorem 2. *Let $L : TM \to \mathbb{R}$ given by $L(x,v) = \frac{1}{2}v^2 - V(x)$ with*

$$\max_x V(x) = c, \quad V^{-1}(c) = \{x_1, \ldots, x_m\}.$$

Suppose that there is $\mu > 0$ such that for any $u \in C(M,\mathbb{R})$ there is $K > 0$ such that (3) holds. Then $(x_i, 0)$, $i = 1, \ldots m$ is a hyperbolic critical point of the Euler-Lagrange flow.

Remark 1. For Theorem 2, we only need that (3) holds for the function $u \equiv 0$.

2. Aubry set and static classes

We recall the definition of the Peierls Barrier ([3]). Define the action of a piecewise C^1 curve $\gamma : [0,T] \to M$ as

$$A(\gamma) = \int_0^T L(\gamma(s),\dot\gamma(s))ds.$$

Given a constant $k \in \mathbb{R}$ and $x_1, x_2 \in M$ let

$$h_T^k(x_1,x_2) = \inf\{A(\gamma) + kT | \gamma : [0,T] \to M \text{ joins } x_1 \text{ and } x_2\},$$

and

$$h^k(x_1,x_2) = \liminf_{T\to\infty} h_T^k(x_1,x_2),$$
$$\Phi^k(x_1,x_2) = \inf_T h_T^k(x_1,x_2).$$

Since time T is not bounded, there is only one possible value of k that will make the function h^k different from being identically $-\infty$ or

∞, this is again $c = c(L)$. We define $\Phi_T = h_T^c$ and the Peierls Barrier $h = h^c$. Mañé action potential Φ^k is identically $-\infty$ for $k < c(L)$ and finite for $k \geq c(L)$. We will also define $\Phi = \Phi^c$. In [3], it is shown that Φ_T actually converges uniformly to h.

Given a fixed $y \in M$, the function $x \mapsto -h(x, y)$ is a forward viscosity solution of (2), whereas $x \mapsto h(y, x)$ is a backward viscosity solution.

We now define as in [3] the Aubry set $\mathcal{A} \subset M$:

$$\mathcal{A} = \{x \in M, h(x, x) = 0\}.$$

(in the reference [3] it was called the Peierls set.)

In close relation to Mather's graph theorem ([4]), it is shown in [5], that the set $\mathcal{A}$ can be lifted, in a unique way, to a set $\tilde{\mathcal{A}} \subset TM$ that is an invariant set of minimizing orbits of the Euler-Lagrange flow. This set projects homeomorphically to $\mathcal{A}$ through the usual projection from TM to M. We also call the set $\tilde{\mathcal{A}}$ "Aubry set".

The "static classes" form a partition of $\mathcal{A}$, defined by the equivalence relation on $\mathcal{A}$: $x \sim y$ if and only if

$$h(x, y) + h(y, x) = 0.$$

If the Aubry set $\tilde{\mathcal{A}}$ is made up of a finite union of periodic orbits of the Euler-Lagrange flow, each static class is a periodic orbit or a critical point.

3. Proof of Theorem 1

Adding a constant to L we may take that $c(L) = 0$. We assume that the Aubry set consists in a finite number of hyperbolic, periodic orbits or critical points Γ_i: $\varphi_t(x_i, v_i) = (\boldsymbol{\gamma}_i(t), \boldsymbol{\gamma}_i'(t))$ $t \in \mathbb{R}$, $1 \leq i \leq m$. In the case of a periodic orbit we denote by T_i its minimal period and in the case of a critical point we put $T_i = 1$

Let $\lambda_{i,j}, j = 1, \ldots, d^*$ be the positive Lyapunov exponents of γ_i where $d^* = d$ if γ_i is a critical point and $d^* = d - 1$ if γ_i is a periodic orbit. Set $\lambda = \min_{i,j} \lambda_{ij}$, $\mathbf{T}_S = T_1 + \cdots + T_m$, $T = \min_{i \in [1,m]} T_i$,

Fix V_i a tubular neighborhood of Γ_i in TM, where the flow is orbit equivalent to its linearizacion. According to a result of Belistskii [8] there is $0 < \alpha < 1$ such that the linearizing map $F_i : B_i \to V_i$ is α-Hölder. We define

$$V = \bigcup_{i=1}^{m} V_i.$$

In [6] it was proved that for any backward viscosity solution v of (2)

$$v(x_i) = \min_{i \in [1,m]} v(x_i) + h(x_i, x). \tag{4}$$

Closely related to this fact we have the following

Proposition 1. *For* $u \in C(M,\mathbb{R})$ *let* $\bar u := \lim_{t\to\infty} \mathcal{L}_t u$. *Then*

$$\bar u(x) = \min_{z\in M} u(z) + h(z,x) \tag{5}$$
$$= \min\{u(z) + h(z,x_i) + h(x_i,x) : i \in [1,m], z \in M\} \tag{6}$$

Proof:

For any $x \in M$ and $t > 0$ there is $y_t(x)$ such that

$$\mathcal{L}_t u(x) = u(y_t(x)) + \Phi_t(y_t(x),x) \le u(z) + \Phi_t(z,x) \quad \forall z$$

Choose $t_n \to \infty$ such that $(y_{t_n}(x))$ converges to some $Y(x)$, then $(\Phi_{t_n}(y_{t_n}(x),x))$ converges to $h(Y(x),x)$ and so

$$\bar u(x) = u(Y(x)) + h(Y(x),x) = \min_{z\in M} u(z) + h(z,x)$$

In particular, for $x = x_i$ there is $y_i \in M$ such that

$$\bar u(x_i) = u(y_i) + h(y_i,x_i) = \min_{z\in M} u(z) + h(z,x_i),$$

and then

$$\begin{aligned} \bar u(x) &= \min_{i\in[1,m]} \bar u(x_i) + h(x_i,x) \\ &= \min\{u(z) + h(z,x_i) + h(x_i,x) : i \in [1,m], z \in M\} \end{aligned}$$

□

Let $u \in C(M,\mathbb{R})$, to prove Theorem 1 we have to stablish to inequalities. We first prove that there is $K > 0$ such that

$$\mathcal{L}_t u - \bar u \le K \exp(-\frac{\lambda T}{2\mathbf{T}_S} t). \tag{7}$$

Given $x \in M$, for every piecewise C^1 curve $\gamma : [0,t] \to M$ with $\gamma(0) = x$

$$\mathcal{L}_t u(x) \le u(\gamma(0)) + \int_0^t L(\gamma,\dot\gamma).$$

For some $i \in [1,m]$ we have that

$$\bar u(x) = \bar u(y_i) + h(y_i,x),$$

and to prove inequality (7) we will construct a curves joining y_i and x with action approximating $h(y_i,x)$.

For $x \in M$ let $i \in [1,m]$ such that

$$\bar u(x) = \bar u(x_i) + h(x_i,x).$$

Since $z \mapsto h(x_i, z)$ is a backward viscosity solution of (2), there is a semistatic curve $\alpha_x :]-\infty, 0] \to M$ with $\alpha_x(0) = x$ such that

$$\int_t^0 L(\alpha_x, \alpha_x') = h(x_i, x) - h(x_i, \alpha_x(t)), \quad t < 0.$$

We may assume that Γ_i is the α-limit of $\{(\alpha_x, \alpha_x')\}$. In fact, let Γ_j be the α-limit of $\{(\alpha_x, \alpha_x')\}$, then we have

$$h(x_i, x) = h(x_i, x_j) + h(x_j, x).$$

Since $\bar{u}(x_j) \leq \bar{u}(x_i) + h(x_i, x_j)$ we have that

$$\bar{u}(x) \leq \bar{u}(x_j) + h(x_j, x) \leq \bar{u}(x_i) + h(x_i, x) = \bar{u}(x)$$

and then $\bar{u}(x) = u(y_j) + h(y_j, x_j) + h(x_j, x)$.

Since $y \mapsto -h(y, x_j)$ is a forward viscosity solution of (2), there is a semistatic curve $\omega_j : [0, \infty[\to M$ such that $\omega_j(0) = y_j$ and

$$\int_0^t L(\omega_j, \omega_j') \;=\; h(y_j, x_j) - h(\omega_j(t), x_j), \quad t > 0$$

Let Γ_k be the ω-limit of $\{(\omega_j, \omega_j')\}$, then we have

$$h(y_j, x_j) = h(y_j, x_k) + h(x_k, x_j).$$

$$d((\omega_j(t), \omega_j'(t)), \varphi_{t+d_1}(x_k, v_k)) \leq C_1 e^{-\lambda t}, \qquad t > \tau(V)$$
$$d((\alpha_x(t), \alpha_x'(t)), \varphi_{t-d}(x_j, v_j)) \leq C_1 e^{\lambda t}, \qquad t < -\tau(V).$$

According to Theorem 3-11.1 in [7] there are $i_1 = k, \ldots, i_l = j$ and semistatic curves $\beta_r : \mathbb{R} \to M$, $r = 2, \ldots, l$ such that $\Gamma_{i_{r-1}}$ and Γ_{i_r} are the α and ω limits of $\{(\beta_r(t), \beta_r'(t)) : t \in \mathbb{R}\}$ respectively. Since all orbits Γ_i are hyperbolic and the semistatic curves β_r are in fact heteroclinic connections we may asssume that

$$d(((\beta_r(t), \beta_r'(t)), \varphi_t(x_{i_{r-1}}, v_{i_{r-1}})) \leq C_1 e^{\lambda t}, \qquad t < -\tau(V)$$
$$d((\beta_r(t), \beta_r'(t)), \varphi_{t+d_r}(x_{i_r}, v_{i_r})) \leq C_1 e^{-\lambda t}, \qquad t > \tau(V).$$

with $0 < d_r < T_{i_r}$ We have

$$\int_s^t L(\beta_r, \beta_r') = h(x_{i_{r-1}}, x_{i_r}) - h(x_{i_{r-}}, \beta_r(s)) - h(\beta_r(t), x_{i_r}).$$

We now define a curve whose action approximates $h(y_j, x)$ that is made of pieces of the heteroclinic connections β_r and some transition curves c_r exponentially close to Γ_{i_r}.

Let $\beta_1 = \omega_j$, $\beta_{l+1} = \alpha_x(t+d)$. For $1 < r \le l+1$ let

$$\mathbf{d}_r = d_1 + \cdots + d_{r-1}, \quad \mathbf{T}_r = T_{i_1} + \cdots + T_{i_{r-1}}$$

$$a_r(n) = \begin{cases} nT_k - d_1 & r = 1 \\ (2n+1)\mathbf{T}_r + nT_{i_r} - \mathbf{d}_{r+1} & 1 < r \le l \\ (2n+1)\mathbf{T}_{l+1} - \mathbf{d}_l - d & r = l+1 \end{cases}$$

Note that $a_{l+1} \le (2n+1)\mathbf{T}$.

There is $\bar\tau(V) > 0$ such that for any $x \in M$, $t \ge \bar\tau(V) - 2\max_i T_i$, we have $\beta_r(t) \in V$, $r = 0, \dots, l+1$.

Consider the curve $\gamma_n : [0, a_{l+1}(n)] \to M$, defined by

$$\gamma_n(s) = \begin{cases} \beta_1(s) & s \in [0, a_1(n)] \\ \beta_r(s - (2n+1)\mathbf{T}_r + \mathbf{d}_r) & s \in [a_{r-1}(n) + T_{i_{r-1}}, a_r(n)], r > 1 \\ c_r(s) & s \in [a_r(n), a_r(n) + T_{i_r}] \end{cases}$$

where $c_r : [a_r(n), a_r(n) + T_{i_r}] \to M$ is defined using tubular coordinates

$$\psi_r : U_r \to \mathbb{S}^1 \times \mathbb{R}^{d-1}, \psi_r(z) = \exp(i\eta_1(z)), \eta_2(z))$$

around γ_{i_r} by the expression

$$\begin{aligned}(\eta_1, \eta_2) \circ c_r(s) &= (1 - \frac{s - a_r(n)}{T_{r-1}})(\eta_1, \eta_2) \circ \beta_{r-1}(s - (2n+1)\mathbf{T}_{r-1} + \mathbf{d}_{r-1}) \\ &+ (\frac{s - a_r(n)}{T_{r-1}}))(\eta_1, \eta_2) \circ \beta_r(s - (2n+1)\mathbf{T}_r + \mathbf{d}_r)\end{aligned}$$

$$\begin{aligned}
\int_0^{a_{l+1}(n)} L(\gamma_n,\gamma_n') &= \int_0^{a_1(n)} L(\beta_1,\beta_1') + \sum_{r=1}^{l}\int_{a_r(n)+T_{i_r}}^{a_{r+1}(n)} L(\beta_{r+1},\beta_{r+1}') + \int_{a_r(n)}^{a_r(n)+T_{i_r}} L(c_r,c_r') \\
&= h(y_j,x_j) - h(a_1(n),x_k) + \sum_{r=1}^{l} h(x_{i_r},x_{i_{r+1}}) \\
&- \sum_{r=2}^{l} h(x_{i_{r-1}},\beta_r(-nT_{i_{r-1}})) + h((\beta_r(nT_{i_r}-d_r),x_{i_r}) \\
&+ \sum_{r=1}^{l}\int_{a_r(n)}^{a_r(n)+T_{i_r}} L(c_r,c_r') + h(x_j,x) - h(x_j,\beta_{l+1}(-nT_j))
\end{aligned}$$

Since $\int_0^{T_j} L(\boldsymbol{\gamma}_j,\boldsymbol{\gamma}_j')=0$ and

$d(c_r(s),\boldsymbol{\gamma}_{i_r}(s-(2n+1)\mathbf{T}_r+\mathbf{d}_r))+|c'(s)-\boldsymbol{\gamma}_{i_r}'(s-(2n+1)\mathbf{T}_r+\mathbf{d}_r)|\le C_2e^{-\lambda nT_{i_r}}$,

we have

$$\begin{aligned}
L_{a_{l+1}(n)}u(x)-\bar u(x) &\le \sum_{r=1}^{l}\int_{a_r(n)}^{a_r(n)+T_{i_r}} L(c_r,c_r') \\
&- \sum_{r=2}^{l} h(x_{i_{r-1}},\beta_r(-nT_{i_{r-1}})) + h((\beta_r(nT_{i_r}-d_r),x_{i_r}) \\
&- h(a_1(n),x_k) - h(x_j,\beta_{l+1}(-nT_j)) \\
&\le C_3e^{-\lambda nT} \le K\exp(-\frac{\lambda T}{2\mathbf{T}}a_{l+1}(n))
\end{aligned}$$

Now we stablish the other inequality.

For $x\in M$, $t>0$ let $\gamma_t:[-t,0]\to M$ be a curve such that

$$\mathcal{L}_tu(x)=u(\gamma_t(-t))+\int_{-t}^{0}L(\gamma_t,\gamma_t');\ \gamma_t(0)=x.$$

For any $s\in[-t,0]$, $i\in[1,m]$ we have

$$\begin{aligned}
\bar u(x) &\le u(\gamma_t(-t))+h(\gamma_t(-t),x_i)+h(x_i,x) \\
&\le u(\gamma_t(-t))+\Phi(\gamma_t(-t),\gamma_t(s)) \\
&\quad +h(\gamma_t(s),x_i)+h(x_i,\gamma_t(s))+\Phi(\gamma_t(s),x) \\
&\le \mathcal{L}_tu(x)+h(\gamma_t(s),x_i)+h(x_i,\gamma_t(s)). \qquad (8)
\end{aligned}$$

The idea of the proof is to choose s for each t sufficiently large such that the last two terms in (8) are $O(e^{-\mu t})$.

Remark 2. According to Lemma 5.3.4 in [3], for each neighborhood W of $\bigcup_i \boldsymbol{\gamma}_i$, there is $T(W) > 0$ such that if $\gamma : [-t, 0] \to M$, $t \geq T(W)$, is a minimizing curve then $(\gamma(s), \gamma'(s)) \in W$ for some $s \in [-t, 0]$.

Following lemma says that if we have a collection of orbits that have points that tend to a periodic orbit and also points that are not close to that orbit, then the time spent to go between those different kind of points tends to infinity.

Lemma 1. *Let W_i be a neigborhood of the orbit Γ_i in TM. Let $\beta_u = \{\varphi_t(q_\nu, w_\nu)\}_{t\in\mathbb{R}} : \nu \geq \nu_0$ be a collection of orbits such that for each $\nu \geq \nu_0$, there are $s_\nu, t_\nu, r_\nu \in \mathbb{R}$ satisfying*

- $\lim\limits_{\nu\to\infty} d(\varphi_{s_\nu}(q_\nu, w_\nu), \varphi_{t_\nu}(x_i, v_i)) = 0$
- $\varphi_{r_\nu}(q_\nu, v_\nu) \notin W_i$

Then $|s_\nu - r_\nu| \to \infty$ as $\nu \to \infty$.

Proof: The proof is immediate for the linear case, the general case follows from α-Hölder linearization. □

Applying Remark 2, for $t > T^*$ we can choose $s_t \in [-t, 0]$, $i(t) \in [1, m]$ such that

$$\lim_{t\to\infty} d((\gamma_t(s_t), \gamma_t'(s_t)), \Gamma_{i(t)}) = 0.$$

This implies by Lemma 1 that $-s_t, s_t + t \to \infty$ as $t \to \infty$.

Claim 1. There are T', s_+ such that for $t > T'$ and $\tau \in [s_t, s_+]$ we have $(\gamma_t(\tau), \gamma_t'(\tau)) \in V_{i(t)}$.

Otherwise, there are sequences $t_n, -\tau_n \to \infty$ such that $\forall n : i(t_n) = k$ and $(\gamma_{t_n}(\tau_n), \gamma_{t_n}'(\tau_n))$ is a sequence of points in ∂V_k converging to (q, v). If $q = \boldsymbol{\gamma}_k(a)$, there is $\delta > 0$ such that $|\gamma_{t_n}'(s + \tau_n) - \boldsymbol{\gamma}_k'(s + a)| \geq \delta$ for $s \in [0, T_k]$ and n sufficiently large. Thus, there is a subsequence that we still denote by t_n such that $\gamma_{t_n}(\tau_n + T_k)$ converges to a point in $M - \boldsymbol{\gamma}_k$. In any case we have a sequence $v_n \to \infty$ such that $\gamma_{t_n}(-v_n)$ converges to $Q \notin \boldsymbol{\gamma}_k$.

Letting $\sigma_n = -s_{t_n}$, we have that $\sigma_n - v_n \to \infty$ and then

$$\begin{aligned}
\lim_{n\to\infty} \int_{-\sigma_n}^{0} L(\gamma_{t_n}, \gamma_{t_n}') &= \lim_{n\to\infty} \Phi_{\sigma_n - v_n}(\gamma_{t_n}(-\sigma_n), \gamma_{t_n}(-v_n)) + \Phi_{v_n}(\gamma_{t_n}(-v_n), x) \\
&= h(x_k, Q) + h(Q, x) \\
&= h(x_k, Q) + h(Q, x_k) + h(x_k, x) > h(x_k, x).
\end{aligned}$$

On the other hand

$$\lim_{t\to\infty}\int_{-\sigma_n}^{0} L(\gamma_{t_n},\gamma_{t_n}') = \lim_{t\to\infty}\Phi_{\sigma_n}(\gamma_{t_n}(-\sigma_n),x) = h(x_k,x). \tag{9}$$

this contradiction proves Claim 1.

Let $r_m \to \infty$ such that $\gamma_{r_m}(-r_m)$ converges to $Y(x)$ and put $\rho_m = s_{r_m}$, then

$$\lim_{m\to\infty}\int_{-r_m}^{\rho} L(\gamma_{r_m},\gamma_{r_m}') = h(Y(x),x).$$

Using the same kind of arguments as in the proof of Claim 1 one proves:

Claim 2. There are T'', s_- such that for $t > T''$ and $\tau \in [s_- - t, s_t]$ we have $(\gamma_t(\tau),\gamma_t'(\tau)) \in V_{i(t)}$.

Taking $T = \max(T',T'')$, for $t > T$ and $\tau \in [s_- - t, s_+]$ we have $(\gamma_t(\tau),\gamma_t'(\tau)) \in V_i$.

For a linear hyperbolic saddle it is easy to see that if a long piece of orbit is contained in a neighborhood then there is a point very near to the fixed point. This holds for both continuous and discrete time. The same holds for a non linear sistem using an α-Hölder linearization. Thus we have the following

Claim 3. There are positive constants C, T and α such that for any $t > T$ there is $\tau_t \in [s_- - t, s_+]$ such that

$$d((\gamma_t(\tau_t),\gamma_t'(\tau_t)),(x_i,v_i)) \le C\exp(-\frac{\alpha}{2}\lambda t)$$

From (8) and the fact that h is Lipschitz in both arguments we get

$$\bar{u}(x) - \mathcal{L}_t u(x) \le K\exp(-\frac{\alpha}{2}\lambda t)$$

□

4. Proof of Theorem 2

Lemma 2. *Let $L : TM \to \mathbb{R}$ given by $L(x,v) = \frac{1}{2}v^2 - V(x)$ with*

$$\max_x V(x) = 0, \quad V^{-1}(0) = \{x_1,\dots,x_m\}.$$

Suppose that there is $\mu > 0$ such that for $u \equiv 0$ there is $K > 0$ such that

$$\|\mathcal{L}_t u - \bar{u}\|_0 \le Ke^{-\mu t} \quad \forall t \ge 0. \tag{10}$$

Then $(x_i, 0)$, $i = 1, \dots m$ *is a hyperbolic critical point of the Euler-Lagrange flow.*

Proof: For any x_j the function $h_j(x) = h(x_j, x)$ is a viscosity solution of the Hamilton Jacobi equation

$$\frac{1}{2}|D\phi(x)|^2 + V(x) = 0.$$

Suppose $(x_i, 0)$ is not hyperbolic, which means that x_i is a degenerate maximum of V. Let $0, -\lambda_1^2, \dots, -\lambda_k^2$, $\lambda_i > 0$, be the eigenvalues of $\operatorname{Hess} V(x_i)$. By the splitting lemma [9], there are local coordinates (y, z) around x_i such that x_i corresponds to the origin and

$$-2V(y,z) = \psi(y) + \lambda_1^2 z_1^2 + \cdots + \lambda_k^2 z_k^2 \tag{11}$$

$$D\psi(0) = 0, \qquad \operatorname{Hess}\psi(0) = 0 \tag{12}$$

Thus, there is $C > 0$ such that

$$|D_z\sqrt{-2V(y,z)}| \leq C, \tag{13}$$

$$\lim_{(y,z)\to 0} D_y\sqrt{-2V(y,z)} = 0 \tag{14}$$

The linearization of the Euler Lagrange flow at $(x_i, 0)$ has eigenvalues $0, \pm\lambda_1, \dots, \pm\lambda_k$. Denote by W^u, W^s, W^c the unstable, stable, and center manifolds at $(x_i, 0)$ respectively.

Claim 4. There exists a calibrated curve $\gamma :]-\infty, 0] \to M$ with α limit x_i such that $(\gamma(t), \dot\gamma(t))$ is not in W^u.

Indeed let 2δ be smaller than the minimum of $h(x_i, x_j)$ for all x_j different from x_i. Let U be the open set of points p such that $h(x_i, p) < \delta$. For any point p in U define the minimizing curve starting at time $-T$ in x_i and ending at p in time 0. The limit curve, as T tends to infinite exists because the velocities are bounded, and it is in fact a minimizer that starts at x_i. So all points p in U have a semistatic curve starting at x_i and ending at p. Some of this points are in the unstable manifold, but since the there are some zero eigenvalues this manifold has positive codimension. This proves the claim. □

Let $\gamma :]-\infty, 0] \to M$ be as in the claim, then there is a trajectory $\varphi_t(w)$ of the Euler Lagrange flow on W^c such that

$$d(\varphi_t(w), (\gamma(t), \dot\gamma(t)) = O(e^{\mu t}), \quad t \to -\infty$$

Since $(\gamma(t), \dot\gamma(t))$ is not on W^u then, writting $\gamma(t) = (\gamma_y(t), \gamma_z(t))$ in local coordinates, we have

$$\lim_{t\to-\infty} \frac{\dot\gamma_z(t)}{|\dot\gamma(t)|} = 0.$$

For the function $u \equiv 0$ we have

$$\lim_{t\to\infty} \mathcal{L}_t u(x) = \min_j h_j(x),$$

and there is a neighborhood W_i of x_i such that for $x \in W_i$

$$\lim_{t\to\infty} \mathcal{L}_t u(x) = h_i(x).$$

Since

$$h_i(\gamma(0)) - h_i(\gamma(-t)) = \int_{-t}^{0} \frac{1}{2}\dot\gamma^2 - V(\gamma) = -\int_{-t}^{0} 2V(\gamma) \geq \mathcal{L}_t u(\gamma(0)),$$

$$\frac{d}{ds} h_i(\gamma(s)) = -2V(\gamma(s)) = \dot\gamma(s)^2, \quad \frac{d}{ds}\log h_i(\gamma(s)) = \frac{-2V(\gamma(s))}{h_i(\gamma(s))}$$

By L'Hopital rule and (13), (14)

$$\begin{aligned}
\lim_{s\to-\infty} \frac{\log h_i(\gamma(s))}{s} &= \lim_{s\to-\infty} \frac{d}{ds}\log h_i(\gamma(s)) \\
&= \lim_{s\to-\infty} \frac{-2DV(\gamma(s))\dot\gamma(s)}{-2V(\gamma(s))} \\
&= -2 \lim_{s\to-\infty} \frac{D_y V(\gamma(s))\dot\gamma_y(s) + D_z V(\gamma(s))\dot\gamma_z(s)}{\sqrt{-2V(\gamma(s))}\quad |\dot\gamma(s)|} \\
&= 0. \qquad (15)
\end{aligned}$$

Assumption (10) gives

$$C\exp(-\mu t) \geq h_i(\gamma(0)) - \mathcal{L}_t u(\gamma(0)) \geq h_i(\gamma(-t))$$

so that

$$\begin{aligned}
-\log C + \mu t &\leq -\log h_i(\gamma(-t)) \\
\mu &\leq \liminf_{t\to\infty} -\frac{\log h_i(\gamma(-t))}{t}, \qquad (16)
\end{aligned}$$

contradicting (15).

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