Bol. Soc. Mat. Mexicana (3) Vol. 11, 2005

ON THE STOCHASTIC AUBRY-MATHER THEORY

RENATO ITURRIAGA AND HÉCTOR SÁNCHEZ-MORGADO

Abstract. In this paper we prove the differentiability of the stochastic analogues of Mather's functions α and β, introduced implicitly by D. Gomes [G]. We also prove that the solution to the viscous Hamilton Jacobi equation associated to α is differentiable in the parameter.

1. Introduction

In the work of Gomes [G] a stochastic analogue of Aubry Mather theory was developed. More precisely let $\mathbb{T}^d$ be the d dimensional torus, $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a strictly convex, C^∞, superlinear Lagrangian. The problem is to find a probability measure $\mu(x, v)$ on $\mathbb{T}^d \times \mathbb{R}^d$ that minimizes the average action

$$A(\mu) = \int L(x, v)d\mu$$

among the probability measures that satisfy

$$\int (d\phi(x)v + \varepsilon \bigtriangleup \phi(x))d\mu = 0 \tag{1.1}$$

for all C^2 functions ϕ. Measures satisfying (1.1) are called stochastic for the following reason. Consider a controlled Markov diffusion

$$dX = vdt + \sigma dW$$

where $W(t)$ is Brownian motion, and the measure μ_T is given by

$$\int_{\mathbb{T}^d} \phi d\mu_T = \frac{1}{T} E\big(\int_0^T \phi(X(t), v(t))dt\big).$$

If μ is a weak limit of a sequence μ_{T_n}, $T_n \to \infty$, using Dynkin's formula we have that μ satisfies (1.1) with $2\varepsilon = \sigma^2$.

For $\varepsilon = 0$, the nonstochastic case, (1.1) is the so-called holonomic condition and the previous fact is the analogous of the fact that weak limits of measures supported on liftings of closed curves are holonomic. According to results of Mather [Ma] and Mañé [M], holonomic minimizing measures are invariant.

Gomes proves that there is only one minimizing measure (Theorem 2). As in the nonstochastic case the theory is enhanced if we define a homology of rotation

---

2000 *Mathematics Subject Classification*: 37J50, 49L25, 70H20.
Keywords and phrases: Hamilton-Jacobi equation, Mather functions.
R. Iturriaga was partially supported by grant CONACYT 36496-E.
H. Sánchez-Morgado was partially supported by DGAPA-UNAM 100505 .

map. Let $\mathcal{M}$ be the space of Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ endowed with the weak topology and define the space

$$\mathcal{N}_\varepsilon = \mathrm{cl}\{\mu \in \mathcal{M} : \forall \phi \in C^2(\mathbb{T}^d) \int (\varepsilon \triangle \phi(x) + d\phi(x)v) d\mu = 0\}$$

Define $\rho : \mathcal{N}_\varepsilon \to H_1(M, \mathbb{R})$ by

$$\langle \rho(\mu), [\omega] \rangle = \int (\varepsilon \delta \omega(x) + \omega(x)v) d\mu \tag{1.2}$$

where δ is the adjoint of d for the flat metric, so that $\triangle \phi = \delta d\phi$.

It is an immediate consequence of the fact that μ belongs to $\mathcal{N}_\varepsilon$ that the integral in (1.2) does not depend on the representative of the cohomolgy class $[\omega]$. Thus, according to the Poincaré duality we can think of $\rho(\mu)$ as a homology class.

The function ρ is onto, see Lemma (2.1) below. Now we can define β_ε and α_ε, the stochastic analogues to Mather's functions β and α. See [Ma1].

$$\beta_\varepsilon(h) = \inf_{\mu \in \mathcal{N}_\varepsilon, \rho(\mu) = h} \int L(x, v) d\mu \tag{1.3}$$

$$\alpha_\varepsilon(\omega) = -\inf_{\mu \in \mathcal{N}_\varepsilon} \int L(x, v) - \omega d\mu \,. \tag{1.4}$$

For each function the infimum is achieved at a unique measure. This is an easy consequence of Theorem 2 in [G]. Due to the convexity of L it is easy to see that the functions α_ε, β_ε are convex and dual of one each another.

For $\varepsilon = 0$, Mather's function β is related to the stable norm for a Riemaniann metric g. Recall that for h in $H_1(M, \mathbb{R})$ the stable norm is defined as

$$||h|| = \inf\{\sum |r_i| l(\sigma_i) : \sum r_i \sigma_i \text{is a 1-cycle representing } h\}$$

where l is the length associated to the metric g. In this setting the stable norm is the square root of β.

Mather's function α coincides with the so called Mañé critical value [CIPP] and the effective Hamiltonian $\bar{H}(P)$ [EG]; that is, the unique number such that the equation

$$H(x, d\phi + P) = \bar{H}(P)$$

has a viscosity solution.

In the case of integrable systems, $\bar{H}(P)$ is a differentiable function which is the value of the Hamiltonian in action-angle variables.

According to Theorem 4 in [G], the function α_ε is characterized by

$$\alpha_\varepsilon(P) = \min_{h \in C^2} \max_x H(x, Dh(x) + P) + \varepsilon \triangle h(x) \,. \tag{1.5}$$

Moreover by Theorems 5, 6, 7 in [G], the value $\alpha_\varepsilon(P)$ is the only number such that

$$H(x, D\varphi(x) + P) + \varepsilon \triangle \varphi(x) = \alpha_\varepsilon(P) \tag{1.6}$$

has C^2 solutions $\varphi : \mathbb{T}^d \to \mathbb{R}$. Moreover, the solution is C^∞ by elliptic regularity and unique besides constants.

Mather's functions α and β are in general not differentiable. It is known since [Ma2] that for time periodic Lagrangians on the circle the function β is differentiable at irrational homologies and generically not differentiable at rational homologies. See also [Ba, Au]. For higher dimensions, to give conditions for the differentiability is still an active research area. See [BIK, Os, Mas, AB]

The first goal of this work is to prove the following.

Theorem (1.7). *For $\varepsilon > 0$ the functions α_ε and β_ε are C^∞.*

Varying P, equation (1.6) defines a family of PDE's having smooth solutions, and the second goal of the paper is to prove that the solution varies smoothly with respect to P.

Theorem (1.8). *If we normalize the solutions $\phi(\cdot, P)$ of (1.6) by fixing its value, say 0 at a point x_0, then $\phi \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$.*

Gomes [G] proved several properties under the assumption that ϕ or α_ε are differentiable. Most notably he proved the stochastic analogue of the fact that, at points ω of differentiability of the Mather α function, ω-minimizing orbits have asymptotic homology $h = D\alpha(\omega)$. More precisely he proved that

$$\lim_{t\to\infty} E\big(\frac{X(t)}{t}\big) = D\alpha_\varepsilon(P)$$

where $X(t)$ is the stochastic process given by

$$dX = \partial_p H(X, d\phi(X, P))dt + \sqrt{2\varepsilon}dW, \quad X(0) = x.$$

To prove the differentiability of the functions β_ε and α_ε we will actually prove that they are strictly convex. Therefore the dual functions, meaning respectively α_ε and β_ε, are differentiable.

The strict convexity of β_ε follows from the uniqueness of a density satisfying a stationary Focker-Plank equation. The strict convexity of α_ε follows from Lemma (2.7) interesting by itself, which claims that subsolutions of (2.5) are actually solutions.

Lemma (2.7) is the counterpart of a result of Fathi [F], saying that any critical subsolution is a solution in the Aubry set. Indeed, for $\varepsilon = 0$ the Aubry set contains the projection of the support of minimizing measures, and by theorem 9 in [G], the projection of minimizing measures for $\varepsilon > 0$ is the whole manifold.

We prove the smoothness results in Theorems (1.7) and (1.8) by induction, applying the implicit function theorem on Sobolev spaces together with some well known facts of elliptic operators.

We thank J. Ize for suggesting the use of the implict function theorem.

2. Differentiability of α_ε and β_ε

From now on we assume ε positive.

Lemma (2.1). *The map ρ is onto.*

The idea of the proof of this lemma is to adapt the proof of the same lemma for $\varepsilon = 0$. Since $\mathcal{N}_\varepsilon$ is convex, and ρ takes convex combinations to convex combinations, it is enough to prove that $me_i \in \rho(\mathcal{N}_\varepsilon)$ for any $m \in \mathbb{Z}$ and any of the standard generators e_i of $H_1(\mathbb{T}^d, \mathbb{Z}) \cong \mathbb{Z}^d$.

In the nonstochastic case one takes a multiple of the lift of a closed curve with homology e_i as a support of a measure. For $\varepsilon > 0$ we follow this idea, taking into account that the associated difusion process concentrates densities according to the stationary Fokker-Planck equation.

Consider $f : \mathbb{T}^{d-1} \to \mathbb{R}$ with a unique maximum and let

$$V = (D_1 f, \ldots, D_{d-1} f, m), \quad k = \int_{\mathbb{T}^{d-1}} \exp(f/\varepsilon).$$

Then $\theta(x) = \frac{1}{k}\exp(f(x_1, \ldots, x_{d-1})/\varepsilon)$ is a solution to the Fokker-Planck equation

$$\varepsilon \triangle \theta - \operatorname{div}(\theta V) = 0. \tag{2.2}$$

Consider the measure $\mu \in \mathcal{N}_\varepsilon$ defined by

$$\int \psi(x, v) d\mu = \int_{\mathbb{T}^d} \psi(x, V(x))\theta(x) dx\,. \tag{2.3}$$

Recalling (1.2), we have

$$\langle \rho(\mu), [\omega] \rangle = \int_{\mathbb{T}^d} \omega(x)(-\varepsilon d\theta(x) + \theta(x)V(x))dx = m \int_{\mathbb{T}^d} \omega_d(x)\theta(x)dx\,.$$

Taking the representative given by $\omega(x)v = Pv$ with $P \in \mathbb{R}^d$, we identify $[\omega]$ with P and have

$$\langle \rho(\mu), P \rangle = mP_d$$

so $\rho(\mu) = me_d$. Similarly for the other e_i's. □

Next we have

PROPOSITION (2.4). *The function β_ε is strictly convex.*

Suppose that there are $h_1, h_2 \in H_1(M, \mathbb{R})$ different such that

$$\beta_\varepsilon(th_1 + (1-t)h_2) = t\beta_\varepsilon(h_1) + (1-t)\beta_\varepsilon(h_2), \quad t \in [0, 1]\,.$$

Let μ_1 and μ_2 be minimizing measures with homologies h_1 and h_2. In particular $\mu_1 \neq \mu_2$.

Consider a supporting hyperplane of the epigraph of β_ε, defined by $P \in \mathbb{R}^d \cong H^1(M, \mathbb{R})$, containing the segment S from h_1 to h_2. Therefore

$$\beta_\varepsilon(h_1) - \langle h_1, P \rangle \leq \beta_\varepsilon(h) - \langle h, P \rangle$$

and the equality holds for $h \in S$. Thus, for any $h \in S$

$$\alpha_\varepsilon(P) = \langle h, P \rangle - \beta_\varepsilon(h).$$

By Theorem 8 in [G], this implies that any minimizing $\mu \in \mathcal{N}_\varepsilon$ with $\rho(\mu) \in S$ is supported on the graph of $V(x) = \partial_p H(x, D\phi(x) + P)$ for any solution ϕ to (1.6) and its density θ is a solution to (2.2), then it is unique and μ is given by (2.3). In particular $\mu_1 = \mu_2$, giving a contradiction. □

A function f_ε is said to be a critical subsolution of

$$H(x, D\phi(x)) + \varepsilon \triangle \phi(x) = c(\varepsilon) := \alpha_\varepsilon(0) \tag{2.5}$$

if

$$H(x, Df_\varepsilon(x)) + \varepsilon \triangle f_\varepsilon(x) \leq c(\varepsilon) \tag{2.6}$$

LEMMA (2.7). *Any critical subsolution f_ε of (2.5) is in fact a solution.*

Let B be the set of points where equality in (2.6) holds. Consider the Hamiltonian

$$\mathbb{H}(x,p) = H(x, p + Df_\varepsilon(x)) + \varepsilon \triangle f_\varepsilon(x)$$

with Lagrangian

$$\mathbb{L}(x,v) = L(x,v) - Df_\varepsilon(x)v - \varepsilon \triangle f_\varepsilon(x).$$

Write the solution ϕ_ε of (2.5) as $\phi_\varepsilon = \psi_\varepsilon + f_\varepsilon$. It follows that ψ_ε is a solution of

$$\mathbb{H}(x, D\psi_\varepsilon(x)) + \varepsilon \triangle \psi_\varepsilon(x) = c(\varepsilon). \tag{2.8}$$

Suppose $D\psi_\varepsilon(z) = 0$, then

$$\varepsilon \triangle \psi_\varepsilon(z) = c(\varepsilon) - \mathbb{H}(z,0) \geq 0.$$

Therefore, if z is a local maximum of ψ_ε, necessarily $\triangle\psi_\varepsilon(z) = 0$ and $\mathbb{H}(z,0) = c(\varepsilon)$ and so $z \in B$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a Brownian motion $W(t) : \Omega \to \mathbb{T}^d$ on the flat d-torus. We denote by $\mathbb{E}$ the expectation with respect to the probability measure $\mathbb{P}$. For $z \in B$, Lax's formula ([FS], Theorem IV 11.1) corresponding to (2.8) gives

$$\psi_\varepsilon(z) = \mathbb{E}\Big(\psi_\varepsilon(X_\varepsilon(T)) - \int_0^T \mathbb{L}(X_\varepsilon(s), u_\varepsilon(X_\varepsilon(s)))ds - c(\varepsilon)T\Big), \tag{2.9}$$

where $u_\varepsilon(x) = \partial_p H(x, D\phi_\varepsilon(x))$ is a maximal control and X_ε is the solution of the stochastic differential equation

$$\begin{cases} dX_\varepsilon(t) &= u_\varepsilon(X_\varepsilon(t))dt + \sqrt{2\varepsilon}\, dW(t)\,, \\ X_\varepsilon(0) &= z. \end{cases} \tag{2.10}$$

For a local maximum z of ψ_ε we have

$$\mathbb{E}\big(\psi_\varepsilon(X_\varepsilon(T)) \leq \psi_\varepsilon(z),$$

$$\mathbb{L}(x,v) + c(\varepsilon) \geq \mathbb{L}(x,v) + \mathbb{H}(x,0) \geq 0.$$

Thus $\psi_\varepsilon(z) = \mathbb{E}(\psi_\varepsilon(X_\varepsilon(T)))$ and

$$\mathbb{P}(\{\omega : \psi_\varepsilon(X_\varepsilon(T,\omega)) \neq \psi_\varepsilon(z)\}) = 0.$$

This implies that ψ_ε is a constant, therefore the subsolution f_ε is in fact a solution. □

PROPOSITION (2.11). *The function α_ε is strictly convex.*

Proof. Assume that there exist P and Q such that for any λ in $[0,1]$ we have $\alpha_\varepsilon(\lambda P + (1-\lambda)Q) = \lambda\alpha_\varepsilon(P) + (1-\lambda)\alpha_\varepsilon(Q)$.

Let f and g be solutions respectively of

$$\begin{aligned} H(x, df + P) + \varepsilon \triangle f &= \alpha_\varepsilon(P)\,, \\ H(x, dg + Q) + \varepsilon \triangle g &= \alpha_\varepsilon(Q)\,. \end{aligned}$$

Defining $h = \lambda f + (1-\lambda)g$ we have

$$
\begin{aligned}
dh + \lambda P + (1-\lambda)Q &= \lambda(df + P) + (1-\lambda)(dg + Q)\,, \\
H(x, dh + \lambda P + (1-\lambda)Q) + \varepsilon \triangle h &\le \lambda(H(x, df + P) + \varepsilon \triangle f) \\
&+ (1-\lambda)(H(x, dg + Q) + \varepsilon \triangle g) \\
&= \lambda \alpha_\varepsilon(P) + (1-\lambda)\alpha_\varepsilon(Q) \\
&= \alpha_\varepsilon(\lambda P + (1-\lambda)Q)\,.
\end{aligned}
\tag{2.12}
$$

Therefore h is a subsolution of

$$H(x, du + \lambda P + (1-\lambda)Q) + \varepsilon \triangle u = \alpha_\varepsilon(\lambda P + (1-\lambda)Q).$$

By Lemma (2.7) we have that h is in fact a solution and so inequality (2.12) is an equality. Since H is assumed to be strictly convex, it follows that $df + P = dg + Q$ at all points, so $P - Q = d(f-g)$ is an exact differential and then $P = Q$. □

LEMMA (2.13). *The functions $\alpha_\varepsilon, \beta_\varepsilon$ are C^1.*

Proof. According to Theorem 26.3 on page 253 of the book [R], the dual of a strictly convex function is differentiable. Morever, a differentiable convex function is always C^1 (Corollary 25.5.1, page 246 of the same book). □

3. Smoothness

Fix $x_0 \in \mathbb{T}^d$ and let $\phi(x, P)$ be the solution to (1.6) with $\phi(x_0, P) = 0$. We know that $\phi(x, P)$ is C^∞ in x and we are going to prove that it is also C^∞ in P. Letting $H^k(\mathbb{T}^d, \mathbb{R}^m)$ be the usual Sobolev space we have that $\phi(\cdot, P)$ belongs to $H^k(\mathbb{T}^d, \mathbb{R})$ for all P and k. Define

$$
\begin{aligned}
V(x, P) &= \partial_p H(x, \partial_x \phi(x, P) + P)\,, \\
M_P &= \varepsilon \triangle + V(x, P) : H^k(\mathbb{T}^d, \mathbb{R}) \to H^{k-2}(\mathbb{T}^d, \mathbb{R})\,,
\end{aligned}
$$

where the vector field $V(\cdot, P)$ acts as a first order differential operator.

We will use the following facts from the theory of elliptic operators. See for example Chapter 5 of [T].

1. A regularity result that says that if f is in H^{k-2} and $u \in H^1$ is a solution to $M_P u = f$, then u is in H^k.
2. $M_P : H^1 \to H^{-1}$ is Fredholm of index zero.
3. $\ker M_P$ is the set of constant functions.

LEMMA (3.1). *We have that $\phi \in C^1(T^d \times \mathbb{R}^d)$ and $\partial_P \phi(\cdot, P) \in C^\infty(\mathbb{T}^d)$.*

Proof. Let $Q \in \mathbb{R}^d$, let $g \in H^{k-2} - M_Q(H^k)$, $2k > d+2$, and define the map

$$
\begin{aligned}
F &: \mathbb{R}^d \times H^k(\mathbb{T}^d, \mathbb{R}) \to H^{k-2}(\mathbb{T}^d, \mathbb{R}) \\
F(P, \varphi) &= H \circ (I, d\varphi + P) + \varepsilon \triangle \varphi - \varphi(x_0) g - \alpha_\varepsilon(P)
\end{aligned}
\tag{3.2}
$$

where I is the identity map on $\mathbb{T}^d$.

Since $\phi(\cdot, P)$ is the solution to (2.5) with $\phi(x_0, P) = 0$, we have $F(P, \phi(\cdot, P)) = 0$. To prove the differentiability of ϕ with respect to P we use the implicit function theorem.

The map F is C^1 by the following facts:

- The map α_ε is C^1;
- Sobolev's inequality

$$\|\varphi\|_{C^{k-[\frac{d}{2}]-1,\frac{1}{2}}} \leq C(d,k)\|\varphi\|_{H^k}\,; \tag{3.3}$$

- If $G:\mathbb{R}^n \to \mathbb{R}^m$ is C^∞, then the map
$$\overline{G}: H^k(\mathbb{T}^d,\mathbb{R}^n) \to H^k(\mathbb{T}^d,\mathbb{R}^m), \quad \overline{G}(\Psi) = G\circ\Psi$$
is C^∞ and $D\overline{G} = \overline{DG}$.

The partial derivatives of F are

$$\begin{aligned} D_1F(P,\varphi) &= \partial_p H\circ(I, d\varphi + P) - d\alpha_\varepsilon\,, \\ D_2F(P,\varphi)\xi &= \partial_p H\circ(I, D\varphi+P)\cdot d\xi + \varepsilon \triangle \xi - \xi(x_0)g\,. \end{aligned}$$

Since $D_2F(Q,\phi(\cdot,Q))\xi = M_Q\xi - \xi(x_0)g$, it is invertible. By the implicit function theorem there is a neighbourhood U of Q such that the map

$$\psi: U \to H^k(\mathbb{T}^d,\mathbb{R}), \quad P \mapsto \phi(\cdot,P)$$

is C^1 and moreover $\Psi(x,P)h = d\psi(P)\cdot h(x)$ satisfies

$$M_P\Psi(x,P)h + V(x,P)\cdot h - (\Psi(x_0,P)h)g(x) - d\alpha_\varepsilon(P)h = 0. \tag{3.4}$$

For $P\in U$ fixed write

$$\phi(x,P+h) = \phi(x,P) + \Psi(x,P)h + \Phi_h(x)|h|$$

so we have that $\lim_{h\to 0}\|\Phi_h\|_{H^k} = 0$. By (3.3)

$$\lim_{h\to 0}\|\Phi_h\|_{C^{k-[\frac{d}{2}]-1,\frac{1}{2}}} = 0.$$

Thus, $\phi \in C^1(\mathbb{T}^d\times U)$ with

$$\partial_P\phi(x,P) = \Psi(x,P)$$

and so $\Psi(x_0,P) = 0$. Letting $\varphi_j = \partial_{P_j}\phi$, we have that φ_j is a solution of the affine PDE

$$M_P(\varphi_j) + V_j(x,P) = \partial_{P_j}\alpha(P) \tag{3.5}$$

and therefore $\varphi_j \in C^\infty(\mathbb{T}^d)$. □

Let $\theta(x,P)$ be the solution to the Fokker-Planck equation (2.2) with $\int\theta dx = 1$. Multiply each equation (3.5) by θ and integrate to obtain

$$\int \theta(x,P)V(x,P)dx = d\alpha_\varepsilon(P)\,. \tag{3.6}$$

We now prove by induction that α_ε and ϕ are smooth.

Lemma (3.7). *For any $r\in\mathbb{N}$, $\alpha_\varepsilon \in C^r(\mathbb{R}^d)$, $\phi\in C^r(\mathbb{T}^d\times\mathbb{R}^d)$ and $\partial_P^r\phi(\cdot,P)\in C^\infty(\mathbb{T}^d)$.*

Proof. Suppose the assertion of the Lemma holds for $r\in\mathbb{N}$. Thus $V \in C^r(\mathbb{T}^d\times\mathbb{R}^d)$. By the theory of elliptic operators quoted above, the adjoint $\mathbf{FP}_P: H^k \to H^{k-2}$ of M_P given by

$$\mathbf{FP}_P\varphi = \varepsilon\triangle\varphi - \operatorname{div}(\varphi V(x,P)) \tag{3.8}$$

satisfies properties (1) and (2) of M_P and $\dim\ker \mathbf{FP}_P = 1$ as well. For $Q \in \mathbb{R}^d$ let $g \in H^{k-2} - \mathbf{FP}_Q(H^k)$ and define

$$J : \mathbb{R}^d \times H^k \to H^{k-2}, \quad J(P,\varphi) = \mathbf{FP}_P\varphi + \left(\int \varphi - 1\right) g$$

so that $J(P,\theta(\cdot,P)) = 0$. The map J is C^r (affine in φ) with

$$D_2J(P,\varphi) = \mathbf{FP}_P + g\int .$$

By construction, any $D_2J(Q,\varphi)$ is invertible. By the implicit function theorem, $\theta \in C^r(\mathbb{T}^d \times U)$ for U a neighbourhood of Q, but Q is arbitrary. By (3.6), $\alpha_\varepsilon \in C^{r+1}(\mathbb{R}^d)$.

The map F defined in (3.2) is now C^{r+1} and so, by the implicit function theorem $\phi \in C^{r+1}(\mathbb{T}^d \times \mathbb{R}^d)$ and $\partial_P^{r+1}\phi(\cdot,P) \in C^\infty(\mathbb{T}^d)$. □

COROLLARY (3.9). *The function β_ε is smooth.*

Since α_ε is strictly convex and smooth, the map $d\alpha_\varepsilon$ has a smooth inverse γ_ε. Then $\beta_\varepsilon(h) = h\gamma_\varepsilon(h) - \alpha_\varepsilon(\gamma_\varepsilon(h))$ is smooth. □

Received May 21, 2004

Final version received February 03, 2005

R. ITURRIAGA
CIMAT
A.P. 402, 3600
GUANAJUATO. GTO, MÉXICO.
renato@cimat.mx

H. SÁNCHEZ-MORGADO
INSTITUTO DE MATEMÁTICAS, UNAM
CD. UNIVERSITARIA
MÉXICO DF 04510
MÉXICO
hector@matem.unam.mx

REFERENCES

[AIPS] N. ANANTHARAMAN, R ITURRIAGA, P. PADILLA, H. SÁNCHEZ-MORGADO, *Physical solutions of the Hamilton-Jacobi equation*, To appear in Discrete Cont. Dyn. Syst.

[Au] S. AUBRY, *The devil staircase transformation in incommensurate lattices*, The Riemann problem, complete integrability and aritmetic applications (Bures-sur-Yvette/New York, 1979/1980), Lecture Notes in Math. Springer **925** (1982), 221–245.

[AB] F. AUER, V. BANGERT, *Differentiability of the stable norm in codimension one*, Preprint http://arxiv.org/abs/math/0403235.

[Ba] V. BANGERT, *Geodesic rays, Busemann Function and monotone twist maps*, Calc. Var. Partial Differential Equations, Springer **2** (1) (1994), 49–63.

[BIK] D. BURAGO, S. IVANOV, B. KLEINER, *On the structure of the stable norm of periodic measures*, Math. Res. Lett. **4** (1997), 6 791–808.

[CIPP] G. CONTRERAS, R. ITURRIAGA, G. PATERNAIN, M. PATERNAIN, *Lagrangian Graphs, Minimizing Measures and Mañés critical values*, GAFA **8** (1998), 788–809.

[CIL] M.G. CRANDALL, H. ITOSHI, P.-L. LIONS, *User guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1), (1992), 1–67.

[EG] L.C. Evans, D. Gomes, *Effective Hamiltonians and averaging for Hamiltonian Dynamics*, Arch. Ration. Mech. Anal. **157** (2001), 1 1–33

[G] D. Gomes, *A stochastic analogue of Aubry-Mather theory*, Nonlinearity **15** (2002), 3, 581–603.

[F] A. Fathi, Weak KAM theorem in Lagrangian dynamics. Book to appear.

[FS] W. Fleming, M. Soner, Controlled Markov Processes and Viscosity Solutions. Springer, New York 1993.

[M] R. Mañé, *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity **9** (1996), 273–310.

[Mas] D. Massart, *Stable norms of surfaces: local structure of the unit ball of rational directions*, Geom. Funct. Anal. **7** (1997), no. 6, 996–1010.

[Ma] Personal communication.

[Ma1] J. Mather, *Action minimizing invariant measure for positive definite Lagrangian systems*, Math. Z. **207** (1991), 169–207.

[Ma2] J. Mather, *Differentiability of the minimal average action as a function of the rotation number*, Bol. Soc. Bras. Mat.**21** (1990), 59–70.

[Os] O. Osuna, *Vertices of Mather's beta function*, To appear in Ergodic Theory & Dynam. Systems.

[R] T. Rockafellar, Convex Analysis. Princeton University Press. Princeton 1972.

[T] M. E. Taylor, Partial Differential Equations, Vol I. Springer, New York 1996.