$\bar{\partial}$-TORSION AND COMPACT ORBITS OF ANOSOV ACTIONS ON COMPLEX 3-MANIFOLDS

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Abstract. In analogy with work of Fried and Laederich we study the relation between $\bar{\partial}$-torsion of a compact complex 3-manifold M and the compact orbits of an Anosov holomorphic action on M.

1. Introduction

In 1968 Milnor [8] pointed out the remarkable similarity between the algebraic formalism of the Reidemeister torsion in topology and zeta functions à la Weil in dynamical systems theory. This theme has been thoroughly investigated by David Fried who devised for any smooth flow and any flat bundle over the underlying manifold a certain zeta function counting the periodic orbits of a flow with appropriate multiplicities. He was able to show for a variety of flows [1] that the zeta function associated to any acyclic flat bundle is actually meromorphic on a neighborhood of $[0, \infty)$, regular at 0, and that its value at 0 coincides with the Reidemeister torsion with coefficients in the given flat bundle and thus is a topological invariant. Because of the analogy with the Lefschetz fixed point formula, Fried used the term "flow with the Lefschetz property" in reference to such a flow. In particular, Fried proved that the geodesic flow of a closed manifold of constant negative curvature has the Lefschetz property [2] and we extended those results to transitive Anosov flows on 3-manifolds [10].

In analogy with their definition of analytic torsion on a Riemannian manifold, Ray and Singer define the $\bar{\partial}$-torsion for complex manifolds. Fried proved that the known connections between torsion and the dynamical features of closed orbits continue to hold in the holomorphic category [3]. He posed also a question about such connections for actions of a noncompact Lie group other than $\mathbb{R}$. Laederich [6] has investigated the case of complex manifolds which fibrate over the torus having a one dimensional holomorphic foliation transverse to the fibers.

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He found a formula relating $\bar{\partial}$-torsion to "theta" functions associated to the compact orbits of the foliation.

In this paper we find a formula (6) relating $\bar{\partial}$-torsion of a compact complex 3-manifold M to special values of zeta functions (3), (5) defined with the compact orbits of an Anosov holomorphic action on M.

2. Main result

2.1. **Holomorphic Anosov actions.** For $(M, ||\cdot||)$ a Hermitian 3-manifold, we follow Ghys [5] and call a holomorphic action $\phi : \mathbb{C}^* \times M \to M$, $(x, T) \mapsto \phi(T)(x)$, Anosov if there exist invariant subbundles E^u, E^s of the real tangent bundle $T_{\mathbb{R}}M$ and constants $c > 0$, $a > 0$, such that

1. $T_{\mathbb{R}}M = E^s \oplus E^u \oplus T\phi$, where $T\phi$ is the bundle tangent to the orbits of the action.
2. For all $T \in \mathbb{C}^*$, $v^s \in E^s$, $v^u \in E^u$ one has

$$\|d\phi(T)(v^s)\| \leq c|T|^{-a}\|v^s\|$$

$$\|d\phi(T)(v^u)\| \leq c|T|^{a}\|v^u\|.$$

In all of this paper G will denote the Lie group $\mathrm{SL}(2, \mathbb{C})$.

The first examples of Anosov actions are holomorphic suspensions. Suppose $A \in G$ preserves a lattice $\Lambda \subset \mathbb{C}^2$ and let $\bar{A}$ be the corresponding diffeomorphism of $\mathbb{C}/\Lambda$. For $\omega \in \mathbb{C} - S^1$ consider the diffeomorphism A_ω of $\mathbb{C}/\Lambda \times \mathbb{C}^*$ given by $A_\omega(x, S) = (\bar{A}(x), \omega S)$, and the properly discontinous and free action of $\mathbb{Z}$ on $\mathbb{C}/\Lambda \times \mathbb{C}^*$ given by $(k, (x, S)) \mapsto A^k_\omega(x, S)$. If the spectrum of A is disjoint from S^1, the action of $\mathbb{C}^*$ given on the quotient manifold M by $(T, [x, S]) \mapsto [x, ST]$ is Anosov.

A second kind of examples of holomorphic Anosov actions comes from the choice of a cocompact discrete subgroup Γ of G with no elliptic elements. Let $M = \Gamma \setminus G$, then the holomorphic $\mathbb{C}^*$ action $\phi(T)(\Gamma g) = \Gamma g \left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1}\end{smallmatrix}\right)$ is Anosov. One proves this fact in exactly the same way as for the corresponding well known examples in the real domain.

Ghys modifies the last examples by the following construction. Let $u : \Gamma \to \mathbb{C}^*$ be a representation and consider the action of Γ on G given by $(\gamma, g) \mapsto \gamma g \left(\begin{smallmatrix} u(\gamma) & 0 \\ 0 & u(\gamma)^{-1}\end{smallmatrix}\right)$. This action commutes with the $\mathbb{C}^*$ action by right translations by $\left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1}\end{smallmatrix}\right)$. If the action of Γ is free, proper and totally discontinous one considers the quotient manifold M with the $\mathbb{C}^*$ action which becomes Anosov.

Ghys proved that any holomorphic Anosov action on a compact complex 3-manifold is up to finite covers holomorphically conjugate to one of the examples described above.

Let Γ be a discrete subgroup of G and $M = \Gamma \setminus G$ with the Anosov action as above. The orbit of Γg is compact iff there are $\gamma \in \Gamma - \{I\}$, $\lambda \in \mathbb{C}^* - \{1\}$ such that

$$g^{-1}\gamma g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \exp(i\theta + l) & 0 \\ 0 & \exp(-i\theta - l) \end{pmatrix} \tag{1}$$

Since the elements of $\Gamma - \{I\}$ are hyperbolic, $\forall \gamma \in \Gamma - \{I\}$ there are a unique $\lambda_\gamma \in \mathbb{C}^*$ with $|\lambda_\gamma| > 1$, and $g \in G$ such that (1) holds. If there is another $h \in G$ such that $h^{-1}\gamma h = \left(\begin{smallmatrix} \lambda_\gamma & 0 \\ 0 & \lambda_\gamma^{-1} \end{smallmatrix}\right)$, then $g = h\left(\begin{smallmatrix} S & 0 \\ 0 & S^{-1} \end{smallmatrix}\right)$.

Let G_γ be the centralizer of γ, then

$$g^{-1}G_\gamma g = \{\left(\begin{smallmatrix} S & 0 \\ 0 & S^{-1} \end{smallmatrix}\right) : S \in \mathbb{C}^*\}.$$

Therefore the orbit of Γg is $\mathcal{O}_\gamma = \{\Gamma x g : x \in G_\gamma\}$ and so it is conformally equivalent to $\Gamma_\gamma \setminus G_\gamma$, where $\Gamma_\gamma = \Gamma \cap G_\gamma$. Moreover

$$\Gamma_\gamma = \left\{ g \begin{pmatrix} \lambda_{\gamma_o}^n & 0 \\ 0 & \lambda_{\gamma_o}^{-n} \end{pmatrix} g^{-1} : n \in \mathbb{Z} \right\}$$

for $\gamma_o \in \Gamma$ prime, and $\mathcal{O}_{\gamma_o} = \mathcal{O}_\gamma$. Writing $\lambda_{\gamma_o} = \exp(l_{\gamma_o} + i\theta_{\gamma_o})$ we have that $\Gamma_{\gamma_o} \setminus G_{\gamma_o} = \Gamma_\gamma \setminus G_\gamma$ is conformally equivalent to the torus $(l_{\gamma_o} + i\theta_{\gamma_o})\mathbb{Z} + 2\pi i\mathbb{Z} \setminus \mathbb{C}$. Note that for $k = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we have

$$(gk)^{-1}\gamma_0^{-1} gk = \begin{pmatrix} \lambda_{\gamma_o} & 0 \\ 0 & \lambda_{\gamma_o}^{-1} \end{pmatrix}.$$

Thus, the cyclic group Γ_γ defines two compact orbits $\mathcal{O}_{\gamma_o}$ and $\mathcal{O}_{\gamma_o^{-1}}$. If Γ_γ, $\Gamma_{\gamma'}$ define the same compact orbit, then there is $\delta \in \Gamma$ such that $\delta\gamma'\delta^{-1} \in \Gamma_\gamma$. Thus, if $[\Gamma]$ denotes the set of Γ-conjugacy classes of elements of Γ, the compact orbits of the Anosov action are parametrized by the classes $[\gamma] \in [\Gamma]$ for prime $\gamma \in \Gamma_0 = \Gamma - \{I\}$.

For $\rho : \Gamma \to \mathrm{U(m)}$ a representation and $\Re z > 0$, we define

$$Z_a(z) = \prod_{[\gamma]\ \text{prime}} \prod_{j,k=1}^{\infty} \prod_{r=\pm 2} \det(I - \rho(\gamma) e^{ir\theta_\gamma} \lambda_\gamma^{-2j+1} \bar{\lambda}_\gamma^{-2k+1} e^{-l_\gamma z}). \tag{2}$$

$$\begin{aligned}
Z_A(z) &= \prod_{[\gamma]\text{ prime}} \prod_{j,k=1}^{\infty} \det(I-\rho(\gamma)\lambda_\gamma^{-2j-k}e^{-l_\gamma z})^{k+3} \\
Z_B(z) &= \prod_{[\gamma]\text{ prime}} \prod_{j,k,r=1}^{\infty} \det(I-\rho(\gamma)\lambda_\gamma^{-2j-k+1}\bar{\lambda}_\gamma^{-2r+1}e^{-l_\gamma z}) \\
Z_C(z) &= \prod_{[\gamma]\text{ prime}} \prod_{j,k=1}^{\infty} \det(I-\rho(\gamma)\bar{\lambda}_\gamma^{-2j}\lambda_\gamma^{-k}e^{-l_\gamma z})^{k-3} \\
Z_\rho(z) &= \frac{Z_A(z)Z_B(z)^6}{Z_C(z)}.
\end{aligned} \tag{3}$$

2.2. **$\bar{\partial}$-torsion.** For a closed Hermitian complex k-manifold M and a representation $\rho : \pi_1(M) \to \mathrm{U(m)}$ we define $\bar{\partial}$-torsion.

First we recall how one defines the determinant of a positive elliptic differential operator D on M. D^{-s} is a trace class operator for $\Re s$ large and the Dirichlet series $\zeta_D(s) = \operatorname{Tr} D^{-s} = \sum_\lambda \lambda^{-s}$ has a meromorphic continuation to $\mathbb{C}$, regular at $s = 0$. One defines $\det D = \exp(-\zeta_D'(0))$.

Next we introduce the $\bar{\partial}$-Laplacian associated to ρ (see [3]). Consider the $\mathbb{C}^m$ valued differential forms ω on the universal cover $\tilde{M}$ of M that are ρ equivariant. That is, if $g \in \pi_1(M)$ acts on $\tilde{M}$ as a deck transformation then the components $\omega_1, \dots, \omega_m$ of ω satisfy

$$g^*\omega_i = \sum_{j=1}^{m} \rho(g)_{ij}\omega_j.$$

The space $\Omega(\rho)$ of such twisted forms has a decomposition

$$\Omega(\rho) = \bigoplus_{0\le p,q\le k} \Omega^{p,q}$$

where the summands fit into a double cochain complex because the derivatives ∂ and $\bar{\partial}$ preserve the equivariance property. Since ρ is unitary, using the Riemannian measure on M and the Hodge star operator one defines an inner product on twisted forms. Taking the adjoint $\bar{\partial}^*$ of $\bar{\partial}$ one forms the $\bar{\partial}$-Laplacian $\bar{\Delta} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on twisted forms. Recalling the bigrading we write $\bar{\Delta} = \bigoplus_{0\le p,q,\le k} \bar{\Delta}^{p,q}$, $\bar{\Delta}^{p,q} : \Omega^{p,q} \to \Omega^{p,q}$.

The Hodge theorem states that $\ker \bar{\Delta}^{p,q}$ is isomorphic to the Dolbeaut group $H^{p,q}(\Omega)$. When all these groups are zero we call the representation **acyclic**. In such a case $D = \bar{\Delta}^{p,q}$ is a positive elliptic operator. In complex differential geometry one computes $\det D$ via the trace of the heat kernel $\operatorname{Tr} e^{-tD}$ as follows. The Mellin transform of $e^{-\lambda t}$ ($\lambda > 0$) is

$$Me^{-\lambda t} = \int_0^\infty t^{s-1}e^{-\lambda t}dt = \lambda^{-s}\int_0^\infty x^{s-1}e^{-x}dx = \lambda^{-s}\Gamma(s).$$

Summing over $\lambda \in$ spec D, gives

$$M\operatorname{Tr} e^{-tD} = \Gamma(s)\zeta_D(s)$$

For an acyclic representation ρ, Ray and Singer [9] defined the $\bar\partial$-torsion $\tau_p \in]0,\infty[$ by

$$\tau_p^2 = \exp \frac{d}{ds}\left(\frac{1}{\Gamma(s)}M\big(\sum_q(-1)^q q \operatorname{Tr} e^{-t\bar\Delta^{p,q}}\big)\right)_{s=0}. \tag{4}$$

The purpose of this paper is to prove the following

THEOREM. *Let Γ be a cocompact discrete subgroup of G and let $M = \Gamma \setminus G$ with the Anosov action as in 2.1. Let $\rho : \Gamma \to U(m)$ be an acyclic representation. Let Z_a, Z_ρ be as in* (2) *and* (3)*, and let $\eta_a = Z_a'/Z_a$. Then $\log Z_\rho$ is analytic for $\Re z > -1$ and η_a has a meromorphic continuation to $\mathbb{C}$ whose poles are simple, are located on $i\mathbb{R}$, and except for the zero pole, have integer residues. If r_ρ denotes the residue of η_a at zero and $c \in \mathbb{C}^*$,*

$$\zeta_\rho(z) = c\ \exp\int_0^z(\eta_a(s) - \frac{r_\rho}{s})ds \tag{5}$$

defines an entire function whose only zeros ocurr at the nonzero poles of η_a. Choosing c such that $\zeta_\rho(1) = Z_a(1)$ we have

$$\tau_p^2 = \left(\frac{\exp(\frac{19}{80}m\, vol(M)\pi^{-3})}{2^{9r_\rho/2}|Z_\rho(0)|^2\zeta_\rho(0)^3}\right)^{\binom{3}{p}} \tag{6}$$

Remarks:

1. The analogous result for the holomorphic suspension examples in 2.1 is a particular case of the theorem of Laederich [6].
2. We have been unable to deal with the modified examples of Ghys.
3. We identified the zeta function Z_a given by the product (2), which is not convergent for $\Re z = 0$, and we found that it was studied by Scott [11].
4. We chose to decompose the function Z_ρ in order to have nice product expansions (3).

3. The $\bar\partial$-Laplacian

Let K =SU$(2,\mathbb{C})$ (maximal compact subgroup of G) and $\mathfrak{g}$ =sl$(2,\mathbb{C})$ (Lie algebra of G). Consider the following $\mathbb{R}$-basis of $\mathfrak{g}$

$E_1 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$, $E_2 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$, $E_3 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$,

$F_1 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & i \\ -i & 0\end{smallmatrix}\right)$, $F_2 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 & i \\ i & 0\end{smallmatrix}\right)$, $F_3 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$.

$\mathfrak{g}$ has a natural Hermitian product $\langle x, y\rangle = \mathrm{Tr}\,(x\bar{y}^t)$ and the above basis is orthonormal for the metric $\Re\langle,\rangle$. Note that E_1, F_2, F_3 is a basis of su$(2,\mathbb{C})$.

Thought as a real Lie algebra, its complexification $\mathfrak{g}_{\mathbb{C}}$ has a basis $\{X_1, X_2, X_3, \overline{X}_1, \overline{X}_2, \overline{X}_3\}$ where

$$X_k = \frac{1}{2}(E_k - \mathbb{J}F_k), \quad \overline{X}_k = \frac{1}{2}(E_k + \mathbb{J}F_k), \quad \mathbb{J}^2 = -\mathbb{I}.$$

Writing $[\overline{X}_l, \overline{X}_j] = \sum_k C^k_{lj}\overline{X}_k$, we have $C^j_{lj} = 0, C^3_{12} = C^1_{32} = C^2_{31} = \sqrt{2}$.

Each element $X \in \mathfrak{g}$ defines a left invariant vector field on the complex manifold G such that

$$X(f)(g) = \frac{d}{ds}\big|_{s=0} f(g\exp(sX))$$

for $f : G \to \mathbb{C}$. The elements of $\mathfrak{g}_{\mathbb{C}}$ define sections of the complexified tangent bundle $T_{\mathbb{C}}G$. Note that

$$\mathbb{J}F_k(f)(g) = i\frac{d}{ds}\big|_{s=0} f(g\exp(sF_k)).$$

Let ω_k, $\overline{\omega}_k$ be the duals of the vector fields X_k, $\overline{X}_k$. Then

$$\omega_k(E_l) = \delta_{kl},\ \omega_k(F_l) = i\delta_{kl},\ \overline{\omega}_k(E_l) = \delta_{kl},\ \overline{\omega}_k(F_l) = -i\delta_{kl}.$$

Any element η of $\wedge^{p,q}(G)$ can be written

$$\eta = \sum_{I,J} \eta_{IJ}\omega_I \wedge \overline{\omega}_J$$

where $I = (i_1, \dots, i_p), i_1 < \cdots < i_p,\ J = (j_1, \dots, j_q), j_1 < \cdots < j_q$,

$$\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}, \quad \overline{\omega}_I = \overline{\omega}_{i_1} \wedge \cdots \wedge \overline{\omega}_{i_p}.$$

All of the above vector fields define vector fields on $M = \Gamma \setminus G$. Define

$$[\overline{\omega}_l, \overline{\omega}_j] = \sum_k C^k_{lj}\overline{\omega}_k.$$

Let $\eta = f\omega_I \wedge \overline{\omega}_J \in \wedge^{p,q}(M)$ then

$$\begin{aligned}\bar\partial\eta =&(-1)^p\sum_j \overline{X}_j(f)\omega_I\wedge\overline{\omega}_j\wedge\overline{\omega}_J\\ &+(-1)^{p+1}\sum_{l<j} f\omega_I\wedge\overline{\omega}_l\wedge\overline{\omega}_j\iota([\overline{X}_l,\overline{X}_j])\wedge\overline{\omega}_J\end{aligned}\tag{7}$$

One computes the adjoint

$$\begin{aligned}\bar\partial^*\eta =&(-1)^{p+1}\sum_j X_j(f)\omega_I\wedge\iota(\overline{X}_j)\overline{\omega}_J\\ &+(-1)^p\sum_{l<j} f\omega_I\wedge[\overline{\omega}_l,\overline{\omega}_j]\wedge\iota(\overline{X}_l)\iota(\overline{X}_j)\overline{\omega}_J,\end{aligned}\tag{8}$$

and so the $\bar\partial$-Laplacian

(9)

$$\begin{aligned}\bar\Delta\eta =& \sum_k -X_k(\overline{X}_k f)\omega_I\wedge\overline{\omega}_J+\sum_{jl}[X_l,\overline{X}_j](f)\omega_I\wedge\overline{\omega}_j\wedge\iota(\overline{X}_l)\overline{\omega}_J\\ &-\sum_{m<n}\sum_{l<j}\sum_k C^k_{mn}C^k_{lj}f\omega_I\wedge\overline{\omega}_m\wedge\overline{\omega}_n\wedge\iota(\overline{X}_l)\iota(\overline{X}_j)\overline{\omega}_J\\ &+\sum_{lj}\big(X_l(f)\omega_I\wedge\overline{\omega}_j\wedge\iota([\overline{X}_l,\overline{X}_j])\overline{\omega}_J+\overline{X}_l(f)\omega_I\wedge[\overline{\omega}_j,\overline{\omega}_l]\wedge\iota(\overline{X}_j)\overline{\omega}_J\big)\\ &+\sum_{m<n} f\omega_I\wedge[\overline{\omega}_m,\overline{\omega}_n]\wedge\iota([\overline{X}_m,\overline{X}_n])\overline{\omega}_J\\ &-\sum_{mnj} f\omega_I\wedge[\overline{\omega}_m,\overline{\omega}_n]\wedge\overline{\omega}_j\wedge\iota(\overline{X}_n)\iota([\overline{X}_m,\overline{X}_j])\overline{\omega}_J\end{aligned}$$

Thus, we have the following

Proposition.

$$\eta=f\omega_I\quad\Rightarrow\quad\bar\Delta^{p,0}\eta=(-\sum_k X_k\overline{X}_k f)\omega_I.\tag{10}$$

Writing $\omega_I\wedge(f_1\overline{\omega}_1+f_2\overline{\omega}_2+f_3\overline{\omega}_3)=\omega_I\otimes(f_1,f_2,f_3)^t$*, we have*

(11) $\bar\Delta^{p,1}:\omega_I\otimes(f_1,f_2,f_3)^t\mapsto$

$$\omega_I\otimes\begin{pmatrix}-\sum_k X_k\overline{X}_k+2\mathbb{I} & -\sqrt{2}\mathbb{J}F_3 & \sqrt{2}\mathbb{J}F_2\\ -\sqrt{2}\mathbb{J}F_3 & -\sum_k X_k\overline{X}_k+2\mathbb{I} & \sqrt{2}E_1\\ \sqrt{2}\mathbb{J}F_2 & -\sqrt{2}E_1 & -\sum_k X_k\overline{X}_k+2\mathbb{I}\end{pmatrix}\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}.$$

Writing $\omega_I\wedge(f_1\overline{\omega}_2\wedge\overline{\omega}_3+f_2\overline{\omega}_3\wedge\overline{\omega}_1+f_3\overline{\omega}_1\wedge\overline{\omega}_2)=\omega_I\otimes(f_1,f_2,f_3)^t$*, we have*

(12) $\bar{\Delta}^{p,2} : \omega_I \otimes (f_1, f_2, f_3)^t \mapsto$

$$\omega_I \otimes \begin{pmatrix} -\sum_k X_k\overline{X}_k + 2\mathbb{I} & \sqrt{2}\mathbb{J}E_3 & -\sqrt{2}\mathbb{J}E_2 \\ \sqrt{2}\mathbb{J}E_3 & -\sum_k X_k\overline{X}_k + 2\mathbb{I} & \sqrt{2}E_1 \\ -\sqrt{2}\mathbb{J}F_2 & -\sqrt{2}E_1 & -\sum_k X_k\overline{X}_k + 2\mathbb{I} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

(13)

$$\eta = f\omega_I \wedge \overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \overline{\omega}_3 \Rightarrow \bar{\Delta}^{p,3}\eta = (-\sum_k X_k\overline{X}_k f)\omega_I \wedge \overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \overline{\omega}_3.$$

4. The trace formula for the heat kernel

Consider an acyclic representation $\rho : \Gamma \to \mathrm{U(m)}$. Let π_ρ be the induced unitary representation of G on

$$L^2(G;\rho) = \{f : G \to \mathbb{C}^m : \int_M |f|^2 < \infty, \quad f(\gamma g) = \rho(\gamma)f(g)\}$$

π_ρ decomposes as $\pi_\rho = \sum_{\omega\in\hat{G}} n_\rho(\omega)\omega$ and $n_\rho < \infty$ for $\omega \in \hat{G}$, where $\hat{G}$ stands for set of all equivalence classes of irreducible unitary representations of G. Let φ be in the Harish-Chandra's L^1 Schwarz space $\mathcal{C}_1(G)$. For $(T_\omega, V_\omega) \in \omega \in \hat{G}$, the operator $T_\omega(\varphi) = \int_G \varphi(x)T_\omega(x)dx$ on V_ω is trace class and $\Theta_\omega(\varphi) = \mathrm{Tr}\, T_\omega(\varphi)$ is the character of ω. $\bar{\Delta}^{p,q}$ extends to

$$L^2(\Omega^{p,q}) = \wedge^{p,q}(\mathfrak{g}) \otimes L^2(G;\rho) = \bigoplus_{\omega\in\hat{G}} n_\rho(\omega) \wedge^{p,q}(\mathfrak{g}) \otimes V_\omega. \tag{14}$$

Denote by $D^{p,q}$ the corresponding operator on $\wedge^{p,q}(\mathfrak{g}) \otimes L^2(G)$. The heat operator $e^{-tD^{p,q}}$ has kernel $h_t^{*p,q}$ in $\mathrm{End}(\wedge^{p,q}(\mathfrak{g})) \otimes \mathcal{C}_1(G)$. Thus the Schwarz kernel for $e^{-t\bar{\Delta}^{p,q}}$ is

$$h_t^{p,q}(\Gamma x, \Gamma y) = \sum_{\gamma\in\Gamma} h_t^{*p,q}(x^{-1}\gamma y) \otimes \rho(\gamma).$$

Setting $\varphi_t^p = \sum_q (-1)^q q \mathrm{Tr}\, h_t^{*p,q}$, one has the trace formula

$$\begin{aligned} H(t) &:= \sum_{q=0}^{3} (-1)^q q \mathrm{Tr}\, e^{-t\bar{\Delta}^{p,q}} = \sum_{\omega\in\hat{G}} n_\rho(\omega)\Theta_\omega(\varphi_t^p) \\ &= \sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}\,\rho(\gamma)\mathrm{vol}\,(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma\setminus G} \varphi_t^p(x^{-1}\gamma x)d\dot{x}. \end{aligned} \tag{15}$$

The last integrals can be expressed in terms of the characters of the representations in the **principal series** which we now define (see [4]).

For $k \in \mathbb{Z}, v \in \mathbb{R}$, let $H_{k,v} = L^2(\mathbb{C})$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we define an operator $T_{k,v}(g)$ in $H_{k,v}$ as follows

$$T_{k,v}(g)f(z) = |bz+d|^{k+iv-2}(bz+d)^{-k} f\left(\frac{az+c}{bz+d}\right).$$

Let $\Theta_{k,v}$ be the character of $T_{k,v}$.

Every element $\gamma \in \Gamma_0$ is conjugate to a matrix

$$\begin{pmatrix} \lambda_\gamma & 0 \\ 0 & \lambda_\gamma^{-1} \end{pmatrix} = \begin{pmatrix} \exp(i\theta_\gamma + l_\gamma) & 0 \\ 0 & \exp(-i\theta_\gamma - l_\gamma) \end{pmatrix}$$

with $l_\gamma > 0$ and one has

(16)
$$\int_{G_\gamma\backslash G} \varphi_t^p(x^{-1}\gamma x) d\dot{x} = \frac{(2\pi)^{-2}}{|\lambda_\gamma - \lambda_\gamma^{-1}|^2} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_{k,v}(\varphi_t^p) e^{ik\theta_\gamma} e^{-ivl_\gamma} dv.$$

For $\gamma = I$ we have the Plancherel formula

$$\varphi_t^p(I) = \frac{1}{32\pi^4} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} (k^2+v^2)\Theta_{k,v}(\varphi_t^p) dv. \tag{17}$$

To give the action of $\mathfrak{g}$ on $H_{k,v}$, let $A = (-k-2+iv)/2$, $B = (k-2+iv)/2$, $X = \left(\begin{smallmatrix} u & w \\ y & -u \end{smallmatrix}\right)$ and $f \in H_{k,v}$, then

$$\begin{aligned} Xf(z) &= [A(wz-u) + B(\overline{wz-u})]f(z) \\ &\quad + \frac{\partial f}{\partial z}(2uz+y-wz^2) + \frac{\partial f}{\partial \bar z}(\overline{2uz+y-wz^2}). \end{aligned} \tag{18}$$

Consider the Casimir elements $\Omega_G = E_2^2 + E_3^2 + F_1^2 - E_1^2 - F_2^2 - F_3^2$, $\Omega_K = -E_1^2 - F_2^2 - F_3^2$. Note that $-4\sum_j X_j \overline{X}_j = -\Omega_G + 2\Omega_K$. From (18), $\Omega_G f = ((A^2+B^2) - 2(A+B))f = ((k^2-v^2)/2 - 2)f = \lambda_{k,v} f$.

The irreducible representations of K are given by (V_n, τ_n), $n \in \mathbb{N} \cup \{0\}$, where V_n is the space of homogeneous polynomials of degree n in two variables and $\tau_n(g)P((z_1,z_2)^t) = P(g^{-1}(z_1,z_2)^t)$.

For $X = \left(\begin{smallmatrix} u & w \\ -\overline{w} & -u \end{smallmatrix}\right)$ and $P_k = z_1^k z_2^{n-k}$ we have

$$XP_k = -kwP_{k-1} + (n-2k)uP_k + (n-k)\overline{w}P_{k+1}.$$

Then $2\Omega_K P = (n^2+2n)P$.

One easily finds that the multiplicity of τ_n in $T_{k,v}|K$ is

$$[\tau_n : T_{k,v}|K] = \begin{cases} 1 & \text{if } n-k \text{ is even and } n \geq |k| \\ 0 & \text{otherwise.} \end{cases}$$

When this multipicity is 1, $\Omega_G - 2\Omega_K|V_n$ is multiplication by $\mu_{k,v,n} = \lambda_{k,v} - n(n+2)$.

$$\omega|K = \sum_{n\in\mathbb{Z}^+} [\tau_n : T_\omega|K](\tau_n, V_n).$$

From (10) and (13), we have that for $q = 0, 3$

$$\bar{\Delta}^{p,q}|V_\omega = I^{p,q} \otimes \frac{1}{4}(-\Omega_G + 2\Omega_K)|V_\omega,$$

and thus

$$\Theta_{k,v}(\operatorname{Tr} h_t^{*p,q}) = \binom{3}{p} \sum_{\substack{n=|k| \\ n-k\,\text{even}}}^{\infty} (n+1)\exp(t\mu_{k,v,n}/4).$$

Let

$$D(k,n) = \begin{pmatrix} n-2(k+1) & 0 & 2(n-k) \\ 0 & -n+2(k-1) & 2k \\ k+1 & n-k-1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda = -2, n, -n-2$. Let $E(k,n,\lambda)$ be the λ-eigenspace of $D(k,n)$ and

$$V_n(\lambda) = \left\{ \sum_k \begin{pmatrix} (a_{k-1}+b_{k+1})P_k \\ (a_{k-1}-b_{k+1})P_k \\ c_k P_k \end{pmatrix} : \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \in E(k,n,\lambda) \right\},$$

$$W_n(\lambda) = \left\{ \sum_k \begin{pmatrix} -(a_{k-1}+b_{k+1})P_k \\ (a_{k-1}-b_{k+1})P_k \\ c_k P_k \end{pmatrix} : \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \in E(k,n,\lambda) \right\}.$$

Then

$$\wedge^{0,1}(\mathfrak{g}) \otimes V_n = V_n(-2) \oplus V_n(n) \oplus V_n(-n-2),$$
$$\wedge^{0,2}(\mathfrak{g}) \otimes V_n = W_n(-2) \oplus W_n(n) \oplus W_n(-n-2).$$

From (11) and (12) we see that

$$\bar{\Delta}^{p,1}| \wedge^{p,0}(\mathfrak{g}) \otimes V_n(\lambda) = I^{p,1} \otimes \frac{1}{4}(-\Omega_G + 2\Omega_K) + (\lambda+2)I|V_n(\lambda),$$

$$\bar{\Delta}^{p,2}| \wedge^{p,0}(\mathfrak{g}) \otimes W_n(\lambda) = I^{p,2} \otimes \frac{1}{4}(-\Omega_G + 2\Omega_K) + (\lambda+2)I|W_n(\lambda).$$

Thus, for $q = 1, 2$ we have

$$\Theta_{k,v}(\operatorname{Tr} h_t^{*p,q}) = \tbinom{3}{p} \sum_{\substack{n=|k| \\ n-k\,\text{even}}}^{\infty} (n+1)\exp(t\mu_{k,v,n}/4)(1 + e^{-t(n+2)} + e^{tn}).$$

Therefore, the characters are given by

$$\Theta_{k,v}(\varphi_t^p) = \tbinom{3}{p} \sum_{\substack{n=|k| \\ n-k \text{ even}}}^{\infty} (n+1)\exp(t\mu_{k,v,n}/4)(e^{-t(n+2)} + e^{tn} - 2) \tag{19}$$

$$= \tbinom{3}{p} e^{t(\lambda_{k,v}+1)/4} \sum_{\substack{n=|k| \\ n-k \text{ even}}}^{\infty} (n+1)(e^{-t(n+3)^2/4} + e^{-t(n-1)^2/4} - 2e^{-t(n+1)^2/4})$$

$$= \tbinom{3}{p} e^{-tv^2/8}((1-|k|)e^{-t(|k|+2)^2/8} + (1+|k|)e^{-t(|k|-2)^2/8}).$$

The function $\Phi_I^p : (0,\infty) \to \mathbb{R}$ defined by

$$\Phi_I^p(t) = \frac{2\binom{3}{p}}{(2\pi)^{7/2}} \sum_{|k|=1}^{\infty} (3+k)\exp(-tk^2/8)((k+2)^2 t^{-1/2} + 4t^{-3/2}))$$

is exponentially small at ∞. By Plancherel formula (17) we have

$$\varphi_t^p(I) = \Phi_I^p(t) + 24(2\pi)^{-7/2}\tbinom{3}{p}(t^{-1/2} + t^{-3/2}), \tag{20}$$

Defining $\Phi_\gamma^p : (0,\infty) \to \mathbb{R}$ by

$$\Phi_\gamma^p(t) = 4\tbinom{3}{p}\sqrt{\frac{\pi}{2t}}\exp(-2l_\gamma^2/t) \sum_{|k|=1}^{\infty} (3+k)\exp(-tk^2/8) 2\cos((k+2)\theta_\gamma,$$

we get

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \Theta_{k,v}(\varphi_t^p) e^{ik\theta_\gamma} e^{-ivl_\gamma} dv \tag{21}$$

$$= \Phi_\gamma^p(t) + 24\tbinom{3}{p}\cos(2\theta_\gamma)\sqrt{\frac{\pi}{2t}}\exp(-2l_\gamma^2/t).$$

5. Zeta functions

This section is concerned with the analyticity of the zeta functions that appear in THEOREM. Recall their definition given by (2), (3).

For each $\gamma \in \Gamma_0$ there are γ_o prime and $n_\gamma \in \mathbb{N}$, such that $\gamma = \gamma_o^{n_\gamma}$. $\Gamma_\gamma \setminus G_\gamma$ is conformally equivalent to $(l_{\gamma_o} + i\theta_{\gamma_o})\mathbb{Z} + 2\pi i\mathbb{Z} \setminus \mathbb{C}$ and so

$\mathrm{vol}\,(\Gamma_\gamma \setminus G_\gamma) = 2\pi l_\gamma/n_\gamma$. Let

$$\begin{aligned}
a_\gamma =& \frac{2\cos(2\theta_\gamma)}{|\lambda_\gamma - \lambda_\gamma^{-1}|^2} = \sum_{j,k=1}^{\infty} (e^{i2\theta_\gamma} + e^{-i2\theta_\gamma})\lambda_\gamma^{-2j+1}\bar{\lambda}_\gamma^{-2k+1},\\
b_\gamma =& \frac{2\lambda_\gamma^{-1}(3\cos(2\theta_\gamma)(1-\lambda_\gamma^{-1}) - i\sin(2\theta_\gamma))}{|\lambda_\gamma - \lambda_\gamma^{-1}|^2(1-\lambda_\gamma^{-1})^2},\\
A_\gamma =& \frac{\lambda_\gamma^{-3}(4-3\lambda_\gamma^{-1})}{(1-\lambda_\gamma^{-2})(1-\lambda_\gamma^{-1})^2} = \sum_{j,k=1}^{\infty} (3+k)\lambda_\gamma^{-2j-k},\\
B_\gamma =& \frac{|\lambda_\gamma|^{-2}\lambda_\gamma^{-1}}{|\lambda_\gamma - \lambda_\gamma^{-1}|^2(1-\lambda_\gamma^{-1})} = \sum_{j,k,r=1}^{\infty} \lambda_\gamma^{-2j-k+1}\bar{\lambda}_\gamma^{-2r+1},\\
C_\gamma =& \frac{\bar{\lambda}_\gamma^{-2}\lambda_\gamma^{-1}(3\lambda_\gamma^{-1}-2)}{(1-\bar{\lambda}_\gamma^{-2})(1-\lambda_\gamma^{-1})^2} = \sum_{j,k=1}^{\infty} (k-3)\bar{\lambda}_\gamma^{-2j}\lambda_\gamma^{-k}.
\end{aligned}$$

Then

$$b_\gamma = A_\gamma + 6B_\gamma - C_\gamma \quad \text{and} \tag{22}$$

$$b_\gamma + \bar{b}_\gamma = |\lambda_\gamma - \lambda_\gamma^{-1}|^{-2} \sum_{|k|=1}^{\infty} (3+k)e^{-|k|l_\gamma} 2\cos((k+2)\theta_\gamma).$$

For $\beta = a, A, B, C$ we have

$$Z_\beta(z) = \exp\bigl(-\sum_{[\gamma]\in[\Gamma_0]} \frac{1}{n_\gamma} \mathrm{Tr}\,\rho(\gamma)\beta_\gamma e^{-l_\gamma z}\bigr).$$

For $\Re z > 0$ and $\beta = a, b, A, B, C$ let

$$\eta_\beta(z) = \frac{Z'_\beta(z)}{Z_\beta(z)} = \sum_{[\gamma]\in[\Gamma_0]} \frac{\mathrm{Tr}\,\rho(\gamma)}{2\pi} \mathrm{vol}\,(\Gamma_\gamma \setminus G_\gamma)\beta_\gamma e^{-l_\gamma z}.$$

By the definition of Z_ρ and (22),

$$\frac{Z'_\rho}{Z_\rho} = \eta_A + 6\eta_B - \eta_C = \eta_b. \tag{23}$$

Let N_T be the number of $[\gamma]$ in $[\Gamma]$ with $l_\gamma \le T$. Margulis [7] has proved that $\lim_{T\to\infty} \log N_T/T = 2$.

For $\beta = A, B, C$, $\lim_{l_\gamma\to\infty} \log|\beta_\gamma|/l_\gamma = -3$ and so the series $\log Z_\beta(z)$ converges uniformly in each set $\Re z \ge -1+\delta$, $\delta > 0$. By (23)

$$\int_0^\infty \eta_b(x)dx = -\log Z_\rho(0).$$

Since $\lim_{l_\gamma\to\infty} \log|a_\gamma|/\, l_\gamma = -2$, the series $\eta_a(z)$ converges uniformly in each set $\Re z \geq \delta > 0$. Scott [11] has shown that $\eta_a(z)$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles at zero and the points $\pm iv \neq 0$ such that $n_\rho(T_{2,v}) \neq 0$, having residue $n_\rho(T_{2,v}) + n_\rho(T_{2,-v})$ at these points, and satisfying the functional equation

$$\eta_a(iz) + \eta_a(-iz) + p(z) = 0 \tag{24}$$

where $p(r) = m\,\mathrm{vol}\,(M)(4 + r^2)/(2\pi)^3$. Let r_ρ be the residue of η_a at zero and define $\psi_\rho(z) = \eta_a(z) - r_\rho/z$. Thus, for any $c \in \mathbb{C}^*$

$$\zeta_\rho(z) = c \exp \int_0^z \psi_\rho(s) ds$$

is a well defined entire function whose only zeros ocurr at the points $z = \pm iv \neq 0$ such that $n_\rho(T_{2,v}) \neq 0$, and are of order $n_\rho(T_{2,v}) + n_\rho(T_{2,-v})$.

6. Proof of formula (6)

Let

$$\begin{aligned}
F(t) =& 24(2\pi)^{-7/2}\tbinom{3}{p} m\mathrm{vol}\,(M)(t^{-1/2} + t^{-3/2}) \\
g(t) =& \tbinom{3}{p} \sum_{[\gamma]\in[\Gamma_0]} \frac{6\cos(2\theta_\gamma)\mathrm{Tr}\,\rho(\gamma)\mathrm{vol}\,(\Gamma_\gamma \setminus G_\gamma)}{\pi^2|\lambda_\gamma - \lambda_\gamma^{-1}|^2} \sqrt{\frac{\pi}{2t}} \exp(-2l_\gamma^2/t) \\
G(t) =& \sum_{[\gamma]\in[\Gamma_0]} \frac{\mathrm{Tr}\,\rho(\gamma)\mathrm{vol}\,(\Gamma_\gamma \setminus G_\gamma)}{4\pi^2|\lambda_\gamma - \lambda_\gamma^{-1}|^2} \Phi_\gamma^p(t).
\end{aligned}$$

Then g is exponentially small at 0^+, and so is G by Poisson summation formula. By (15), (16), (20), (21) we have

$$H(t) - m\,\mathrm{vol}\,(M)\Phi_I^p(t) - \chi_{(0,1]}F(t) = \chi_{(1,\infty)}F(t) + g(t) + G(t)$$

the left hand side is exponentially small at ∞ while the right hand side is exponentially small at 0^+. We define an entire function $h(s)$ by taking the Mellin transform of both sides. For $\Re s > 3/2$, $(t^{-1/2} + t^{-3/2})\chi_{(0,1]}$ has Mellin transform $(s - \frac{1}{2})^{-1} + (s - \frac{3}{2})^{-1}$. For $\Re s < 1/2$, $(t^{-1/2} + t^{-3/2})\chi_{(1,\infty)}$ has Mellin transform $-(s - \frac{1}{2})^{-1} - (s - \frac{3}{2})^{-1}$. Both have meromorphic continuation to $\mathbb{C}$. Therefore

$$h(s) + 24(2\pi)^{-7}\tbinom{3}{p} m\,\mathrm{vol}\,(M) \left(\frac{1}{s - 1/2} + \frac{1}{s - 3/2}\right)$$

gives a meromorphic continuation of both $MH(s) - m\,\mathrm{vol}\,(M)M\Phi_I^p(s)$ and $Mg(s) + MG(s)$. Thus

$$MH(s) = m\,\mathrm{vol}\,(M)M\Phi_I^p(s) + Mg(s) + MG(s). \tag{25}$$

We have

$$M\Phi_I^p(s) = \binom{3}{p} 8^{s-1}\pi^{2s-5}(\frac{7s-9}{\pi^2}\Gamma(2-s)\zeta(4-2s)+12\Gamma(1-s)\zeta(2-2s)),$$

$$M\Phi_I^p(0) = \binom{3}{p} 19/80\pi^3. \tag{26}$$

For $\Re s < 0$ we use the substitution

$$t^{s-1} = \frac{8^{s-1}}{\Gamma(1-s)} \int_0^\infty (x(x+2|k|))^{-s} e^{-x(x+2|k|)t/8}(2x+2|k|)dx, \tag{27}$$

switch the order of integration in the Mellin transform, integrate term by term using

$$\int_0^\infty e^{-(x+|k|)^2 t/8}(2x+2|k|)\frac{1}{\sqrt{2\pi t}}e^{-2l_\gamma^2/t}dt = 4e^{-l_\gamma(x+|k|)},$$

and let $s \uparrow 0$ to get

$$MG(0) = \binom{3}{p} \int_0^\infty (\eta_b(x) + \overline{\eta_b(x)})dx = -\binom{3}{p} \log |Z_\rho(0)|^2. \tag{28}$$

For $g(t)$ we have

$$Mg(s) = 3\binom{3}{p} \frac{8^s}{\Gamma(1-s)} \int_0^\infty x^{-2s}\eta_a(x)dx.$$

Since

$$\int_0^\infty x^{-2s}\eta_a(x)dx = \int_0^1 x^{-2s}\psi_\rho(x)dx + \int_0^1 r_\rho x^{-2s-1}dx + \int_1^\infty x^{-2s}\eta_a(x)dx,$$

we have

$$\lim_{s\uparrow 0}\left(\int_0^\infty x^{-2s}\eta_a(x)dx + \frac{r_\rho}{2s}\right) = \log\left(\frac{\zeta_\rho(1)}{\zeta_\rho(0)}\right) - \log Z_a(1) = -\log\zeta_\rho(0).$$

Thus

$$\begin{aligned} \frac{d}{ds}\left(\frac{Mg(s)}{\Gamma(s)}\right)_{s=0} &= -3\binom{3}{p}\left(\log\zeta_\rho(0) + \frac{d}{ds}\left(\frac{8^s r_\rho \sin\pi s}{2\pi s}\right)_{s=0}\right) \\ &= -3\binom{3}{p}\left(\log\zeta_\rho(0) + \frac{r_\rho}{2}\log 8\right). \end{aligned} \tag{29}$$

From the definition (4), equations (25), (26), (28) and (29) give (6).

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