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LIMIT OF THE INFINITE HORIZON DISCOUNTED HAMILTON-JACOBI EQUATION

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Abstract. Motivated by the infinite horizon discounted problem, we study the convergence of solutions of the Hamilton Jacobi equation when the discount vanishes. If the Aubry set consists in a finite number of hyperbolic critical points, we give an explicit expression for the limit. Additionaly, we give a new characterization of Mañé's critical value as for wich the set of viscosity solutions is equibounded.

1. **Introduction.** Let $\mathbb{T}^d$ be the d-dimensional torus. A Lagrangian is a C^3 real function on $\mathbb{T}^d \times \mathbb{R}^d$ that is convex and superlinear in the fibers. The associated Hamiltonian is the function $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$H(x,p) = \max_v p.v - L(x,v),$$

which is also convex and superlinear in the fibers. We further assume that there are constants $A, B > 0$ such that

$$|pH_p(x,p)| \leq AH(x,p) + B \quad \forall (x,p).$$

We address the problem of convergence of viscosity solutions of the infinite horizon discounted Hamilton-Jacobi equation

$$\lambda u + H(x, Du) = c \tag{1}$$

as λ tends to zero. It is known that under some assumptions, for positive λ there is a unique solution u_λ of (1), see books [2, 3]. D. Gomes [12] has proved this result in our setting. For fixed c we prove in Lemma 2.4 that the family $\{u_\lambda\}$ es equiLipschitz.

It is known that limits of solutions of (1) are viscosity solutions of

$$H(x, Du) = c \tag{2}$$

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It is also known that there is only one value $c = c(L)$, the so called critical value, such that (2) has a viscosity solution. So it is natural to consider equation (1) only when c is the critical value. In section 3 we use the existence of a strict critical subsolution of the Hamilton Jacobi equation (see [11, 4]), to give another characterization of the critical value: The family $\{u_\lambda\}$ of solutions of (1) is equibounded if and only if $c = c(L)$. Thus, for $c = c(L)$ there are always sequences u_{λ_n} converging to a viscosity solution of (2).

If there is only one viscosity solution of (2), the complete convergence of the sequence is trivial. The number of solutions has to do with the set of equivalence classes of the Aubry set. See for example [7], [5] or the brief review in the next section. Our main result establishes the convergence of the solutions of (1) to a special solution of (2) under an assumption on the Aubry set.

Main Theorem. *Assume that the Aubry set consists in a finite collection of hyperbolic critical points of the Euler-Lagrange flow* $(x_k, 0)$ $k = 1, \dots N$*. Then the uniform limit* $u = \lim_{\lambda\to 0} u_\lambda$ *exists and*

$$u(x) = \min_k h(x_k, x) \tag{3}$$

where h *is the Peierls barrier.*

In [1] another selection criteria is provided, in that case the Hamilton Jacobi equation is perturbed by adding an elliptic term. Under the same hypothesis on the Aubry set and the possibility of choosing one critical point, according a criterion involving the Lyapunov exponents, the convergence of the sequence is proved. In general the selection criteria for this two methods do not agree.

In section 2 we give some preliminaries about properties of the solution of (1). For convenience of the reader we also provide the definitions of the Peierls barrier, Aubry set, and state some known results. In section 3 we give the characterization of the critical value. In section 4 we prove Main Theorem.

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2. Preliminaries.

2.1. Aubry Mather theory.

The constant c in equation (2) can be characterized as $\alpha(0)$ where α is Mather's function (see [14], [13]):

$$c = \alpha(0) = -\inf_\nu\{\int_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)d\nu(x,v)\},$$

where the inf is taken over the set of probability measures ν on $\mathbb{T}^d \times \mathbb{R}^d$ which are invariant under the Euler-Lagrange flow of L.

We recall the definition of Peierls barrier ([8]) $h : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$. Define the action of a piecewise C^1 curve $\gamma : [0, T] \to M$ as

$$A(\gamma) = \int_0^T L(\gamma(s), \dot\gamma(s))ds.$$

Given a constant $k \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{T}^d$ let

$$h^k_T(x_1, x_2) = \inf\{A(\gamma) + kT | \gamma : [0, T] \to \mathbb{T}^d \text{ joins } x_1 \text{ and } x_2\},$$

and

$$\begin{aligned} h^k(x_1, x_2) &= \liminf_{T\to\infty} h^k_T(x_1, x_2) \\ \Phi^k(x_1, x_2) &= \inf_T h^k_T(x_1, x_2). \end{aligned}$$

Since time T is not bounded, there is only one possible value of k that will make the function h^k different from being identically $-\infty$ or ∞, this is again $c = c(L)$. We define $h_T = h^c_T$ and the Peierls Barrier $h = h^c$. Mañé's action potential Φ^k is identically $-\infty$ for $k < c(L)$ and finite for $k \geq c(L)$. We will also define $\Phi = \Phi^c$. In [10], it is shown that h_T actually converges uniformly to h. Recall that given a fixed $y \in \mathbb{T}^d$, the function $x \mapsto h(y, x)$ is a viscosity solution of (2).

Set $c = c(L)$. A piecewise C^1 curve $\gamma : [a, b] \to \mathbb{T}^d$ is called *semistatic* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = \Phi(\gamma(t_0), \gamma(t_1)), \tag{4}$$

for all $a < t_0 \leq t_1 < b$; and it is called *static* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = -\Phi(\gamma(t_1), \gamma(t_0)) \tag{5}$$

for all $a < t_0 \leq t_1 < b$. Since $\forall k \geq c(L)$

$$\Phi^k(x_1, x_2) + \Phi^k(x_2, x_1) \geq \Phi^k(x_1, x_1) \geq 0,$$

static curves are semistatic.

We now define as in [9] the Aubry set $\mathcal{A} \subset \mathbb{T}^d$:

$$\mathcal{A} = \{x \in \mathbb{T}^d, h(x, x) = 0\}.$$

(in the reference [9] it was called the Peierls set.)

In close relation to Mather's graph theorem ([14]), it is shown in [9], that the set $\mathcal{A}$ can be lifted, in a unique way, to a homeomorphic invariant set $\tilde{\mathcal{A}} \subset \mathbb{T}^d \times \mathbb{R}^d$.

In [7] it is proven that a curve is static if and only if it is part of the projection of a curve in $\tilde{\mathcal{A}}$. It can also be shown that alpha and omega limits of lifted semistatic curves belong to $\tilde{\mathcal{A}}$.

If u is a viscosity solution of (2), for an arbitrary curve (x_t) we have

$$u(x_T) - u(x_0) \leq \int_0^T L(x_t, \dot{x}_t)dt + cT$$

for all T, see [7]. A crucial property with respect to the Aubry set is the following: if (x_0, v_0) is an element of $\tilde{\mathcal{A}}$, and $(x_t, v_t)_{t\in\mathbb{R}}$ is its orbit under the Euler-Lagrange flow, then

$$u(x_T) - u(x_0) = \int_0^T L(x_t, v_t)dt + cT. \tag{6}$$

The "static classes" form a partition of $\mathcal{A}$, defined by the equivalence relation on $\mathcal{A}$: $x \sim y$ if and only if

$$h(x, y) + h(y, x) = 0.$$

In this work we assume that the Aubry set $\tilde{\mathcal{A}}$ is made up of a finite number of hyperbolic critical points of the Euler-Lagrange flow. This implies in particular that each static class is a critical point. In particular we will use the following fact, the no loop condition: for every collection $x_{i_1}, x_{i_2}...x_{i_s} = x_{i_1}$ of critical points the action of the cycle, $\sum h(x_{i_j}, x_{i_{j+1}})$, is positive .

Viscosity solutions are completely determined giving one value in one point for each static class, as shown in [5]:

Denote the static classes $S_i, 1 \le i \le m$ and choose one point x_i in each static class. For each $i \in [1, m]$, assign a value $\phi_i \in \mathbb{R}$. Because of the general properties recalled above, if there exists a viscosity solution $\phi : \mathbb{T}^d \to \mathbb{R}$ of (2) such that $\phi(x_i) = \phi_i$ for all $i \in [1, m]$, we must have $\phi_j - \phi_i \le h(x_i, x_j)$ for all i, j. Conversely, if this necessary condition is satisfied, then there is a unique viscosity solution ϕ of the Hamilton-Jacobi equation having these prescribed values. In fact it is given by (see [5])

$$\phi(x) = \min_i \phi_i + h(x_i, x). \tag{7}$$

2.2. **The value function.** Solutions of (1) are given by the value function

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda t}(L(\gamma, \dot\gamma) + c)dt \tag{8}$$

where the infimum is taken over the piecewise C^1 curves $\gamma :]-\infty, 0] \to \mathbb{T}^d$, with $\gamma(0) = x$.

In Optimal Control, see books [2, 3], it is shown that under assumptions such as the Lagrangian is bounded, the value function has some properties, for instance that it is a viscosity solution of (1). We did not find in the literature the crucial property for us that for any $x \in \mathbb{T}^d$ there is a curve γ where the infimum in (8) is achieved. A main ingredient for the proof that we provide is the fact that minimizers of the Lagrangian $e^{\lambda t}L(x, v)$ have bounded velocities.

Theorem 2.1. Dynamic Programming Principle. *The value function u_λ satisfies*

$$u_\lambda(x) = \inf\{u_\lambda(\gamma(-T))e^{-\lambda T} + \int_{-T}^{0} e^{\lambda s}(L(\gamma(s), \dot\gamma(s)) + c)ds : \\ \gamma : [-T, 0] \to \mathbb{T}^d \text{ piecewise } C^1, \gamma(0) = x\}. \tag{9}$$

The proof is standard, see [2, 3].

We note that the Euler-Lagrange equation for the time dependent Lagrangian $e^{\lambda t}(L(x, v) + c)$ is

$$\frac{d}{dt}L_v(x, x') + \lambda L_v(x, x') = L_x(x, x'). \tag{10}$$

Using the Legendre transformation $(x, v) \mapsto L_v(x, v)$ and the Hamiltonian $H(x, p)$ associated to L, equation (10) is transformed to the system

$$\begin{aligned} x' &= H_p(x, p) \\ p' &= -H_x(x, p) - \lambda p. \end{aligned} \tag{11}$$

Thus, along solutions $(x(t), p(t))$ of (11) we have $\frac{d}{dt}H = -\lambda p H_p$. Since

$$|pH_p(x, p)| \le AH(x, p) + B \quad \forall (x, p)$$

we have that along solutions of (11), H is bounded on finite intervals, and so is $p(t)$ by the superlinearity of H. Therefore solutions are defined for all t. Let ϕ_t^λ be the corresponding flow for (10).

Definition 2.2. Given $x, y \in \mathbb{T}^d$ and $a < b$ we will call λ - minimizer a minimizer of $\int_a^b e^{\lambda t}(L(\gamma(t), \dot\gamma(t)) + c)dt$ among all curves $\gamma : [a, b] \to \mathbb{T}^d$ such that $\gamma(a) = x, \gamma(b) = y$.

Lemma 2.3 and Proposition 1 below, as well as their proofs, are adaptations of the corresponding results in [7] for $\lambda = 0$, which can be recovered letting λ go to 0.

Lemma 2.3. *Let $T > 0$, there exists a constant C_T such that for each $a \in \mathbb{R}$ and $x, y \in \mathbb{T}^d$ there is a C^∞ curve $\gamma : [a, a+T] \to \mathbb{T}^d$ such that $\gamma(a) = x, \gamma(a+T) = y$ and for all $\lambda > 0$*

$$\int_a^{a+T} e^{\lambda t}(L(\gamma(t), \dot\gamma(t)) + c)dt \le C_T e^{\lambda a}\frac{e^{\lambda T} - 1}{\lambda}. \tag{12}$$

Proof. Let $\gamma : [a, a+T] \to \mathbb{T}^d$ be a geodesic between x and y with length $d(x, y)$. Thus $\|\dot\gamma(t)\| = \dfrac{d(x,y)}{T} \le \dfrac{\sqrt{d}}{T}$ for any $t \in [a, a+T]$. Letting $C_T = \max\{L(x, v) + c : (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \|v\| \le \dfrac{\sqrt{d}}{T}\}$ we get (12). □

Proposition 1. *For $T > 0$ there exists a compact set $K_T \subset \mathbb{T}^d \times \mathbb{R}^d$ such that for every $0 < \lambda \le 1$ and any λ - minimizer $\gamma : [-S, 0] \to \mathbb{T}^d$ with $S \ge T$ we have that*

$$\forall t \in [-S, 0], \quad (\gamma(t), \dot\gamma(t)) \in K_T$$

Proof. For any $a \in [-S, -T]$ and any λ-minimizer $\gamma : [-S, 0] \to \mathbb{T}^d$ we have

$$\int_a^{a+T} e^{\lambda t}(L(\gamma(t), \dot\gamma(t)) + c)dt \le C_T e^{\lambda a}\frac{e^{\lambda T} - 1}{\lambda}$$

Since $t \mapsto (L(\gamma(t), \dot\gamma(t)) + c)$ is continuous and $e^{\lambda t} > 0$, by the mean value theorem for integrals, there is $t_0 \in [a, a+T]$ such that

$$\int_a^{a+T} e^{\lambda t}(L(\gamma(t), \dot\gamma(t)) + c)dt = (L(\gamma(t_0), \dot\gamma(t_0)) + c)\int_a^{a+T} e^{\lambda t}dt$$

and so

$$L(\gamma(t_0), \dot\gamma(t_0)) + c \le C_T.$$

The set $D_T = \{(x, v) \in \mathbb{T}^d \times \mathbb{R}^d : L(x, v) + c \le C_T\}$ is compact. By continuity of $\phi_t^\lambda(x, v)$ in all its variables, the set

$$K_T = \bigcup\{\phi_s^\lambda(D_T) : |s| \le T, \lambda \in [0, 1]\}$$

is compact. Thus, for any λ -minimizer $\gamma : [-S, 0] \to \mathbb{T}^d$ and any $t \in [a, a+T]$,

$$(\gamma(t), \dot\gamma(t)) \in \phi_{t-t_0}^\lambda(B) \in K_T$$

□

Lemma 2.4. *The family of value functions $u_\lambda : \mathbb{T}^d \to \mathbb{R}, \lambda \in [0, 1]$ is equiLipschitz.*

Proof. Let $x, y \in \mathbb{T}^d$, $|x - y| < \sqrt{d}$. Let K^* be the $\sqrt{d}$ neighbourhood of K_1 and

$$M = \max\{|L_x(x, v)| + |L_v(x, v)| : (x, v) \in K^*\}.$$

Given $\varepsilon > 0$ let $\gamma :]-\infty, 0] \to \mathbb{T}^d$ with $\gamma(0) = x$ be such that

$$\int_{-\infty}^0 e^{\lambda s}(L(\gamma(s), \dot\gamma(s)) + c)ds < u_\lambda(x) + \varepsilon.$$

We may assume that $\gamma|_{[-1,0]}$ is a λ -minimizer and then $(\gamma(t),\dot\gamma(t))\in K_1$ for $t\in[-1,0]$. Let $\alpha(t)=\gamma(t)+(t+1)(y-x)$ for $t\in[-1,0]$, then

$$\begin{aligned} u_\lambda(y)-u_\lambda(x) &< \int_{-1}^{0} e^{\lambda s}(L(\alpha(s),\alpha'(s))-L(\gamma(s),\dot\gamma(s))ds+\varepsilon \\ &\le \int_{-1}^{0} e^{\lambda s}M|x-y|+\varepsilon=\frac{M(1-e^{-\lambda})}{\lambda}|x-y|+\varepsilon. \end{aligned} \tag{13}$$

Since ε is arbitrary

$$u_\lambda(y)-u_\lambda(x)\le \frac{M(1-e^{-\lambda})}{\lambda}|x-y|\le M|x-y|.$$

□

Lemma 2.5. *Let $T>0$, $x\in\mathbb{T}^d$ be given and $u_\lambda:\mathbb{T}^d\to\mathbb{R}$ be the value function. There exists a curve $\gamma:[-T,0]\to\mathbb{T}^d$ such that $\gamma(0)=x$ and*

$$u_\lambda(x)=u_\lambda(\gamma(-T))e^{-\lambda T}+\int_{-T}^{0} e^{\lambda s}(L(\gamma(s),\dot\gamma(s))+c)ds. \tag{14}$$

The curve γ is a λ - minimizer. In particular, we have

$$\forall s\in[-T,0],(\gamma(s),\dot\gamma(s))\in K_T,$$

where $K_T\subset\mathbb{T}^d\times\mathbb{R}^d$ is the compact set given by Proposition 1

The proof is standard, see [2, 3].

Lemma 2.6. *For $T>0$, $x\in\mathbb{T}^d$ given and $u_\lambda:\mathbb{T}^d\to\mathbb{R}$ the value function, let $\gamma:[-T,0]\to\mathbb{T}^d$ be the curve with $\gamma(0)=x$ satisfying (14). Then, for all $t\in[0,T]$*

$$u_\lambda(x)=u_\lambda(\gamma(-t))e^{-\lambda t}+\int_{-t}^{0} e^{\lambda s}(L(\gamma(s),\dot\gamma(s))+c)ds \tag{15}$$

Proof. By the dynamic programming principle, for all $t\in[0,T]$

$$u_\lambda(x)-u_\lambda(\gamma(-t))e^{-\lambda t}\le\int_{-t}^{0} e^{\lambda s}(L(\gamma(s),\dot\gamma(s))+c)ds \tag{16}$$

$$u_\lambda(\gamma(-t))e^{-\lambda t}-u_\lambda(\gamma(-T))e^{-\lambda T}\le\int_{-T}^{-t} e^{\lambda s}(L(\gamma(s),\dot\gamma(s))+c)ds. \tag{17}$$

Adding (16) and (17)

$$u_\lambda(x)-u_\lambda(\gamma(-T))e^{-\lambda T}\le\int_{-T}^{0} e^{\lambda s}(L(\gamma(s),\dot\gamma(s))+c)ds,$$

which by (14) is an equality. It follows that inequalities (16),(17) are in fact equalities. □

The following Proposition and its proof are standard, see [2, 3].

Proposition 2. *The value function is a viscosity solution of (1).*

Proposition 3. *For $x \in \mathbb{T}^d$ given and $u_\lambda : \mathbb{T}^d \to \mathbb{R}$ the value function, there exists $\gamma :]-\infty, 0] \to \mathbb{T}^d$ such that $\gamma(0) = x$ and for any $t \geq 0$*

$$u_\lambda(x) = u_\lambda(\gamma(-t))e^{-\lambda t} + \int_{-t}^{0} e^{\lambda s}(L(\gamma(s), \dot{\gamma}(s)) + c)ds. \tag{18}$$

Proof. For $T > 1$ and $x \in \mathbb{T}^d$ given, by Lemma 2.5 there exists an extremal curve $\gamma_T : [-T, 0] \to \mathbb{T}^d$ with $\gamma_T(0) = x$ and satisfying (14). By Lemma 2.6, for any $t \in [0, T]$

$$u_\lambda(x) = u_\lambda(\gamma_T(-t))e^{-\lambda t} + \int_{-t}^{0} e^{\lambda s}(L(\gamma_T(s), \gamma_T'(s)) + c)ds. \tag{19}$$

We know that

$$\forall s \in [-T, 0], (\gamma(s), \dot{\gamma}(s)) \in K_1,$$

where $K_1 \subset \mathbb{T}^d \times \mathbb{R}^d$ is the compact set given by Proposition 1. Thus, we can find a sequence $T_n \nearrow \infty$ such that $(\gamma_{T_n}(0), \gamma'_{T_n}(0)) = (x, \gamma'_{T_n}(0))$ converges to (x, v_∞). The negative orbit $\phi_s^\lambda(x, v_\infty)$ is of the form $(\gamma(s), \dot{\gamma}(s))$. If $t \geq 0$, for n large enough, $\phi_s^\lambda(x, \gamma_{T_n}(0)) = (\gamma_{T_n}(s), \gamma'_{T_n}(s))$ is defined for $s \in [-t, 0]$ and by the continuity of the flow ϕ_s^λ, this sequence converges uniformly on the compact interval $[-t, 0]$ to $(\gamma, \dot{\gamma})$. Passing to the limit in (19) we get (18). □

Corollary 1. *For $x \in \mathbb{T}^d$ and $u_\lambda : \mathbb{T}^d \to \mathbb{R}$ the value function, let $\gamma :]-\infty, 0] \to \mathbb{T}^d$ be such that $\gamma(0) = x$ and for any $t \geq 0$ satisfies (18), then*

$$u_\lambda(x) = \int_{-\infty}^{0} e^{\lambda s}(L(\gamma(s), \dot{\gamma}(s)) + c)ds \tag{20}$$

$$u_\lambda(\gamma(-t)) = \int_{-\infty}^{0} e^{\lambda s}(L(\gamma(s-t), \dot{\gamma}(s-t)) + c)ds \quad \forall t > 0 \tag{21}$$

Proof. Taking limit as $t \to \infty$ in (18) we get (20). Thus

$$u_\lambda(\gamma(-t))e^{-\lambda t} = \int_{-\infty}^{-t} e^{\lambda s}(L(\gamma(s), \dot{\gamma}(s)) + c)ds$$

for any $t > 0$, and changing variable s by $s - t$ we get (21). □

Corollary 2. *Let $u_\lambda : \mathbb{T}^d \to \mathbb{R}$ be the value function. Suppose that $u = \lim_{k\to\infty} u_{\lambda^k}$ for a sequence $\lambda^k \to 0$. Let $\gamma_\lambda :]-\infty, 0] \to \mathbb{T}^d$ be a curve such that for any $t \geq 0$*

$$u_\lambda(\gamma_\lambda(0)) = u_\lambda(\gamma(-t))e^{-\lambda t} + \int_{-t}^{0} e^{\lambda s}(L(\gamma_\lambda(s), \gamma_\lambda'(s)) + c)ds.$$

Then there is a subsequence $\lambda_n \to 0$ and a curve $\gamma :]-\infty, 0] \to \mathbb{T}^d$ such for any $t > 0$, $\gamma_{\lambda_n}|_{[-t,0]}$ converges uniformly to $\gamma|_{[-t,0]}$ and

$$u(x) = u(\gamma(-t)) + \int_{-t}^{0} (L(\gamma(s), \dot{\gamma}(s)) + c)ds = u(\gamma(-t)) + \Phi(\gamma(-t), x). \tag{22}$$

where $x = \lim_{n\to\infty} \gamma_{\lambda_n}(0)$, *and* Φ *is the Mañé's potential.*

Proof. By Proposition 1 there is a compact set K_1 such that $\dot\gamma_\lambda(0) \in K_1$ for all $0 < \lambda < 1$. Therefore there is a subsequence $\lambda_n \to 0$ such that $(\gamma_{\lambda_n}(0), \gamma'_{\lambda_n}(0))$ converges to $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$. The negative orbit $\phi_s(x, v)$ is of the form $(\gamma(s), \dot\gamma(s))$. If $t > 0$, $\gamma_{\lambda_n}|_{[-t,0]}$ converges uniformly to $\gamma|_{[-t,0]}$ and taking limit as $\lambda_n \to 0$ in (18) we obtain (22). □

3. Solutions of the Hamilton Jacobi equation.

3. Solutions of the Hamilton Jacobi equation. We now give a new characterization of $c(L)$

Proposition 4. *For fixed c, the family* $\{u_\lambda : \lambda > 0\}$ *of solutions of* (1) *is uniformly bounded if and only if* $c = c(L)$.

Proof. If u_λ is the family of solutions of (1) then $v_\lambda = u_\lambda + \dfrac{d-c}{\lambda}$ is the family of solutions of

$$\lambda v + H(x, Dv) = d.$$

Hence, there is at most one value of c such that the family of solutions of (1) is uniformly bounded.

On the other hand according to an extension of [11] in [4], for $c = c(L)$ there is $f : \mathbb{T}^d \to \mathbb{R}$ a $C^{1,1}$ strict critical subsolution of (2) so that for any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$

$$L(x, v) + c(L) - df(x)v \geq 0. \tag{23}$$

For $\gamma :]-\infty, 0] \to \mathbb{T}^d$, integration by parts along $(\gamma, \dot\gamma)$ gives

$$\int_{-\infty}^{0} e^{\lambda t} df(\gamma)\dot\gamma \, dt = e^{\lambda t} f \circ \gamma(t)|_{-\infty}^{0} + \int_{-\infty}^{0} e^{\lambda t} \lambda f \circ \gamma \, dt$$

$$\left| \int_{-\infty}^{0} e^{\lambda t} df(\gamma)\dot\gamma \, dt \right| \leq \|f\| + \int_{-\infty}^{0} e^{\lambda t} \lambda \|f\| = 2\|f\|.$$

From (23)

$$\int_{-\infty}^{0} e^{\lambda t}(L(\gamma, \dot\gamma) + c(L)) \, dt \geq \int_{-\infty}^{0} e^{\lambda t} df(\gamma)\dot\gamma \, dt \geq -2\|f\|.$$

For $x \in \mathbb{T}^d$ we then have

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda t}(L(\gamma, \dot\gamma) + c(L)) \, dt \geq -2\|f\|.$$

Moreover, since $h(x, \cdot)$ is a viscosity solution of (2) there is a calibrating curve $\gamma :]-\infty, 0] \to \mathbb{T}^d$ with $\gamma(0) = x$. Choose a sequence $s_n \to -\infty$ such that $x_i =$

$\lim_{n\to\infty} \gamma(s_n)$ exists. Then

$$\begin{aligned}
\int_{-\infty}^{0} e^{\lambda t}(L(\gamma,\dot\gamma)+c(L))\,dt &= \int_{-\infty}^{0} e^{\lambda t}(L(\gamma,\dot\gamma)+c(L)-df(\gamma)\dot\gamma)\,dt + \int_{-\infty}^{0} e^{\lambda t} df(\gamma)\dot\gamma \\
&\le \lim_{n\to\infty}\int_{s_n}^{0}(L(\gamma,\dot\gamma)+c(L)-df(\gamma)\dot\gamma)\,dt + 2\|f\| \\
&= \lim_{n\to\infty} -h(x,\gamma(s_n)) + f(\gamma(s_n)) - f(x) + 2\|f\| \\
&= -h(x,x_i) + f(x_i) - f(x) + 2\|f\|
\end{aligned}$$

Thus

$$u_\lambda(x) \le -h(x,x_i) + 4\|f\|$$

□

Therefore, for $c = c(L)$ the family $\{u_\lambda : \lambda > 0\}$ of solutions of (1) has convergent subsequences. From now on we will assume that $c(L) = 0$.

4. **The limit solution.** We assume the Aubry set consists in a finite collection of hyperbolic critical points $(x_i, 0)$ $i = 1, \dots N$.

Proposition 5. *Suppose that* $u = \lim_{r\to\infty} u_{\lambda_r}$ *for a sequence* $\lambda_r \to 0$. *Then*

$$u(x) = \min_k h(x_k, x).$$

Fix the sequence λ_r such that $u = \lim_{r\to\infty} u_{\lambda_r}$. From the definition of the value function we have

$$u_{\lambda_r}(x_k) \le \int_{-\infty}^{0} e^{\lambda_r t} L(\varphi_t(x_k,0))dt = 0.$$

Letting $r\to\infty$ we have $u(x_k) \le 0$. Since $u(x) = \min_k u(x_k) + h(x_k, x)$ by (7), we have

$$u(x) \le \min_k h(x_k, x) \tag{24}$$

To prove the opposite inequality we observe that it is enough to do it for the points in the Aubry set. Indeed, assuming that it holds for the points in the Aubry set, we have for $x \in \mathbb{T}^d$

$$u(x) = \min_j u(x_j) + h(x_j, x) \ge \min_{i,j} h(x_i, x_j) + h(x_j, x) \ge \min_i h(x_i, x).$$

We define a directed graph G_u that depends on the solution u with vertices at the points $x_1, x_2, ...x_N$ of the Aubry set, and with a directed segment from x_i to x_k if and only if

$$u(x_k) = u(x_i) + h(x_i, x_k).$$

We call a point x_k a root of the graph if there is no segment arriving to this point. Since in general

$$u(x_k) - u(x_i) \le h(x_i, x_k),$$

that x_k is a root means that for all $i \ne k$

$$u(x_k) - u(x_i) < h(x_i, x_k).$$

Proposition 6. *The graph G_u contains no cycles.*

Suppose there is a cycle $x_{i_1}, \cdots, x_{i_s} = x_{i_1}$ with

$$u(x_{i_{j+1}}) - u(x_{i_j}) = h(x_{i_j}, x_{i_{j+1}}).$$

Then

$$0 = \sum u(x_{i_{j+1}}) - u(x_{i_j}) = \sum h(x_{i_j}, x_{i_{j+1}})$$

This is a contradiction with the no loop condition we mentioned in the Preliminaries.

Proposition 7. *For any $j = 1, \ldots, N$*

$$u(x_j) \geq \min_i h(x_i, x_j) \tag{25}$$

Claim 1. It is enough to prove inequality (25) for roots

In fact, if x_{j_0} is not a root then there exists an x_{j_1} such that

$$u(x_{j_0}) = u(x_{j_1}) + h(x_{j_1}, x_{j_0}).$$

If x_{j_1} is a root we have

$$u(x_{j_0}) \geq \min_i h(x_i, x_{j_1}) + h(x_{j_1}, x_{j_0}) \geq \min_i h(x_i, x_{j_0}).$$

If x_{j_1} is not a root then there exists an x_{j_2} such that

$$u(x_{j_1}) = u(x_{j_2}) + h(x_{j_1}, x_{j_2}).$$

Continuing this process we find $x_{j_1}, \ldots, x_{j_s}$ with x_{j_s} a root, since the number of points x_i is finite and there are no cycles in the graph.

We obtain

$$\begin{aligned} u(x_{j_0}) &= u(x_{j_s}) + h(x_{j_s}, x_{j_{s-1}}) + \ldots + h(x_{j_1}, x_{j_0}) \\ &\geq \min_i h(x_i, x_{j_s}) + h(x_{j_s}, x_{j_{s-1}}) + \ldots + h(x_{j_1}, x_{j_0}) \\ &\geq \min_i h(x_i, x_{j_0}). \end{aligned}$$

We now prove (25) for roots. Let x_k be a root.

Let $\alpha_\lambda :]-\infty, 0] \to \mathbb{T}^d$ be a curve such that $\alpha_\lambda(0) = x_k$ and satisfies (18) for any $t \geq 0$.

Claim 2. For $\varepsilon > 0$, there is n such that if $r > n$ then $(\alpha_{\lambda_r}(t), \alpha'_{\lambda_r}(t))$ remains in the ε-neighbourhood of $(x_k, 0)$ for $t \leq 0$.

Otherwise there are subsequences $r_m \to \infty$, $s_m > 0$ such that $(p_m, v_m) = (\alpha_{\lambda_{r_m}}(-s_m), \alpha'_{\lambda_{r_m}}(-s_m))$ is in the boundary of the ε ball of $(x_k, 0)$. We may assume that for $t \in (-s_m, 0)$, $(\alpha_{\lambda_{r_m}}(t), \alpha'_{\lambda_{r_m}}(t))$ is in $B_\varepsilon(x_k, 0)$. There is a subsequence of (p_m, v_m), that we still denote by (p_m, v_m), converging to a point (p, v) .

For each m the curve $\alpha_{\lambda_{r_m}}$ satisfies for any $t \geq 0$.

$$u_{\lambda_{r_m}}(x_k) = u_{\lambda_{r_m}}(\alpha_{\lambda_{r_m}}(-t)) e^{-\lambda_{r_m} t} + \int_{-t}^{0} e^{\lambda_{r_m} s} L(\alpha_{\lambda_{r_m}}(s), \alpha'_{\lambda_{r_m}}(s)) ds. \tag{26}$$

Case 1. The sequence s_m of first exit times is bounded.

By Corollary 2, there is a subsequence $\alpha_{\lambda_{r_{m_q}}}$ that converges to a curve $\gamma :]-\infty, 0] \to \mathbb{T}^d$, uniformly in compact intervals, this implies that for some s we have $p = \gamma(-s)$ is in the boundary of $B_\varepsilon(x_k, 0)$. Moreover for any t

$$u(x_k) = u(\gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))ds = u(\gamma(-t)) + \Phi(\gamma(-t), x_k). \tag{27}$$

Since γ is semistatic, the α-limit of (γ, γ') is a point $(x_i, 0)$ in the Aubry set. Letting $t \to \infty$ we obtain

$$u(x_k) = u(x_i) + h(x_i, x_k)$$

Since x_k is a root this impossible for $i \neq k$. For $i = k$ the nontrivial curve γ lies in the Aubry set, which is assumed to be finite, giving a contradiction.

Case 2. There is a subsequence of s_m, that we still denote by s_m diverging to infinity.

For each m the curve $\gamma_m :]-\infty, s_m] \to \mathbb{T}^d$ given by $\gamma_m(t) = \alpha_{\lambda_{r_m}}(t - s_m)$, satisfies for any $t_1 < t_2 \leq s_m$.

$$u_{\lambda_{r_m}}(\gamma_m(t_2)) = u_{\lambda_{r_m}}(\gamma_m(t_1))e^{\lambda_{r_m} t_1} + \int_{t_1}^{t_2} e^{\lambda_{r_m} s} L(\gamma_m(s), \dot\gamma_m(s))ds. \tag{28}$$

By Corollary 2 there is a subsequence γ_{m_q} that converges to a curve $\gamma : \mathbb{R} \to \mathbb{T}^d$, uniformly in compact intervals. Moreover for any $t_1 < t_2$

$$u(\gamma(t_2)) = u(\gamma(t_1)) + \int_{t_1}^{t_2} L(\gamma(s), \dot\gamma(s))ds = u(\gamma(t_1)) + \Phi(\gamma(t_1), \gamma(t_2)). \tag{29}$$

Since γ is a semistatic the α and ω limits of (γ, γ') belong to the Aubry set.

Given any positive T the curve (γ, γ') in $[0, T]$ is the uniform limit of curves contained in the ε ball around $(x_k, 0)$ so the ω limit of (γ, γ') is $(x_k, 0)$. As before the α limit is a point $(x_i, 0)$

Leting $t_1 \to -\infty$ and $t_2 \to \infty$ we obtain

$$u(x_k) = u(x_i) + h(x_i, x_k)$$

and get the same contradiction as in Case 1.

Observe that we have not used yet the hyperbolicity assumption.

Since the flow ϕ_t has hyperbolic critical points $(x_i, 0)$ $i = 1, \ldots, N$, by analytic continuation, for $\lambda > 0$ small enough the flow ϕ_t^λ has hyperbolic critical points $(x_i^\lambda, 0)$, $i = 1, \ldots, N$ converging to $(x_i, 0)$ as $\lambda \to 0$.

Taking $\varepsilon > 0$ small enough in Claim 2, it follows that for r large, the orbit $(\alpha_{\lambda_r}, \alpha'_{\lambda_r})$ lies on the unstable manifold of $(x_k^{\lambda_r}, 0)$.

Moreover the flow ϕ_t^λ and x_i^λ depend smoothly on λ. Thus the local unstable manifold of x_k^λ converges to that of x_k and there is a $\mu > 0$ such that for large enough r we have:

$$d(\alpha_{\lambda_r}(t), x_k^{\lambda_r}) + |\alpha'_{\lambda_r}(t)| \leq C_1 e^{\mu t}.$$

We will prove that

$$u(x_k) = 0 = h(x_k, x_k) \geq \min_i h(x_i, x_k)$$

by estimating $|u_{\lambda_r}(x_k^{\lambda_r})|$. For this we first estimate $u_\lambda(\alpha_{\lambda_r}(-n))$.

We will use

$$\int_{-\infty}^{-n} e^{\lambda t} L(x_k^\lambda, 0)dt = \frac{e^{-n\lambda}}{\lambda} L(x_k^\lambda, 0) \tag{30}$$

and that for $\lambda = \lambda_r$ we have

$$\begin{aligned} \int_{-\infty}^{-n} e^{\lambda t}|L(\alpha_\lambda(t), \alpha'_\lambda(t)) - L(x_k^\lambda, 0)|dt &\leq \int_{-\infty}^{-n} Ce^{(\lambda+\mu)t}dt \\ &\leq \frac{Ce^{-(\lambda+\mu)n}}{\lambda+\mu}, \end{aligned} \tag{31}$$

Since

$$\begin{aligned} u_\lambda(\alpha_\lambda(-n)) &= \int_{-\infty}^{0} e^{\lambda t} L(\alpha_\lambda(t-n), \alpha'_\lambda(t-n))dt \\ &= e^{n\lambda} \int_{-\infty}^{-n} e^{\lambda t} L(\alpha_\lambda(t), \alpha'_\lambda(t))dt \\ &= e^{n\lambda} \int_{-\infty}^{-n} e^{\lambda t}(L(\alpha_\lambda(t), \alpha'_\lambda(t)) - L(x_k^\lambda, 0) + L(x_k^\lambda, 0))dt, \end{aligned} \tag{32}$$

using (30), (31), we have

$$|u_\lambda(\alpha_\lambda(-n))| \leq \frac{Ce^{-\mu n}}{\lambda+\mu} + \frac{L(x_k^\lambda, 0)}{\lambda}. \tag{33}$$

Then, letting n tend to infinity we obtain

$$|u_\lambda(x_k^\lambda)| \leq \frac{L(x_k^\lambda, 0)}{\lambda}. \tag{34}$$

Taking limit as $r \to \infty$ and using L'Hopital's rule

$$|u(x_k)| \leq L_x(x_k, 0)\frac{dx_k^\lambda}{d\lambda}\Big|_{\lambda=0} = 0.$$

□

Proof of Main Theorem. Let λ^m be any sequence of positive numbers converging to zero, since the family u_λ is equibounded and equiLipschitz there is a subsequence λ_r of λ^m such that u_{λ_r} converges uniformly to $u : \mathbb{T}^d \to \mathbb{R}$. By Proposition 5

$$u(x) = \min_k h(x_k, x)$$

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