A MINIMAX SELECTOR FOR A CLASS OF HAMILTONIANS ON COTANGENT BUNDLES*

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We construct a minimax selector for eventually quadratic Hamiltonians on cotangent bundles. We use it to give a relation between Hofer's energy and Mather's action minimizing function. We also study the local flatness of the set of twist maps.

1. Introduction

One purpose of this paper is to have some understanding of the relation between two approaches to the study of Hamiltonian systems, namely the dynamical point of view of Mather [8] and the geometric point of view of Hofer and Zehnder [5].

The main result concerning this purpose is an extension to arbitrary cotangent bundles of a result given by Siburg [11] for the cotangent bundle of an n-torus. See Theorem 1.3 below.

To do so we follow the suggestion of Bialy and Polterovich [1] and construct a minimax selector for a certain class of Hamiltonians. They conjectured that a selector could be constructed for symplectic manifolds admitting a nice Floer homology. This program has been carried out by Schwarz [9] for symplectic compact manifolds using Floer homology.

Since our interest is mainly on convex superlinear Hamiltonians defined on cotangent bundles, we found very appealing to use the methods developed by Golé [4] to find periodic orbits for Hamiltonians that are not necessarily convex but quadratic outside of a neighbourhood of the zero section.

More precisely, let $(M, \langle\,,\rangle)$ be a compact connected orientable Riemannian manifold, such that there are no nontrivial contractible closed geodesics. Let

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$\pi : T^*M \to M$ be its cotangent bundle endowed with its natural symplectic structure $\Omega = -d\lambda$, and let

$$U_r = \{(q,p) : |p| \le r\}\,.$$

Let $\mathcal{H}_0$ be the set of smooth functions on $T^*M \times \mathbb{S}^1$ with support contained in $U_1 \times \mathbb{S}^1$. Let $\mathcal{H}_\mathcal{R}$ be the set of smooth functions on $T^*M \times \mathbb{S}^1$ such that $H = \frac{R}{2}(|p|^2 - 1)$ for $|p| \ge 1$. For $H \in \mathcal{H}_0 \cup \mathcal{H}_R$, let $\varphi_t(x) = \varphi_t^H(x)$ be the solution to the Hamiltonian system $\dot{x} = X_H(x,t)$ with $\varphi_0(x) = x$ and define $\varphi_s^t(x) = \varphi_t \circ \varphi_s^{-1}(x)$.

Letting

$$\mathcal{D}_0 = \{\varphi_1^H : H \in \mathcal{H}_0\}\,, \quad \mathcal{D}_R = \{\varphi_1^H : H \in \mathcal{H}_R\}\,,$$

for any $\psi \in \mathcal{D}_\mathcal{R}$ we have $\mathcal{D}_R = \{\psi \circ \phi : \phi \in \mathcal{D}_0\}$.

For $H \in \mathcal{H}_0 \cup \mathcal{H}_R$, let

$$E^+(H) = -\int_0^1 \min_{x \in U_1} H(x,t)\,dt\,, \quad E^-(H) = -\int_0^1 \max_{x \in U_1} H(x,t)\,dt\,,$$

and

$$\|H_t\| = \max_{x \in U_1} H(x,t) - \min_{x \in U_1} H(x,t)\,.$$

If $H \in \mathcal{H}_0 \cup \mathcal{H}_\mathcal{R}$, then $H = 0$ on ∂U_1 and so $E^+(H) \ge 0 \ge E^-(H)$.

For $H \in \mathcal{H}_\mathcal{R}$, define the *contractible action spectrum*

$$\sigma_c(H) = \left\{ \int_\Gamma \lambda - H dt : \Gamma \text{ is a contractible 1-periodic orbit of } \varphi_t^H \right\}.$$

Theorem 1.1. *For any nonzero $v \in H^*(M,\mathbb{R})$ and $H \in \mathcal{H}_\mathcal{R}$ there is $c_v(H) \in \mathbb{R}$ such that*

(1) $c_v(H) \in \sigma_c(H)\,.$
(2) *For $K \in \mathcal{H}_R$, $E^-(K-H) \le c_v(K) - c_v(H) \le E^+(K-H)$.*

In particular, for $H \in \mathcal{H}_R$ define the selectors

$$\gamma_-(H) = C_1(H),\ \gamma_+(H) = C_\mu(H)\,,$$

where μ is the orientation form.

These selectors have the following key properties:

Theorem 1.2. *Let $H, K \in \mathcal{H}_R$. Then*

(1) *If $\varphi_1^H = \varphi_1^K$ then $\gamma_\pm(H) = \gamma_\pm(K)$.*
(2) *If $H < 0$ in the interior of U_1 then $\gamma_-(H) > 0$.*
(3) *$E^-(H) \le \gamma_-(H) \le \gamma_+(H) \le E^+(H)$.*
(4) *If $H \le 0$ on U_1 then $\gamma_-(H) \ge 0$.*

If φ is generated by $H \in \mathcal{H}_R$ i.e. $\varphi_1^H = \varphi$, define $\gamma_\pm(\varphi) = \gamma_\pm(H)$.

For $\varphi \in \mathcal{D}_R$ define its energy as

$$E(\varphi) = \inf \left\{ \int_0^1 \|H_t\|\,dt : H \in \mathcal{H}_R\,, \varphi_1^H = \varphi \right\}.$$

For $\phi \in \mathcal{D}_0$ let

$$\|\phi\| = \inf\left\{\int_0^1 \|K_t\|\,dt : K \in \mathcal{H}_0,\ \varphi_1^K = \phi\right\}.$$

If $\varphi, \psi \in \mathcal{D}_R$ define $d(\varphi,\psi) = \|\psi^{-1}\varphi\|$.

As proved by Lalonde and Mc Duff in [7], $\|\varphi\| = 0$ only when φ is the identity and then d is a metric in $\mathcal{D}_{\mathcal{R}}$.

For $L : TM \times \mathbb{S}^1 \to \mathbb{R}$ a convex superlinear Lagrangian with complete Euler-Lagrange flow, let $\beta = \beta_L : H_1(M,\mathbb{R}) \to \mathbb{R}$ be the Mather's beta function. The following theorem gives a connection between the points of view of Mather and Hofer-Zehnder.

Theorem 1.3. *If φ is generated by a convex Hamiltonian $H \in \mathcal{H}_R$, then*

$$\beta_L(0) \le E(\varphi)$$

where L is the Legendre transform of H.

A *twist map* F is a diffeomorphism of a neighborhood U of the zero section in T^*M onto itself satisfying the following:

- If $F(q,p) = (Q,P)$, then the map $\Phi(q,p) = (q,Q)$ is an embedding of U in $M \times M$.
- F is *exact symplectic*, that is, $F^*\lambda - \lambda = d(S \circ \Phi)$ for some real function S on $\Phi(U)$.

The function S is called a *generating function* for F.

Fix $r > 0$ smaller than the injectivity radius of the given metric. Let $\mathcal{T}$ be the set of twist maps $\varphi|U_1$ with $\varphi \in \mathcal{D}_r$. Any map in $\mathcal{T}$ has a generating function satisfying

- $\frac{\partial^2 S}{\partial q \partial Q}$ is negative definite,
- $S(q,Q) = \frac{d(q,Q)^2}{2r}$ for $d(q,Q) \ge r$.

The following proposition gives the local flatness of Hofer's metric for twist maps and is a generalization of Theorem 1 in Siburg's paper [10].

Proposition 1.1. *For any $\varphi \in \mathcal{T}$ there is a C^1 neighborhood $\mathcal{O}$ of φ such that if φ_0, $\varphi_1 \in \mathcal{O}$ and S_0, S_1 are their generating functions as above, then*

$$d(\varphi_0,\varphi_1) = \|S_0 - S_1\|.$$

In the appendix we consider a general convex superlinear Hamiltonian $H : T^*M \times \mathbb{S}^1 \to \mathbb{R}$ and its Legendre transform $L : TM \times \mathbb{S}^1 \to \mathbb{R}$. Let $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ be the convex dual of β.

Recall that the image of a one-form in M is a Lagrangian submanifold of T^*M if and only if the form is closed. Following Bialy and Polterovich [1] we define

$$c(H) = \inf\left\{\int_0^1 \max_{q\in M} H(q,\theta(q),t)\,dt : \theta \text{ is a closed 1-form}\right\}.$$

More generally we can define for any cohomology class $[\omega] \in H^1(M, \mathbb{R})$

$$c(H, [\omega]) = \inf \left\{ \int_0^1 \max_{q \in M} H(q, \theta(q), t)\, dt : \theta \in [\omega] \right\}.$$

When H is autonomous this reduces to

$$c(H, [\omega]) = \inf_{\theta \in [\omega]} \max_{q \in M} H(q, \theta(q))$$

and it was proved in [2] that $c(H, [\omega]) = \alpha([\omega])$. In the non-autonomous case we recover the following inequality

Proposition 1.2.

$$\alpha([\omega]) \leq c(H, [\omega])\,. \tag{1}$$

Taking infimum on both sides of inequality (1), we obtain

Corollary 1.1.

$$-\beta(0) \leq c(H)\,.$$

For the last application we go back to our special family $\mathcal{H}$

Corollary 1.2. *Let $H \in \mathcal{H}$ be convex and suppose there is a Lagrangian section in U_1 consisting of fixed points of φ_t^H. Then*

$$\hat{E}(\varphi_1^H) = \beta(0) = -c(H)\,.$$

The plan of the article is as follows, in Sec. 2, we introduce the generating families whose critical values belong to the contractible action spectrum. In Sec. 3, we prove Theorems 1.1 and 1.2. In Sec. 4, we recall the definition of Mather's beta function and prove Theorem 1.3. In Sec. 5, we apply the results of Lalonde and Mc Duff [6] describing the geodesics of Hofer's metric for certain classes of symplectic manifolds that include cotangent bundles of compact manifolds and prove Proposition 1.1.

2. Generating Families

Let g_t be the geodesic flow on T^*M, d be the distance in M defined by the metric and fix $r > 0$ smaller than the injectivity radius. The map $g_r|U_1$ is twist since $G_r(q, p) = (q, \exp_q(rp))$ is an embedding of U_1 in $M \times M$ and $S(q, Q) = d(q, Q)^2/2r$ is a generating function. Write $g_{-r} = (Q_g, P_g)$.

For $H \in \mathcal{H}_R$, let $\varphi_t(x) = \varphi_t^H(x)$. Since the C^1 distance from φ_s^t to the identity is $O(s - t)$ on U_1, there is N such that

$$\Phi_s^t(q, p) = (q, Q_g \circ \varphi_s^t(q, p))$$

is an embedding of U_1 in $M \times M$ for $|t - s| \leq 1/N$. For $0 \leq j < N$, let $s_j = j/N$,

$$\varphi^j = \varphi_{s_j}^{s_{j+1}},\ F_{2j} = g_{-r} \circ \varphi^j,\ F_{2j+1} = g_r,\ \Phi_{2j} = \Phi_{s_j}^{s_{j+1}},\ \Phi_{2j+1} = G_r\,.$$

Let $\Gamma_x^j(t) = \varphi_{s_j}^t(x)$, $t \in [s_j, s_{j+1}]$, and define the *action* $A_j : U_1 \to \mathbb{R}$ by

$$A_j(x) = \int_{\Gamma_x^j} \lambda - H dt\,, \tag{2}$$

then $dA_j = (\varphi^j)^*\lambda - \lambda$. Thus, defining

$$S_{2j} = (A_j - S \circ G_r \circ F_{2j}) \circ \Phi_{2j}^{-1}\,,$$

we have

$$\begin{aligned} d\,S_{2j} \circ \Phi_{2j} &= dA_j - F_{2j}^*\, d\,S \circ G_r \\ &= (\varphi^j)^*\lambda - \lambda - (\varphi^j)^* g_{-r}^*(g_r^*\lambda - \lambda) \\ &= F_{2j}^*\lambda - \lambda\,. \end{aligned} \tag{3}$$

Therefore F_{2j} is a twist map with generating function S_{2j}. Let $S_{2j+1} = S$ and

$$\mathcal{M}_N = \{(q_0, q_1, \ldots, q_{2N}) : (q_j, q_{j+1}) \in \Phi_j(U_1),\ q_{2N} = q_0\}\,.$$

The function $W_N : \mathcal{M}_N \to \mathbb{R}$ given by

$$W_N : \mathbf{q} = (q_0, q_1, \ldots, q_{2N}) \mapsto \sum_{j=0}^{2N-1} S_j(q_j, q_{j+1})\,.$$

is called a generating family. Letting $(q_j, p_j) = \Phi_j^{-1}(q_j, q_{j+1})$, $p_{2N} = p_0$, $(q_{j+1}, P_{j+1}) = F_j(q_j, p_j)$, $P_{2N} = P_0$, we have by (3)

$$dW_N = \sum_{j=0}^{2N-1} (P_j - p_j)\, dq_j\,. \tag{4}$$

Therefore $\mathbf{q}$ is a critical point of W_N if and only if $(q_{j+1}, p_{j+1}) = F_j(q_j, p_j)$ for $0 \le j < 2N$, so $(q_{2j}, p_{2j}) = \varphi_{j/N} = (q_0, p_0)$, (q_0, p_0) is a fixed point of φ_1, and $\Gamma(t) = \varphi_t(q_0, p_0)$ is a 1-periodic orbit in U_1. In such a case

$$W_N(\mathbf{q}) = \sum_{j=0}^{N-1} A_j(q_j, p_j) = \int_\Gamma \lambda - H dt\,.$$

Proposition 2.1. *Given $\varepsilon > 0$ there is $N(\varepsilon)$ such that for $N \ge N(\varepsilon)$ we have*

$$\left\| \frac{\partial P_{2j+1}}{\partial q_{2j+1}} - \frac{\partial p_{2j+1}}{\partial q_{2j+1}} \right\| < \varepsilon \tag{5}$$

for any j such that $F_{2j} \circ \Phi_{2j}^{-1}(q_{2j}, q_{2j+1}) = G_r^{-1}(q_{2j+1}, q_{2j+2})$, and

$$\left\| \frac{\partial P_{2j}}{\partial q_{2j}} - \frac{\partial p_{2j}}{\partial q_{2j}} \right\| < \varepsilon \tag{6}$$

for any j such that $g_r \circ G_r^{-1}(q_{2j-1}, q_{2j}) = \Phi_{2j}^{-1}(q_{2j}, q_{2j+1})$.

Proof. (a) From $(q_{2j+1}, P_{2j+1}) = F_{2j} \circ \Phi_{2j}^{-1}(q_{2j}, q_{2j+1})$, $F_{2j} = g_{-r} \circ \varphi^j$ one gets

$$\frac{\partial P_{2j+1}}{\partial q_{2j+1}} = \frac{\partial(P_g \circ \varphi^j)}{\partial p} \circ \Phi_{2j}^{-1} \left(\frac{\partial(Q_g \circ \varphi^j)}{\partial p} \circ \Phi_{2j}^{-1} \right)^{-1} .$$

(b) From $(q_{2j+1}, p_{2j+1}) = G_r^{-1}(q_{2j+1}, q_{2j+2})$ one gets

$$\frac{\partial p_{2j+1}}{\partial q_{2j+1}} = \frac{\partial P_g}{\partial p} \circ g_r \circ G_r^{-1} \left(\frac{\partial Q_g}{\partial p} \circ g_r \circ G_r^{-1} \right)^{-1} .$$

(c) From $q_{2j+1} = Q_g \circ \varphi^j(q_{2j}, p_{2j})$ one gets

$$\frac{\partial p_{2j}}{\partial q_{2j}} = - \left(\frac{\partial(Q_g \circ \varphi^j)}{\partial p} \circ \Phi_{2j}^{-1} \right)^{-1} \frac{\partial(Q_g \circ \varphi^j)}{\partial q} \circ \Phi_{2j}^{-1} .$$

(d) From $q_{2j-1} = Q_g(q_{2j}, P_{2j})$ one gets

$$\frac{\partial P_{2j}}{\partial q_{2j}} = - \left(\frac{\partial Q_g}{\partial p} \circ g_r \circ G_r^{-1} \right)^{-1} \frac{\partial Q_g}{\partial q} \circ g_r \circ G_r^{-1} .$$

Suppose that $F_{2j} \circ \Phi_{2j}^{-1}(q_{2j}, q_{2j+1}) = G_r^{-1}(q_{2j+1}, q_{2j+2})$ so that

$$\varphi^j \circ \Phi_{2j}^{-1}(q_{2j}, q_{2j+1}) = g_r \circ G_r^{-1}(q_{2j+1}, q_{2j+2}) ,$$

then

$$\begin{aligned} &D(g_{-r} \circ \varphi^j)(\Phi_{2j}^{-1}(q_{2j}, q_{2j+1})) \\ &\quad = Dg_{-r}(g_r \circ G_r^{-1}(q_{2j+1}, q_{2j+2})) D\varphi^j(\Phi_{2j}^{-1}(q_{2j}, q_{2j+1})) . \end{aligned}$$

Suppose $g_r \circ G_r^{-1}(q_{2j-1}, q_{2j}) = \Phi_{2j}^{-1}(q_{2j}, q_{2j+1})$, then

$$D(g_{-r} \circ \varphi^j)(\Phi_{2j}^{-1}(q_{2j}, q_{2j+1})) = D(g_{-r} \circ \varphi^j)(g_r \circ G_r^{-1}(q_{2j-1}, q_{2j})) .$$

Taking N sufficiently large, one makes any $\varphi^j|U_1$ as C^1-close to the identity as one wants. Thus, (5) follows from items (a) and (b), and (6) follows from items (c) and (d). □

As observed by Golé [4] one can define a path $\sigma_k(q, Q)$ between q and $Q \in \Phi_k(U_1 \cap T_q^* M)$ that for k odd concides with the unique geodesic between these points. Therefore to each $\mathbf{q} \in \mathcal{M}_N$ we associate a polygonal loop $c(\mathbf{q})$. We can work with the component $\mathcal{M}_N^e$ of $\mathcal{M}_N$ consisting of points $\mathbf{q}$ such that $c(\mathbf{q})$ is homotopically trivial. Thus, $\sigma_c(H)$ is the set of critical values of $W_N : \mathcal{M}_N^e \to \mathbb{R}$ and so it is nowhere dense.

Golé proved that the gradient flow of W_N has the index pair (B_N, B_N^-),

$$B_N = \{\mathbf{q} \in \mathcal{M}_N^e : d(q_j, q_{j+1}) \leq |a_j|\}, ,$$

$$B_N^- = \{\mathbf{q} \in B_N : d(q_j, q_{j+1}) = |a_j| \quad \text{for some even } j\} ,$$

where

$$a_j = \begin{cases} r & j \text{ odd} \\ \dfrac{R}{N} - r & j \text{ even} \end{cases}$$

Lemma 2.1 (Floer [3]). *Let ζ^τ be a one parameter family of flows on a manifold $\mathcal{M}$. Suppose that Σ^0 is a compact submanifold invariant under ζ^0. Assume moreover that Σ^0 is* normally hyperbolic *i.e. there is a decomposition*

$$T_{\Sigma^0}\mathcal{M} = \mathcal{T}\Sigma' \oplus E^+ \oplus E^-$$

invariant under the covariant linearization of the vector field V_0 generating ζ^0 with respect to some metric $\langle\, ,\rangle$, so that for some $m > 0$:

$$\langle \xi, DV_0\xi\rangle \leq -m\langle \xi,\xi\rangle\,, \quad \xi \in E^-$$

$$\langle \xi, DV_0\xi\rangle \geq m\langle \xi,\xi\rangle\,, \quad \xi \in E^+\,.$$

Suppose that there is a retraction $\alpha : \mathcal{M} \to \Sigma'$ and that there is an index pair (B, B^-) for all ζ^τ. Then

- *There is $u \in H^*(B, B^-)$ of dimension* $\dim E^+$ *so that the map*

  $$T : H^*(\Sigma^0) \to H^*(B, B^-),\, T(v) = (\alpha|_B)^* v \cup u$$

  is an isomorphism.
- *If Σ^τ denotes the maximal invariant set for the flow ζ^τ, the homomorphism in Čech cohomology $(\alpha|\Sigma^\tau)^* : H^*(\Sigma^0) \to H^*(\Sigma^\tau)$ is injective.*

3. Critical Values of the Action

Let $H \in \mathcal{H}_R$, and define $H_0 = \frac{R}{2}(|p|^2 - 1)$. Let $H_\tau = H_0 + \tau(H - H_0)$ and φ_t^τ be the corresponding flow. Let N be sufficiently large to have a decomposition of all φ_1^τ in $2N$ twist maps. Let W_N^τ be the corresponding generating family and ζ^τ its gradient flow.

Lemma 3.1 (Golé [4]). *Let $\Sigma^0 = \{\mathbf{q} \in \mathcal{M}_N : q_k = q_0, \forall k\}$. Then Σ^0 is a normally hyperbolic invariant set for ζ^0 and it is a retract of $\mathcal{M}_N^e$.*

By Lemma 2.1, $T_N : H^*(M) \to H^{*+k}(B_N, B_N^-)$ is an isomorphism.

Let $B_N^a = \{\mathbf{q} \in B_N : W_N(\mathbf{q}) \leq a\}$, $j_a : B_N^a \hookrightarrow B_N$ and consider the induced map $j_a^* : H^*(B_N, B_N^-) \to H^*(B_N^a, B_N^-)$.

For N of the form $N = 2^m$, we follow Viterbo [12] and define for any nonzero $v \in H^*(M) = H^*(\Sigma^0)$

$$c_v(H, m) = \inf\{a : j_a^* T_N(v) \neq 0\}\,.$$

so that $c_v(H, m)$ is a critical point of W_N.

Proposition 3.1. *There is* $\mathbf{m}$ *such that for any nonzero* $v \in H^*(M)$ *and* $m \geq \mathbf{m}$, $c_v(H, m) = c_v(H, \mathbf{m})$.

Proof. Represent the points of $\mathcal{M}_{2N}$ in the form

$$\mathbf{q} = (\eta_0, \xi_0, \xi_1, \eta_1, \dots, \eta_{2N-2}, \xi_{2N-2}, \xi_{2N-2}, \eta_{2N-1}, \eta_{2N} = \eta_0)$$

so that

$$\begin{aligned} W_{2N}(\mathbf{q}) &= W_{2N}(\eta, \xi) \\ &= \sum_{j=0}^{N-1} S_{4j}(\eta_{2j}, \xi_{2j}) + S(\xi_{2j}, \xi_{2j+1}) \\ &\quad + S_{4j+2}(\xi_{2j+1}, \eta_{2j+1}) + S(\eta_{2j+1}, \eta_{2j+2})\,. \end{aligned}$$

Let

$$h_j = G_r \circ g_{-R} \circ \varphi_{j/N}^{2j+1/2N} \circ (\Phi_{j/N}^{j+1/N})^{-1}$$

and define $h : \mathcal{M}_N \to M^{2N}$ by

$$h(\eta) = (h_0(\eta_0, \eta_1), \dots, h_{N-1}(\eta_{2N-2}, \eta_{2N-1}))\,.$$

Then $W_{2N}(\eta, h(\eta)) = W_N(\eta)$ and

$$\frac{\partial W_{2N}}{\partial \xi_j}(\eta, h(\eta)) = 0\,.$$

Define $f : \mathcal{M}_{2N} \to \mathbb{R}$ by $f(\eta, \xi) = W_{2N}(\eta, \xi) - W_N(\eta)$, then

$$\frac{\partial f}{\partial \xi_{2j+i}} = \frac{\partial W_{2N}}{\partial \xi_{2j+i}} = P_{4j+1+i} - p_{4j+1+i}\,, \quad i = 0, 1\,, \tag{7}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial W_{2N}}{\partial \eta} - \frac{\partial W_{2N}}{\partial \eta}(\eta, h(\eta)) - \frac{\partial W_{2N}}{\partial \xi} Dh\,. \tag{8}$$

Thus $Df|\operatorname{graph} h = 0$ and

$$\frac{\partial^2 f}{\partial(\xi_{2j}, \xi_{2j+1})} = \begin{bmatrix} \dfrac{\partial(P_{4j+1} - p_{4j+1})}{\partial \xi_{2j}} & \dfrac{\partial^2 S}{\partial q \partial Q}(\xi_{2j}, \xi_{2j+1}) \\[2ex] \dfrac{\partial^2 S}{\partial q \partial Q}(\xi_{2j}, \xi_{2j+1}) & \dfrac{\partial(P_{4j+2} - p_{4j+2})}{\partial \xi_{2j+1}} \end{bmatrix}.$$

Since $\frac{\partial^2 S}{\partial q \partial Q}$ is negative definite on $G_r(U_1)$, we have that $\frac{\partial^2 f}{\partial(\xi_{2j}, \xi_{2j+1})}|\text{graph } h$ is invertible if $\frac{\partial(P_{4j+1+i} - p_{4j+1+i})}{\partial \xi_{2j+i}}|\text{graph } h$, $i = 0, 1$ are sufficiently small.

We know from Proposition 2.1 that this holds if $N = 2^m$ is sufficiently large. In such a case, $\operatorname{graph} h$ is a nondegenerate critical manifold of f. By the generalized Morse's lemma there is a tubular neighborhood $\psi : E \to \mathcal{M}_{2N}$, with $E = E^+ \oplus E^-$ a vector bundle over $\mathcal{M}_N$, such that $f(\psi(\eta, \zeta)) = |\zeta_-|^2 - |\zeta_+|^2$ and so

$$W_{2N}(\psi(\eta, \zeta)) = W_N(\eta) + |\zeta_-|^2 - |\zeta_+|^2\,. \tag{9}$$

Consider the commutative diagram

$$\begin{array}{ccccc} & T_{2N} & H^{*+k+l}(B_{2N},B_{2N}^-) & \xrightarrow{j_a^*} & H^{*+k+l}(B_{2N}^a,B_{2N}^-) \\ & \nearrow & \downarrow \psi^* & & \downarrow \psi^* \\ H^*(M) & & H^{*+k+l}(\psi^{-1}(B_{2N},B_{2N}^-)) & \xrightarrow{J_a^*} & H^{*+k+l}(\psi^{-1}(B^a_{2N},B^-_{2N})) \\ & \searrow & \uparrow \text{Thom} & & \uparrow \cong \\ & T_N & H^{*+k}(B_N,B_N^-) & \xrightarrow{j_a^*} & H^{*+k}(B_N^a,B_N^-) \end{array}$$

where the upwards isomorphisms are as in [12].

The maximal invariant set Σ_a for the gradient flow of W_{2N} in $B^a_{2N} - B^-_{2N}$ consists of critical points and heteroclinic orbits. The critical points belong to graph h by (7) and so do the heteroclinic orbits by (9). Thus (B^a_{2N}, B^-_{2N}) and $(B^a_{2N}, B^-_{2N}) \cap \psi(E)$ are index pairs for Σ_a and then

$$\psi^* : H^*(B^a_{2N}, B^-_{2N}) \cong H^*(\psi^{-1}(B^a_{2N}, B^-_{2N}))\,.$$

Therefore $j_a^* T_{2N}(v) = 0$ if and only if $j_a^* T_N(v) = 0$ and so

$$c_v(H, m) = c_v(H, m+1)\,. \qquad \square$$

Lemma 3.2 (Viterbo [12]). *Let S_τ be a smooth family of smooth functions. Let $c(\tau) = S_\tau(x_\tau)$ be a critical value obtained by minimax as above. Assume that $dS_\tau(x) = 0$ implies $\frac{\partial S_\tau(x)}{\partial \tau} \geq 0$: (≤ 0). Then $c(\tau)$ is increasing (decreasing).*

The following lemma completes the proof of Theorem 1.1.

Lemma 3.3. *For $v \in H^*(M)$, H, $K \in \mathcal{H}_R$*

$$E^-(K-H) \leq c_v(K) - c_v(H) \leq E^+(K-H)\,.$$

Proof. For $\tau \in [0,1]$, define $H_\tau = H + \tau(K-H)$.

Let $S_\tau = W_\tau - E^-(K-H)\tau$ then $c_v(H_\tau) - E^-(K-H)\tau$ is a critical value of S_τ and

$$c_v(H_\tau) = W_\tau(x_\tau) = \int_{S^1} \psi_\tau^*(\lambda - H_\tau\, dt)\,,$$

where $\psi_\tau(t) = \varphi_t^\tau(x_\tau)$. Then

$$\begin{aligned} \frac{\partial W_\tau}{\partial \tau}(x_\tau) &= \int_{S^1} d\psi_\tau^* \lambda \left(\frac{\partial {\psi_t}^\tau}{\partial \tau}(x_\tau)\right) + \psi_\tau^* i \left(\frac{\partial {\psi_t}^\tau}{\partial \tau}(x_\tau)\right) d\lambda \\ &\quad - \int_0^1 \left(\frac{\partial H_\tau}{\partial \tau}(\psi_\tau(t), t) + dH_\tau(\psi_\tau(t), t) \frac{\partial {\psi_t}^\tau}{\partial \tau}(x_\tau)\right) dt \\ &= \int_0^1 (H-K)(\varphi_t^\tau(x_\tau), t)\, dt \geq E^-(K-H)\,. \end{aligned}$$

By Lemma 3.2, $c_v(H_\tau)-E^-(K-H)\tau$ is increasing. Similarly $c_v(H_\tau)-E^+(K-H)\tau$ is decreasing. Then

$$c_v(K)-E^+(K-H)\le c_v(H)\le c_v(K)-E^-(K-H)\,. \qquad \square$$

Corollary 3.1. *If $H\le K$, then $c_v(H)\ge c_v(K)$ for any nonzero $v\in H^*(M)$.*

Proof of Theorem 1.2.

Lemma 3.4. *If $H,K\in\mathcal{H}_R$ are such that the corresponding time one maps φ_1 and ψ_1 are equal then $\sigma_c(H)=\sigma_c(K)$.*

Proof. Let x_0 in the border of U_1. Let x_1 in $\mathrm{Fix}(\varphi_1)=\mathrm{Fix}(\psi_1)$. Then $\varphi_t(x_0)=\psi_t(x_0)$ for all t. Let β be a curve on T^*M such that $\beta(0)=x_0$ and $\beta(1)=x_1$. Define $\sigma_1,\sigma_2:[0,1]^2\to T^*M\times\mathbb{R}$ by

$$\sigma_1(s,t)=(\varphi_t(\beta(s)),t)\,,$$

$$\sigma_2(s,t)=(\psi_t(\beta(s)),t)\,.$$

Then

$$0=\int_{\sigma_1}d(\lambda-Hdt)=\int_{\varphi_t(x_1)}\lambda-Hdt-\int_{\varphi_t(x_0)}\lambda-Hdt-\int_{\varphi_1(\beta)}\lambda+\int_\beta\lambda$$

and

$$0=\int_{\sigma_2}d(\lambda-Kdt)=\int_{\psi_t(x_1)}\lambda-Kdt-\int_{\psi_t(x_0)}\lambda-Kdt-\int_{\psi_1(\beta)}\lambda+\int_\beta\lambda\,.$$

So

$$\int_{\varphi_t(x_1)}\lambda-Hdt=\int_{\psi_t(x_1)}\lambda-Kdt\,.$$

It remains to prove that $\varphi_t(x_1)$ is contractible if and only if $\psi_t(x_1)$ is. To see that, define the path

$$h_t(x)=\begin{cases}\varphi_t(x) & \text{if} \quad t\in[0,1]\\ \psi_{2-t}(x) & \text{if} \quad t\in[1,2]\,.\end{cases}$$

Let $\Omega(U_1)$ be the free loop space of U_1, and define the continuous function $\chi:U_1\to\Omega(U_1)$ by $\chi(x)=(h_t(x))_{t\in[0,2]}$.

Since M is connected all the loops in $\chi(U_1)$ are homotopic to $\chi(x_0)$ which is homotopically trivial. Thus $\varphi_t(x_1)$ is contractible if and only if $\psi_t(x_1)$ is and then

$$\sigma_c(H)=\sigma_c(K)\,. \qquad \square$$

Let $H\in C^\infty(T^*M\times\mathbb{S}^1\times[0,1])$ be such that $H_s(p,t)=H(p,t,s)$ belongs to $\mathcal{H}_{\mathcal{R}}$ for any $s\in[0,1]$. Assume now that $\varphi_1^{H_s}=\psi$ for all $s\in[0,1]$, then $\gamma_\pm(H_s)\in\sigma_c(H_s)=\sigma_c(\psi)$. Since $\sigma_c(\psi)$ is nowhere dense, the continuous function $s\mapsto\gamma_\pm(H_s)$ must be constant and so $\gamma_\pm(H_0)=\gamma_\pm(H_1)$.

For $K\in\mathcal{H}_{\mathcal{R}}$, $L\in\mathcal{H}_0$ let $K\#L(p,t)=K(p,t)+L((\varphi_t^K)^{-1}(p),t)$, then $H=K\#L\in\mathcal{H}_{\mathcal{R}}$ and $\varphi_t^H=\varphi_t^K\varphi_t^L$. Taking $H_0=\frac{R}{2}(|p|^2-1)$, we can write any $H\in\mathcal{H}_{\mathcal{R}}$ as $H=H_0\#L$, where $L(p,t)=(H-H_0)(g_t(p),t)$ belongs to $\mathcal{H}_0$.

Proposition 3.5. *Let $H, K \in \mathcal{H}_{\mathcal{R}}$ with $\varphi_1^H = \varphi_1^K$. Then $\gamma_\pm(H) = \gamma_\pm(K)$.*

Proof. We follow the argument of Hofer and Zehnder. Let $H, K \in \mathcal{H}_{\mathcal{R}}$ such that $\varphi_1^H = \varphi_1^K$. Write $H = H_0\#L, K = H_0\#F$ with $L, F \in \mathcal{H}_0$ and let $\psi = \varphi_1^L = \varphi_1^F \in \mathcal{D}_0$. By reparametrizing the time, one first homotope φ_t^L to an arc ψ_t in $\mathcal{D}$, that is the identity for $t \in [0, 1/4]$ and is equal to ψ for $t \in [3/4, 1]$, and do the same for φ_t^F. Hence we can assume that $L(p, t) = F(p, t) = 0$ for $|t|_{\text{mod}\,1} < 1/4$. For $0 < s \leq 1$ one defines

$$L_s(p, t) = sL(p/s, t),\ F_s(p, t) = sF(p/s, t).$$

Then

$$\varphi_t^{L_s}(p) = s\varphi_t^L(p/s),\ \varphi_t^{F_s}(p) = s\varphi_t^F(p/s).$$

For $t \in [3/4, 1]$ one has

$$\varphi_t^{L_s}(p) = s\varphi_t^L(p/s) = s\varphi_t^F(p/s) = \varphi_t^{F_s}(p).$$

Take $\beta : [3/4, 1] \to [0, 1]$ a smooth which is 0 near to $3/4$ and is 1 near to 1. Define

$$\varphi_{s,t}(p) = \begin{cases} \varphi_t^{L_s}(p) & t \in [0, 3/4] \\ (s + (1-s)\beta(t))\psi((s + (1-s)\beta(t))^{-1}p) & t \in [3/4, 1] \end{cases}$$

and similarly $\psi_{s,t}$ by replacing L_s for F_s. Then $\varphi_{s,1} = \psi_{s,1} = \psi$ for $0 < s \leq 1$.

If $\hat{L}_s, \hat{F}_s$ generate $\varphi_{s,t}, \psi_{s,t}$ respectively we have $\gamma_\pm(H) = \gamma_\pm(H_0\#\hat{L}_s)$, $\gamma_\pm(K) = \gamma_\pm(H_0\#\hat{F}_s)$. Note that

$$\hat{L}_s(p, t) = \hat{F}_s(p, t) \quad t \in [3/4, 1]$$
$$\hat{L}_s(p, t) = L_s(p, t) \quad t \in [0, 3/4]$$
$$\hat{F}_s(p, t) = F_s(p, t) \quad t \in [0, 3/4].$$

Thus

$$\begin{aligned} |\gamma_\pm(H) - \gamma_\pm(K)| &= |\gamma_\pm(H_0\#\hat{L}_s) - \gamma_\pm(H_0\#\hat{F}_s)| \\ &\leq E^+(\hat{L}_s - \hat{F}_s) - E^-(\hat{L}_s - \hat{F}_s) \leq s(E^+(L) - E^-(L) + E^+(F) - E^-(F)) \end{aligned}$$

for all $0 < s \leq 1$ and then $\gamma_\pm(H) = \gamma_\pm(K)$. □

For any $\varepsilon > 0$ let $\mathcal{K}_\varepsilon$ be the set of functions $f(|p|)$ in $\mathcal{H}_{\mathcal{R}}$ such that f is convex, $f(s) = -\varepsilon$ for $s \leq 1 - 2\varepsilon/R$.

We claim that $\gamma_\pm(K) = \varepsilon$ for any $K \in \mathcal{K}_\varepsilon$. In fact, the only contractible 1-periodic orbits on U_1 for K are the constants and these have action ε.

Item 2. Since H is negative for $|p| \leq 1$ and equals $\frac{R}{2}(|p|^2 - 1)$ for $|p| \geq 1$, we can find $\varepsilon > 0$ small enough and $K \in \mathcal{K}_\varepsilon$ such that $K \geq H$ and then by Corollary 3.1 we have $\gamma_-(H) \geq \gamma_-(K) > 0$.

Item 3. By Lemma 3.3, for $K_\varepsilon \in \mathcal{K}_\varepsilon$ we have

$$E^-(H - K_\varepsilon) \leq \gamma_\pm(H) - \gamma_\pm(K_\varepsilon) \leq E^+(H - K_\varepsilon)\,.$$

Now let $\varepsilon \to 0$.

Item 4. Since $H \leq 0$, $E^-(H) = 0$. By Item 3, $\gamma_\pm(H) \geq 0$. □

The following corollary is an immediate consequence of Item 3 of Theorem 1.2.

Corollary 3.2. *If* $\varphi \in \mathcal{D}$

$$\gamma_-(\varphi) \le \gamma_+(\varphi) \le E(\varphi)\,, \quad \gamma_+(\varphi) - \gamma_-(\varphi) \le E(\varphi)\,.$$

4. Proof of Theorem 1.3

Let us recall the main concepts introduced by Mather in [8]. Let $L : TM \times \mathbb{S}^1 \to \mathbb{R}$ be a convex superlinear Lagrangian with complete Euler-Lagrange flow. Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra of TM that have compact support and are invariant under the Euler-Lagrange flow ϕ_t. Let $H_1(M,\mathbb{R})$ be the first real homology group of M. Given a closed one-form ω on M and $\rho \in H_1(M,\mathbb{R})$, let $\langle \omega, \rho \rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its *rotation vector* is defined as the unique $\rho(\mu) \in H_1(M,\mathbb{R})$ such that

$$\langle \omega, \rho(\mu) \rangle = \int \omega \, d\mu\,,$$

for all closed one-forms on M. The integral on the right-hand side is with respect to μ with ω considered as a function $\omega : TM \to \mathbb{R}$. The function $\rho : \mathcal{M}(L) \to H_1(M,\mathbb{R})$ is surjective [8]. The *action* of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L \, d\mu\,.$$

Finally we define the function $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$ by

$$\beta(\gamma) = \inf\{A_L(\mu) : \rho(\mu) = \gamma\}\,.$$

The function β is *convex* and *superlinear* and the infimum can be shown to be a *minimum* [8] and the measures at which the minimum is attained are called *minimizing measures*. In other words, $\mu \in \mathcal{M}(L)$ is a minimizing measure iff

$$\beta(\rho(\mu)) = A_L(\mu)\,.$$

Each contractible 1-periodic orbit is the support of an invariant probability measure with zero homology and the same action. So

$$\beta(0) \le \gamma_-(H)\,.$$

By Corollary 3.2, $\gamma_-(\varphi) \le E(\varphi)$, and this concludes the proof of Theorem 1.3.

5. A Hofer's Distance and Twist Maps

For $\phi \in \mathcal{D}_0$ let

$$\|\phi\| = \inf\left\{ \int_0^1 \|L_t\| \, dt : L \in \mathcal{H}_0, \varphi_1^L = \phi \right\}.$$

If $\varphi, \psi \in \mathcal{D}_{\mathcal{R}}$ define $d(\varphi,\psi) = E(\psi^{-1}\varphi)$.

If H, $K \in \mathcal{H}_{\mathcal{R}}$ generate φ, ψ respectively, then $L(x,t) = (H-K)(\psi_t(x),t)$ belongs to $\mathcal{H}_0$ and generates $\psi^{-1}\varphi$. Reciprocally, suppose that $K \in \mathcal{H}_{\mathcal{R}}$ generates ψ. If $L \in \mathcal{H}_0$ generates $\psi^{-1}\varphi$ then $H(x,t) = K(x,t) + L(\psi_t^{-1}(x),t)$ belongs to $\mathcal{H}_{\mathcal{R}}$ and generates φ.

Thus, for any ψ, $\varphi \in \mathcal{D}_{\mathcal{R}}$ and $K \in \mathcal{D}_{\mathcal{R}}$ generating ψ we have

$$\begin{aligned}\{\|(H-J)_t\| : H,\, J \in \mathcal{H}_{\mathcal{R}},\, \varphi = \varphi_1^H,\, \psi = \varphi_1^J\} \\ \subset \{\|L_t\| : L \in \mathcal{H}_0,\, \psi^{-1}\varphi = \varphi_1^L\} \\ \subset \{\|(H-K)_t\|\, dt : H \in \mathcal{H}_{\mathcal{R}},\, \varphi = \varphi_1^H\}\,.\end{aligned}$$

Therefore

$$\begin{aligned} d(\varphi,\psi) &= \inf\left\{\int_0^1 \|(H-J)_t\|\, dt : H,\, J \in \mathcal{H} : \varphi = \varphi_1^H,\, \psi = \varphi_1^J\right\} \\ &= \inf\left\{\int_0^1 \|(H-K)_t\|\, dt : H \in \mathcal{H} : \varphi = \varphi_1^H\right\}. \end{aligned}$$

Corollary 5.1. *Let $\phi \in \mathcal{D}_0$ and define*

$$\begin{aligned} E^+(\phi) &= \inf\{E^+(L) : L \in \mathcal{H}_0,\, \varphi_1^L = \phi\} \\ E^-(\phi) &= \sup\{E^-(L) : L \in \mathcal{H}_0,\, \varphi_1^L = \phi\}\,. \end{aligned}$$

Let $\psi \in \mathcal{D}_{\mathcal{R}}$. Then

$$E^-(\phi) \le \gamma_\pm(\psi \circ \phi) - \gamma_\pm(\psi) \le E^+(\phi)\,.$$

Proof. Let $L \in \mathcal{H}_0$, $K \in \mathcal{H}$ with $\varphi_1^L = \phi$ and $\varphi_1^K = \psi$. As above $H(x,t) = K(x,t) + L(\psi_t^{-1}(x),t)$ belongs to $\mathcal{H}$ and generates $\psi \circ \phi$. By Item 2 in Theorem 1.1

$$E^-(L) \le \gamma_\pm(H) - \gamma_\pm(K) \le E^+(L)\,,$$

from which Corollary 5.1 follows. □

Definition 5.1. A function $H \in \mathcal{H}_0$ is called quasi-autonomous if there are x_-, $x_+ \in U_1$ such that

$$H(x_-,t) = \min H_t\,, \quad H(x_+,t) = \max H_t$$

for all $t \in \mathbb{S}^1$.

Recall that $G_r(q,p) = (q, \exp_q(rp))$. Let $\mathcal{G}$ be the set of functions

$$S : G_r(U_{1+\delta}) \to \mathbb{R}$$

such that

- $\frac{\partial^2 S}{\partial q \partial Q}$ is negative definite,
- $S(q,Q) = \frac{d(q,Q)^2}{2r}$ for $d(q,Q) \geq r$.

Proposition 5.1. *Let S_t, $t \in [a,b]$ be a path in $\mathcal{G}$. Then the corresponding path φ_t in $\mathcal{T}$ is generated by a Hamiltonian $H \in \mathcal{H}_0$ satisfying the Hamilton–Jacobi equation*

$$\frac{\partial S_t}{\partial t}(q,Q) + H\left(Q, \frac{\partial S_t}{\partial Q}(q,Q), t\right) = 0\,. \tag{10}$$

Moreover, H is quasi-autonomous if and only if $\frac{\partial S_t}{\partial t}$ is.

The proof of Eq. (10) is standard and the proof of the last statement is the same as the given by Bialy and Polterovich [1].

Consider the length function $\mathcal{L}$ on the fundamental group of $\mathcal{D}_0$

$$\mathcal{L}([\varphi_t]) = \inf_{\psi_t \in [\varphi_t]} \text{length}(\psi_t)\,.$$

We claim that the image of $\mathcal{L}$ is $\{0\}$. In fact, given any loop $\varphi_t = (Q_t, P_t)$ in $\mathcal{D}_0$ with generating Hamiltonian $H \in \mathcal{H}_0$, we can define the loop $\psi : \mathbb{S}^1 \to \mathcal{D}_0$ by $\psi_t(q,p) = (Q_t(q,p/\varepsilon), \varepsilon P_t(q,p/\varepsilon))$ with generating Hamiltonian $F(q,p,s) = \varepsilon H(q,p/\varepsilon,s)$. Therefore

$$\text{length}(\psi) = \int_0^1 \|F_t\|\,dt = \varepsilon \int_0^1 \|H_t\|\,dt\,.$$

Thus, for any class $[\varphi] \in \pi_1(\mathcal{D}_0)$ we have $\mathcal{L}([\varphi]) = 0$.

Proposition 5.2. *There is a C^2 neighborhood $\mathcal{U}$ of zero in $\mathcal{H}_0$ such that*

$$\|\varphi_1^H\| = \int_0^1 \|H_t\|\,dt$$

for any quasi-autonomous $H \in \mathcal{U}$.

Proof. By Remark 3.3 and Item (ii) of Theorem 1.3 in [6] II, there is a C^2 neighborhood $\mathcal{U}$ of zero in $\mathcal{H}_0(U_1)$ such that for any quasi-autonomous $H \in \mathcal{U}$ the path $\varphi^H_{t\in[0,1]}$ is length-minimizing amongst all paths homotopic (with fixed end points) to $\varphi^H_{t\in[0,1]}$.

Suppose that ϕ is other path in $\mathcal{D}_0$ with the same end points and length$(\phi) <$ length(φ^H). Choose a loop $\psi \in [-\phi * \varphi^H]$ such that

$$\text{length}(\psi) < \text{length}(\varphi^H) - \text{length}(\phi)\,.$$

the path $\phi * \psi$ is homotopic to φ^H and shorter: a contradiction. □

Proof of Proposition 1.1. For φ_0, $\varphi_1 \in \mathcal{T}$, let S_0, $S_1 \in \mathcal{G}$ be their generating functions. For $t \in [0,1]$, $S_t = (1-t)S_0 + tS_1$ defines a map $\varphi_t \in \mathcal{T}$. Consider the path $\varphi_t\varphi_0^{-1}$ in $\mathcal{D}_0$ and its generating Hamiltonian H. As in [1], one easily shows that

$\frac{\partial S_t}{\partial t} = S_1 - S_0$ is quasi-autonomous. By Proposition 5.1, H is quasi-autonomous and

$$\|H_t\| = \|S_0 - S_1\| .$$

By Proposition 5.2, there is a C^1 neighborhood $\mathcal{O}$ of φ such that if φ_0, $\varphi_1 \in \mathcal{O}$, the length of the path $\varphi_t \varphi_0^{-1}$ equals $d(\varphi_0, \varphi_1)$. □

Appendix

Let θ be a closed 1-form in M such that $\theta \in [\omega]$. Let $\Gamma : [0, n] \to T^*M$ be a holonomic curve meaning that for $\gamma = \pi \circ \Gamma$ one has $\dot\gamma(t) = \partial H/\partial p(\Gamma(t), t)$. For each $(q, p, t) \in T^*M \times \mathbb{S}^1$, there is $\xi \in T_q^*M$ such that

$$H(q, \theta(q), t) = H(q, p, t) + (\theta(q) - p) \frac{\partial H}{\partial p}(q, p, t) + \frac{1}{2} p \frac{\partial^2 H}{\partial p^2}(q, \xi, t)\, p ,$$

so

$$(p - \theta(q)) \frac{\partial H}{\partial p}(q, p, t) - H(q, p, t) = -H(q, \theta(q), t) + \frac{1}{2} p \frac{\partial^2 H}{\partial p^2}(q, \xi, t) p .$$

Let $L : TM \times \mathbb{S}^1 \to \mathbb{R}$ be the Legendre transform of H. Since H is convex, for each $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}\int_0^n (L(\gamma, \dot\gamma, t) - \theta(\gamma)\dot\gamma)\, dt &= \int_\Gamma \lambda - \pi^*\theta - H dt \\ &\geq - \int_0^n H(\gamma(t), \theta(\gamma(t)), t)\, dt \geq - \int_0^n \max_{q \in M} H(q, \theta(q), t)\, dt \\ &= -n \int_0^1 \max_{q \in M} H(q, \theta(q), t)\, dt .\end{aligned}$$

As the left hand side does not depend on the representant of the class $[\omega]$, we have

$$\frac{1}{n} \int_0^n (L(\gamma, \dot\gamma, t) - \omega(\gamma)\dot\gamma)\, dt \geq \sup_{\theta \in [\omega]} - \int_0^1 \max_{q \in M} H(q, \theta(q), t)\, dt = -c(H, [\omega]) .$$

Since this holds for any holonomic curve, it follows from Proposition 1.1 in [8] that

$$-\alpha([\omega]) = \min \left\{ \int (L - \omega)\, d\mu : \mu \in \mathcal{M}(L) \right\} \geq -c(H, [\omega]) .$$

So

$$\alpha([\omega]) \leq c(H, [\omega]) .$$

Proof of Corollary 1.2. Since H is convex, for each t we have that

$$H_t|_L = \min_{x \in U_1} H(x, t) , \quad 0 = \max_{x \in U_1} H(x, t) .$$

Therefore

$$\hat{E}(\varphi_1^H) \le \int_0^1 \|H_t\|\, dt = -\int_0^1 H_t|_L\, dt \le -C_H \le \beta(0) \le \hat{E}(\varphi_1^H)\,. \qquad \square$$

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