# REIDEMEISTER TORSION AND MORSE - SMALE FLOWS

HÉCTOR SÁNCHEZ MORGADO

Abstract. For several types of stable flows $\phi$ and representations $\rho$ of the fundamental group of the underlying manifold, R-torsion for $\rho$ can be computed from the periodic orbits of $\phi$. However, there are counterexamples and the purpose of this paper is to describe the role of heteroclinic orbits in such counterexamples.

## 0. Introduction

Let $M$ be a compact smooth manifold. A representation $\rho : \pi_1 M \to$ Gl(m,$\mathbb{C}$) defines a flat $\mathbb{C}^m$ bundle $E$. When the twisted cohomology $H^*(M; E)$ vanishes, the representation and the flat bundle are called *acyclic*. If moreover $|\det(\rho)| = 1$ one defines the positive number $\tau_\rho(M)$ called *R-torsion*.

Let $\phi$ be an axiom A flow on $M$. Taking into account the position in $\pi_1 M$ of its prime periodic orbits $\gamma$ one defines the torsion function

$$R_{\rho,\phi}(z) = \prod_\gamma \det(I - \Delta(\gamma) e^{-zl(\gamma)} \rho(\gamma))^{(-1)^{u(\gamma)}}$$

where $l(\gamma)$ is the period of $\gamma, u(\gamma)$ is the rank of the unstable bundle $E^u(\gamma)$ and $\Delta(\gamma) = \pm 1$ according to whether or not $E^u(\gamma)$ is orientable.

For a nonsingular Smale flow $\phi$ on $M$ one can construct an index filtration $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_N = M$. When $E|(M_i, M_{i-1})$ is acyclic for each ı, one can compute R-torsion as

$$\tau_\rho(M) = \prod_{i=1}^{N} \tau_\rho(M_i, M_{i-1})$$

2000 *Mathematics Subject Classification.* 58G10, 58F23.

Partially supported by DGAPA-UNAM.

and then prove the Lefschetz formula $\tau_\rho(M) = |R_{\rho,\phi}(0)|$. Our interest in this paper is to study the situation when not all $E|(M_i, M_{i-1})$ are acyclic and the Lefschetz formula may not hold. To do so we consider the spectral sequence defined by the filtration and following Witten's ideas we bring into play the orbits connecting the periodic orbits.

## 1. R-torsion and Spectral Sequences

Let $W$ be a finite dimensional vector space with basis $\mathbf{w} = \{w_1, \dots, w_n\}$, then $\wedge\mathbf{w} = w_1 \wedge \cdots \wedge w_n$ is a generator of $\lambda(W) = \wedge^n W$. If dim $V = 0$ set $\lambda(V) = \mathbb{C}$. If $V, W$ are finite dimensional vector spaces we let

$$\frac{\lambda(W)}{\lambda(V)} := \lambda(W) \otimes \lambda(V)^* \cong Hom(\lambda(V), \lambda(W)).$$

Consider a cochain complex of finite dimensional vector spaces

$$0 \to V^0 \xrightarrow{d} V^1 \to \cdots \to V^m \xrightarrow{d} 0 \tag{1.1}$$

Let $V^+ = \bigoplus_i V^{2i}$, $V^- = \bigoplus_i V^{2i+1}$ and

$$\lambda(V) = \frac{\lambda(V^-)}{\lambda(V^+)} \cong \lambda(V^-) \otimes \lambda(V^+)^*.$$

Let $Z^\pm = V^\pm \cap \ker\, d$, $B^\mp = d(V^\pm)$, $H^\pm = Z^\pm / B^\pm$, $T^\pm = V^\pm / Z^\pm$.

We now define the *torsion isomorphism* $\lambda(d) : \lambda(H) \to \lambda(V)$. Pick ordered relative bases $\mathbf{h}_\pm$ for $(Z^\pm, B^\pm)$ and $\mathbf{t}_\pm$ for $(V_\pm, Z_\pm)$, then $d\mathbf{t}_\mp$ is a basis for $B_\pm$. $\lambda(d)$ is given by

$$\frac{\wedge\mathbf{h}_-}{\wedge\mathbf{h}_+} \mapsto \frac{\wedge(\mathbf{t}_-, \mathbf{h}_-, d\mathbf{t}_+)}{\wedge(d\mathbf{t}_-, \mathbf{h}_+, \mathbf{t}_+)}. \tag{1.2}$$

*NOTATION.* Consider the cochain complex

$$0 \to V \xrightarrow{A} V \to 0$$

We have $H^0 = \ker A$, $H^1 =$coker$A$ and denote $\lambda(A) := \lambda(d)$. When $A$ is an isomorphism, $\lambda(A)$ is the coordinate free version of $\det(A)$.

**Proposition 1.** [5] *Let* $0 = F^i_{N+1} \subset F^i_N \subset \cdots \subset F^i_0 = V^i$ *be a filtration of the cochain complex (1.1) such that* $d^i(F^i_p) \subset F^{i+1}_p$*. Let* $\{E_r, d_r\}$ *be the corresponding spectral sequence. Then*

$$\lambda(d) = \lambda(d_0)\lambda(d_1)\cdots\lambda(d_N)$$

We will be interested in the first terms of the spectral sequence which we now describe. The filtration defines the associated graded complex $G^i = \bigoplus_p G^i_p$ where $G^i_p = F^i_p/F^i_{p+1}$. The coboundary $d$ induces a map $d^p_0 : G^i_p \to G^{i+1}_p$ whose cohomology defines the term $E_1$ by

$$E^{p,q}_1 := H^{p+q}(F_p/F_{p+1})$$

and induces the first differential $d^p_1 : E^{p,q}_1 \to E^{p+1,q}_1$ as the coboundary map for the short exact sequence

$$0 \to F_{p+1}/F_{p+2} \to F_p/F_{p+2} \to F_p/F_{p+1} \to 0$$

i.e. the map $d_1$ in the long exact sequence

(1.3)
$$\xrightarrow{j_p} H^{p+q}(F_p/F_{p+2}) \xrightarrow{k_p} H^{p+q}(F_p/F_{p+1}) \xrightarrow{d_1} H^{p+q+1}(F_{p+1}/F_{p+2})$$

The term $E^{p,q}_2$ is defined by

$$E^{p,q}_2 := \frac{\ker(d_1 : E^{p,q}_1 \to E^{p+1,q}_1)}{\operatorname{im}(d_1 : E^{p-1,q}_1 \to E^{p,q}_1)}.$$

From (1.3) we have $\ker(d_1)$ =$\operatorname{im}(k_p)$ and $\operatorname{im}(d_1) = \ker(j_{p-1})$, and thus $E^{p,q}_2 = \operatorname{im}(k_p)/\ker(j_{p-1})$.

Consider the commutative diagram

$$\begin{array}{ccccc} H^{p+q}(F_p/F_{p+2}) & \xrightarrow{k_p} & H^{p+q}(F_p/F_{p+1}) & \xrightarrow{j_{p-1}} & H^{p+q}(F_{p-1}/F_{p+1}) \\ \delta_0 & \nearrow & \delta_1 & \nearrow & \\ H^{p+q-1}(F_{p-2}/F_p) & \xrightarrow{k_{p-2}} & H^{p+q}(F_{p-2}/F_{p-1}) & \xrightarrow{j_{p-3}} & H^{p+q-1}(F_{p-3}/F_{p-1}), \end{array}$$

the second differential $d^{p-2}_2 : E^{p-2,q+1}_2 \to E^{p,q}_2$ is given as the composite map

$$\operatorname{im}(k_{p-2})/\ker(j_{p-3}) \xrightarrow{\delta_1} \operatorname{im}(j_{p-1}) \xrightarrow{j^{-1}_{p-1}} \operatorname{im}(k_p)/\ker(j_{p-1}) \tag{1.4}$$

Further terms of the spectral sequence $E_r^{p,q}$ are obtained as cohomology of the previous term and the differentials $d_r^p : E_r^{p,q} \to E_r^{p+r,q+r-1}$ are the maps induced by the original $d$.

Let now $K$ be a finite CW-complex. Let $p : \tilde{K} \to K$ be the universal covering and $\rho : \Gamma \to \mathrm{Gl(m,}\mathbb{C})$ be a representation of the fundamental group $\Gamma$ of $K$ which defines a flat vector bundle $E := \tilde{K} \times_\Gamma \mathbb{C}^m$. Lifting cells to $\tilde{K}$ we obtain a $\Gamma$-invariant CW complex structure on $\tilde{K}$. The space of $\rho$-equivariant cochains

$$C^*(K;E) = \{\xi \in C^*(\tilde{K};\mathbb{C}^m) : \xi \circ \gamma = \rho(\gamma) \circ \xi \quad \forall \gamma \in \Gamma\}$$

is preserved by $d^j : C^j(\tilde{K};\mathbb{C}^m) \to C^{j+1}(\tilde{K};\mathbb{C}^m)$ and so $\{C^*(K;E), d(K;E)\}$ forms a subcomplex. Its cohomology $H^*(K;E)$ is called the $\rho$-twisted cohomology of $K$. As usual $H^*(K;E)$ is subdivision invariant and we have a torsion isomorphism

$$\lambda(d(K;E)) : \lambda(H^*(K;E)) \to \lambda(C^*(K;E)).$$

Order the $j$-cells $\sigma$ and choose an oriented lift $\tilde{\sigma}$ for each $\sigma$. This gives an isomorphism $C^j(K;E) \cong \oplus_\sigma \mathbb{C}^m$ and determines a preferred generator $w_K^\rho$ of $\lambda(C^*(K;E))$ up to multiplication by an element of the subgroup

$$U_\rho = (\pm 1)^m \det \rho \subset \mathbb{C}^*, \quad \text{where} \quad \det \rho = \{\det \rho(\gamma) : \gamma \in \Gamma\}$$

The orbit $U_\rho w_K^\rho \subset \lambda(C^*(K;E))$ is invariant under subdivision, so we can define *R-torsion* of $K$ at $\rho$ as the $U_\rho$ orbit

$$\tau(K;E) = (U_\rho w_K^\rho)^* \lambda(d(K;E)) \subset \lambda(H^*(K;E))^* \tag{1.5}$$

which is invariant under subdivision. When $\rho$ is acyclic, i.e. when $H^*(K;E) = 0$, we have $\lambda(H^*(K;E)) = \mathbb{C}$ and we can identify $\tau$ as an element of $\mathbb{C}^*/U_\rho$. If moreover $\det(\rho) \subset S^1$, all elements in $\tau(K;E)$ have the same modulus which we denote by $\tau_\rho(M)$.

The previous definitions can be extended to relative pairs. Let $L$ be a subcomplex of K. For each $j$ we have the relative space of cellular $j$-cochains

$$C^j(K, L; \mathbb{C}) = \bigoplus_{\sigma \in K \setminus L} H^j(\sigma, \partial\sigma; \mathbb{C})$$

Let $\tilde{K}$ and $\rho$ be as above and let $\tilde{L} = p^{-1}(L)$. We can define the space of relative $\rho$-equivariant cochains $C^*(K, L; E) \subset C^*(\tilde{K}, \tilde{L}; \mathbb{C}^m)$ with coboundary $d(K, L; E)$ and then we get a torsion isomorphism

$$i(d(K, L; E)) : H^*(K, L; E) \to C^*(K, L; E).$$

Thus, choosing preferred basis as before we obtain a $U_\rho$ orbit

$$\tau(K, L; E) \in \lambda(H^*(K, L; E))^*$$

which is invariant under subdivision.

*Remark* 1. Another name for the twisted cohomology is cohomology with local coefficients. One chooses a point on each cell of $K$ and a path from a fixed point to each chosen point. In this way any path $c$ between chosen points defines a closed path $\gamma_c$ and then a matrix $\rho(c) := \rho(\gamma_c)$ which gives the relation between the coefficients at the ends of the path.

## 2. Morse-Smale flows

In this section we describe the contribution of heteroclinic orbits of a Morse- Smale flow in the computation of R-torsion. We will observe that the spectral sequence stops at $E_3$. Depending on the representation the sequence can stop before.

**Definition 1.** A smooth flow $\phi$ on a compact n-manifold $M$ is called Morse- Smale if

(1) The chain recurrent set $\mathcal{R}(\phi)$ consists of finitely many hyperbolic critical points and periodic orbits $\gamma_1, \ldots, \gamma_N$.

(2) For each pair $i, j$, $W^s(\gamma_i)$ intersects $W^u(\gamma_j)$ transversally. Hence, $C(i,j) = W^s(\gamma_i) \cap W^u(\gamma_j)$ is a submanifold of $M$.

By results of Conley [1] and Wilson [6], there is a smooth function $g : M \to \mathbf{R}$ called *Lyapunov function* such that $\frac{d}{dt} g \circ \phi_t(x) < 0$ if $x \notin \mathcal{R}(\phi)$ and $g$ is constant on each $\gamma_i$. If $C(i,j) \neq \emptyset$ for $i \neq j$ then $g(\gamma_j) > g(\gamma_i)$. $g$ can be chosen such that it takes different values on different components of $\mathcal{R}(\phi)$ and so we can order the components by their $g$-values. In such a case $[\gamma_1, \dots, \gamma_N]$ is an *admissible ordering of a Morse decomposition of M*. The rank $u_i$ of the unstable bundle $E^u(\gamma_i)$ is called the index of $\gamma_i$. If $C(i,j) \neq \emptyset$ for $i \neq j$ then $u_j \geq u_i$ with strict inequality if $\gamma_j$ is a critical point.

**Assumption.** $\phi$ will denote a Morse-Smale flow on an oriented manifold such that $u_j > u_i$ for $\gamma_i \neq \gamma_j$ with $C(i,j) \neq \emptyset$.

For $i < j$ define

$$M_{ij} = \{x \in M : \omega(x), \alpha(x) \subset \gamma_i \cup \cdots \cup \gamma_{j-1}\}$$

then $[\gamma_i, \dots, \gamma_{j-1}]$ is an admissible ordering of a Morse decomposition of $M_{ij}$.

For any sequence $1 = i_0 < i_1 < \cdots < i_l = N+1$, $[M_{i_0 i_1}, \dots, M_{i_{l-1} i_l}]$ is an admissible ordering of a Morse decomposition of $M$. Choosing numbers $g(\gamma_{i_1 - 1}) < c_1 < \dots < c_{l-1} < g(\gamma_{i_l - 1})$ and letting $M_j = g^{-1}(-\infty, c_j]$ we get an *index filtration*

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

for the Morse decomposition such that $\phi$ is transverse inwards on $\partial M_p$ and $M_p \setminus M_{p-1}$ is an *isolating neighborhood* of $M_{i_{p-1} i_p}$, meaning that

$$\bigcap_{t \in \mathbf{R}} \phi_t(M_p \setminus M_{p-1}) = M_{i_{p-1} i_p}.$$

Let $k(p)$ be the number of components of $\mathcal{R}(\phi)$ with index $p = 0, \dots, n$. We choose the sequence $i_p = 1 + k(0) + \cdots + k(p-1)$, $p = 1, \dots, n+1$ to get a filtration $M_0 \subset M_1 \subset \cdots \subset M_{n+1}$ such that

$u_i = p \Leftrightarrow \gamma_i \subset M_{p+1} \setminus M_p$. We also consider a filtration $K_0 \subset K_1 \subset \cdots \subset K_N$ for the Morse decomposition $[\gamma_1, \dots, \gamma_N]$.

Fix a representation $\rho : \pi_1(M) \to \mathrm{Gl(m,\mathbb{C})}$. All cochain complexes and cohomology groups will have coefficients in the flat bundle $E$ defined by $\rho$. Consider the filtration

$$F_n \subset \cdots \subset F_0$$

of $C^*(M)$ given by $F_p = \ker(C^*(M) \to C^*(M_p)$. The associated graded complex is given by

$$G_p = F_p / F_{p+1} \cong C^*(M_{p+1}, M_p)$$

with 0-differential $d_0 = \oplus_p d_0^p$ where $d_0^p : G_p \to G_p$, and torsion isomorphism $\lambda(d_0) = \otimes_p \lambda(d_0^p)$ where

$$\lambda(d_0^p) : \lambda(C^*(M_{p+1}, M_p)) \to \lambda(C^*(M_{p+1}, M_p))$$

**Lemma 1.** *Let $i < j < k$ and suppose there are no trajectories from $M_{jk}$ to $M_{ij}$, i.e. $C(s,l) = \emptyset$ for $i \le s < j \le l < k$. Take any index filtration*

$$\emptyset \subset N_1 \subset N_2 \subset N_3 \subset M$$

*with $M_{ij} \subset N_2 \setminus N_1$ and $M_{jk} \subset N_3 \setminus N_2$. Then*

$$H^*(N_3, N_1) \cong H^*(N_3, N_2) \oplus H^*(N_2, N_1)$$

*and the coboundary map of the triad $(N_3, N_2, N_1)$ vanishes.*

**Proof.** Since there are no trajectories from $M_{jk}$ to $M_{ij}$, we can replace $N_1, N_2$ by $N_1', N_2'$ to get an index filtration such that

$$cl(N_3 \setminus N_2') \cap cl(N_2' \setminus N_1') = \emptyset \tag{2.1}$$

By Conley's theory, the homotopy types of $(N_3, N_2)$ and $(N_2, N_1)$ do not change with the replacement . (2.1) implies that the couple $((N_3 \setminus N_2') \cup N_1', N_2')$ is excisive. Therefore

$$H^*((N_3 \setminus N_2') \cup N_1', N_1') \cong H^*(N_3, N_2').$$

This and the Mayer-Vietoris sequence

$$\to H^*(N_3, N_1') \xrightarrow{(j,k)} H^*(N_2', N_1') \oplus H^*((N_3 \setminus N_2') \cup N_1', N_1') \to$$
$$\to H^*(N_1', N_1') = 0$$

give $H^*(N_3, N_1') \cong \ker(k) \oplus \ker(j) \cong H^*(N_2', N_1') \oplus H^*(N_3, N_2')$. Consider now the exact sequence of the triad $(N_3, N_2', N_1')$

$$\xrightarrow{\delta} H^*(N_3, N_2') \to H^*(N_3, N_1') \xrightarrow{j} H^*(N_2, N_1') \xrightarrow{\delta}$$

$\ker(\delta) = \mathrm{im}(j) = H^*(N_2', N_1')$ and so $\delta = 0$. □

**Corollary 1.**

$$H^*(M_{p+1}, M_p; E) = \bigoplus_{u_j = p} H^*(K_j, K_{j-1}) \tag{2.2}$$

$$\lambda(d_0^p) = \bigotimes_{u_j = p} \lambda(d_0^{p,j}) \tag{2.3}$$

*where*

$$\lambda(d_0^{p,j}) = \begin{cases} 1 & \text{if } \gamma_j \text{ is a critical point} \\ \lambda(I - \Delta(\gamma_j)\rho(\gamma_j))^{(-1)^p} & \text{if } \gamma_j \text{ is a periodic orbit} \end{cases}$$

**Proof.** Since there are no trajectories from $\gamma_j$ to $M_{i_p j}$ for $i_p < j \leq i_{p+1} - 1$, we have

$$H^*(K_j, M_p) \cong H^*(K_j, K_{j-1}) \oplus H^*(K_{j-1}, M_p).$$

and so (2.2) follows by induction on $j$. Thus we have the decomposition (2.3) where

$$\lambda(d_0^{p,j}) : \lambda(H^*(K_j, K_{j-1})) \to \lambda(C^*(K_j, K_{j-1})$$

Let $D_j^u$ be the total space of the unstable disc bundle over $\gamma_j$, then $(K_j, K_{j-1})$ has the same simple homotopy type as $(D_j^u, \partial D_j^u)$.

If $\gamma_j$ is a critical point, $C^k(D_j^u, \partial D_j^u) = 0$ if $k \neq p$ and so $d_0^{p,j} = 0$.

If $\gamma_j$ is a periodic orbit,

$$C^k(D_j^u, \partial D_j^u) = 0 \quad k \neq p, p+1$$
$$C^p(D_j^u, \partial D_j^u) = \{\xi \in \prod_{l \in \mathbf{Z}} \mathbb{C}^m : \xi_l = \rho(\gamma_j)^l \xi_0\}$$
$$C^{p+1}(D_j^u, \partial D_j^u) = \{\eta \in \prod_{l \in \mathbf{Z}} \mathbb{C}^m : \eta_l = \rho(\gamma_j)^l \eta_0\}$$

Let $\xi = (\xi_l)_{l \in \mathbf{Z}} \in C^p(D_j^u, \partial D_j^u)$, then

$$(d\xi)_0 = (I - \Delta(\gamma_j)\rho(\gamma_j))\xi_0$$

where $\Delta(\gamma_j) = 1$ when $D_j^u$ is orientable and $-1$ otherwise. Hence the cochain complex $\{C^*(K_j, K_{j-1}), d_0^{p,j}\}$ reduces to

$$0 \to \mathbb{C}^m \xrightarrow{I - \Delta(\gamma_j)\rho(\gamma_j)} \mathbb{C}^m \to 0$$

and so $\lambda(d_0^{p,j}) = \lambda(I - \Delta(\gamma_j)\rho(\gamma_j))^{(-1)^p}$. □

By corollary (1), the computation of the map

$$d_1^p : H^*(M_p, M_{p-1}) \to H^{*+1}(M_{p+1}, M_p)$$

reduces to computing for each $i, j$ with $u_i = p, u_j = p+1$ its component $F_{ij}^* : H^*(K_i, K_{i-1}) \to H^{*+1}(K_j, K_{j-1})$.

Let $g : M \to \mathbf{R}$ be a Lyapunov function for $\phi$. The orientation of $M$ and the flow define an orientation of $L_c = g^{-1}(c)$ for each regular value $c$. Giving an orientation to the unstable subspace $E^u(x)$ for each critical point $x$ of $\phi$, and using the orientation of $M$ we also get an orientation of $E^s(x)$. Then we have orientations of $W^u(x)$ and $W^s(x)$. Let $x, y$ be critical points of $\phi$ of indices $p, p+1$ respectively and let $c$ be a regular value with $g(x) < c < g(y)$. Then $S^u(y) = W^u(y) \cap L_c$ and $S^s(x) = W^s(x) \cap L_c$ are oriented transverse submanifolds of $L_c$ with dimensions $p$ and and $n - p - 1$ respectively. Therefore $S^s(x) \cap S^u(y)$ is a finite set. For each $q \in S^s(x) \cap S^u(y)$ denote by $I_p$ the intersection number.

Henceforth we denote a critical point $\gamma_i$ of $\phi$ by $z_i$. The next lemma gives $F_{ij}^*$ for a pair $z_i, z_j$. The proof given by Floer [2], for the untwisted case can be readily adapted to our twisted situation. The only new ingredients are the matrices $\rho(\alpha)$ for nonclosed paths $\alpha$ used to define cohomology with local coefficients.

**Lemma 2.** *Let $z_i, z_j$ be critical points of $\phi$ of indices $p, p+1$. For each $q \in S(i,j) = S^s(z_i) \cap S^u(z_j)$ let $\alpha_q(t) = \phi_{\cot(\pi t)}(q) : t \in [0,1]$. We have*

$$F^p_{ij} = \sum_{q \in S(i,j)} I_q \rho(\alpha_q)$$

**Proposition 2.** [3] *Let $X$ be a Morse-Smale vector field on an oriented manifold and let $\gamma$ be a periodic orbit of index $p$. Given a small neighborhood $U$ of $\gamma$ there is a new More-Smale vector field $Y$ which agrees with $X$ outside $U$ and has rest points $x$, $y$ of indices $p, p+1$ in $U$ and no other chain recurrent point in $U$. For $Y$, $S^s(x) \cap S^u(y) = \{q_1, q_2\}$ with $I_{q_1} = -\Delta(\gamma) I_{q_2}$. Moreover $W^u(\gamma) = W^u(x) \cup W^u(y)$.*

We use this proposition to obtain a Morse-Smale flow $\psi$ such that each periodic orbit $\gamma_i$ is replaced by a pair of critical points $x_i, y_i$.

$M_0 \subset M_1 \subset \cdots \subset M_{n+1}$ is an index filtration for $\psi$ that we refine by choosing $N_1, \ldots, N_{n+1}$ such that $N_{p+1} \backslash M_p$ contains $\{z_j : u_j = p\} \cup \{x_i : u_i = p\}$ and $M_{p+1} \setminus N_{p+1}$ contains $\{y_i : u_i = p\}$. Thus, $H^k(N_{p+1}, M_p)$ and $H^k(M_p, N_p)$ vanish for $k \neq p$. Therefore the long exact sequence for the triad $(M_{p+1}, N_{p+1}, M_p)$ reduces to

$$0 \to H^p(M_{p+1}, M_p) \xrightarrow{\alpha_p} H^p(N_{p+1}, M_p) \xrightarrow{D_p}$$
$$H^{p+1}(M_{p+1}, N_{p+1}) \xrightarrow{\beta_p} H^{p+1}(M_{p+1}, M_p) \to 0$$

Hence $H^p(M_{p+1}, M_p) \cong \ker(D_p)$ and $H^{p+1}(M_{p+1}, M_p) \cong \mathrm{coker}(D_p)$. Note that $N_{p+1} \setminus N_p$ contains the critical points of $\psi$ of index $p$ and that the long exact sequence of the triad $(N_{p+1}, M_p, N_p)$ reduces to

$$0 \to H^p(N_{p+1}, M_p) \xrightarrow{l_p} H^p(N_{p+1}, N_p) \xrightarrow{m_p} H^p(M_p, N_p) \to 0$$

Consider the map $d_1^* : H^p(N_{p+1}, N_p) \to H^{p+1}(N_{p+2}, N_{p+1})$ corresponding to the flow $\psi$. Chasing diagrams we have $l_{p+1}\alpha_{p+1}d_1 = d_1^* l_p \alpha_p$, $\beta_{p+1} m_{p+2} d_1^* = d_1 \beta_p m_{p+1}$. Therefore $d_1$ on $H^p(M_{p+1}, M_p)$ and

$H^{p+1}(M_{p+1}, M_p)$ are maps induced by the components $d'_p : H^p(N_{p+1}, M_p) \to H^{p+1}(N_{p+2}, M_{p+1})$ and $d''_p : H^{p+1}(M_{p+1}, N_{p+1}) \to H^{p+2}(M_{p+2}, N_{p+2})$ of $d^*_1$ respectively. By corollary (1) we have (2.2) and

$$H^p(N_{p+1}, M_p) = \bigoplus_{u_j=p} \mathbb{C}^m z_j \oplus \bigoplus_{u_i=p} \mathbb{C}^m x_i,$$

$$H^{p+1}(M_{p+1}, N_{p+1}) = \bigoplus_{u_i=p} \mathbb{C}^m y_i.$$

Thus the computation of $d''_{p+1}$ reduces to computing its components $G^{p+1}_{y_i y_j}$ for $u_i = p$, $u_j = p+1$ and similarly for $d'_p$. The sequence of the triad $(M_{p+1}, N_{p+1}, M_p)$ decomposes as the sum of terms

$$0 \to H^p(K_i, K_{i-1}) \xrightarrow{\alpha_p} \mathbb{C}^m x_i \xrightarrow{D_p} \mathbb{C}^m y_i \xrightarrow{\beta_p} H^{p+1}(K_i, K_{i-1}) \to 0$$

for periodic orbits of $\phi$ and

$$0 \to H^p(K_j, K_{j-1}) \to 0 \to 0 \to H^{p+1}(K_j, K_{j-1}) \to 0$$

for critical points of $\phi$.

**Theorem 1.** *Let $\gamma_i, \gamma_j$ be periodic orbits of $\phi$ of indices $p, p+1$ respectively, replaced with pairs of critical points $x_i, y_i$ and $x_j, y_j$. Let $S_p(i,j) = S^s(x_i) \cap S^u(x_j)$ and let $S_{p+1}(i,j) = S^s(y_i) \cap S^u(y_j)$. For $k = p, p+1$ we have*

$$F^k_{ij} = \sum_{q \in S_k(i,j)} I_q \rho(\alpha_q)$$

*where $\alpha_q$ is as in Lemma (2).*

**Proof.** According to the decomposition, $l_{p+1}\alpha_{p+1}F^p_{ij} = G^p_{x_i x_j} l_p \alpha_p$ and $\beta_{p+1} m_{p+2} G^{p+1}_{y_i y_j} = F^{p+1}_{ij} \beta_p m_{p+1}$. Thus the maps $F^p_{ij}$ and $F^{p+1}_{ij}$ are induced by $G^p_{x_i x_j}$ and $G^{p+1}_{y_i y_j}$. Hence, Theorem 1 follows from Lemma 2. □

We note that

$$E^{p,q}_1 = H^{p+q}(M_{p+1}, M_p) \cong \bigoplus_{u_j=p} H^{p+q}(K_j, K_{j-1})$$

vanishes for $q \neq 0, 1$ and so does $E_2^{p,q}$. Therefore $d_2 : E_2^{p,q+1} \to E_2^{p+1,q}$ is nonzero only for $q = 0$ and $d_r = 0$ for $r \geq 3$, i.e. the sequence stops at $E_3$. By section (1) the map $d_2^{p-1}$ is given as the composition

$$\mathrm{im}(k_{p-1})/\ker(j_{p-2}) \xrightarrow{\delta} \mathrm{im}(j_p) \xrightarrow{j_p^{-1}} \mathrm{im}(k_{p+1})/\ker(j_p)$$

where $k_p : H^p(M_{p+2}, M_p) \to H^p(M_{p+1}, M_p)$,
$j_p : H^{p+1}(M_{p+2}, M_{p+1}) \to H^{p+1}(M_{p+2}, M_p)$, and
$\delta : H^p(M_p, M_{p-1}) \to H^{p+1}(M_{p+2}, M_p)$.

Consider the commutative diagram

$$\begin{array}{ccccc}
H^p(M_p, M_{p-1}) & \xrightarrow{\delta} & H^{p+1}(M_{p+2}, M_p) & \xleftarrow{j_p} & H^{p+1}(M_{p+2}, M_{p+1}) \\
\beta_{p-1} \uparrow & & k \downarrow & & \alpha_{p+1} \downarrow \\
H^p(M_p, N_p) & \xrightarrow{\Delta} & H^{p+1}(N_{p+2}, M_p) & \xleftarrow{j} & H^{p+1}(N_{p+2}, M_{p+1}) \\
m_p \uparrow & & h \uparrow & \swarrow & l_{p+1} \\
H^p(N_{p+1}, N_p) & \xrightarrow{d_1^*} & H^{p+1}(N_{p+2}, N_{p+1}) & &
\end{array}$$

Suppose $\delta\beta_{p-1}(\xi) = j_p(\eta)$, then $\Delta(\xi) = kj_p(\eta) = j\alpha_{p+1}(\eta)$.
Let $\zeta \in m_p^{-1}(\xi)$, then $hd_1^*(\zeta) = hl_{p+1}\alpha_{p+1}(\eta)$.
Thus there is a $\chi \in H^p(N_{p+1}, M_p)$ such that
$d_1^*(\zeta) = l_{p+1}\alpha_{p+1}(\eta) + d_1^* l_p(\chi)$.
Letting $\zeta' = \zeta + l_p(\chi)$ we have $\delta\beta_{p-1}m_p(\zeta') = j_p(\eta)$ and
$d_1^*(\zeta') = l_{p+1}\alpha_{p+1}(\eta)$.
Thus $\delta$ :ker$(k_{p-1})$ →im$(j_p)$ is induced by the component
$d''' : H^p(M_p, N_p) \to H^{p+1}(N_{p+2}, M_{p+1})$ of $d_1^*$ and we obtain

**Theorem 2.** *The map $d_2^{p-1} : E_2^{p,1} \to E_2^{p+1,0}$ is induced by the map*

$$d''' : \bigoplus_{u_k=p-1} \mathbb{C}^m y_k \to \bigoplus_{u_j=p+1} \mathbb{C}^m z_j \oplus \bigoplus_{u_i=p+1} \mathbb{C}^m x_i$$

*with components $G_{y_k z_j}, G_{y_k x_i}$ given as in Lemma (2).*

## 3. Example: Seifert manifolds

We now give an instance of our theorems on Seifert manifolds, where Fried

[4] has found counterexamples to Lefschetz formulae.

Let $M$ be an oriented closed Seifert 3-manifold, i.e. $M$ carries a locally free circle action. Denote by $\gamma_1, \dots, \gamma_n$ the singular orbits

and let $N_1, \dots, N_n$ be neighborhoods of these orbits, invariant under the

circle action. Then $N_j \cong D^2 \times S^1$ with the circle action given by $e^{it}(z, e^{is}) = (ze^{its_j}, e^{i(s+n_jt)}))$ where $s_j, n_j$ are relatively

prime integers. $E_N = cl(M \setminus \cup_{j=1}^n N_j)$ is a circle bundle over a compact oriented surface $\Sigma$ with nonempty boundary, thus

$H^2(\Sigma, \mathbf{Z}) = 0$ and then $E_N = \Sigma \times S^1$. Let $J : \Sigma \to M$

be the inclusion $J(x) = (x, 1)$. Let $T \in \pi_1 M$ be the homotopy class of the general Seifert orbit, then $\pi_1 M$ is generated by $T$ and $J_*(\pi_1 M)$.

$T = \gamma_j^{n_j}$ and $c_j = \partial N_j \cap \Sigma \times \{1\}$ form a basis of

$\pi_1(\partial N_j)$, thus there are integers $r_j, m_j$ such that

$n_j r_j - m_j s_j = 1$ and $c_j = \gamma_j^{m_j}$.

Let $\phi$ be a nonsingular Morse-Smale flow on $M$ commuting with the circle action. Thus its periodic orbits are Seifert orbits as well. We assume that the singular Seifert orbits are periodic orbits of $\phi$ of index zero. Notice that $\phi$ defines a Morse-Smale flow $\phi^*$ on the orbifold $M/S^1$ with the periodic orbits of $\phi$ corresponding to critical points $z_j$ of $\phi^*$.

We have $W^u(\gamma_j)/S^1 = W^u(z_j)$. If $i > n$ then $C(i,j) = (W^u(z_j) \cap W^s(z_i)) \times S^1$. For $j \leq n$, $H^0(K_j, K_{j-1}) = \ker(I - \rho(\gamma_j))$ and $H^1(K_j, K_{j-1}) = \operatorname{coker}(I - \rho(\gamma_j))$. For $j < n$ and $u_j = p$, $H^p(K_j, K_{j-1}) = \ker(I - \rho(T))$ and $H^{p+1}(K_j, K_{j-1}) = \ker(I - \rho(T))$.

We replace periodic orbits $\gamma_j$ with pairs of critical points $x_i, y_i$ as follows. If $j > n$ let $x_j = (z_j, 1)$, $y_j = (z_j, -1)$ where $z_j \in \Sigma$ and so $W^u(x_j) \cap E_N = (W^u(z_j) \cap \Sigma) \times \{1\}$. If $j < n$ let $x_j = (0,1), y_j = (0,-1) \in D^2 \times S^1$ so that $S^s(y_j) = W^s(y_j) \cap \partial N_j = \partial D^2 \times \{-1\}$.

For $i > n, u_i = p, u_j = p + 1$, let $S^*(i,j) = S^s(z_i) \cap S^u(z_j)$, then $S_p(i,j) = S^* \times \{1\}$ and $S_{p+1}(i,j) = S^*(i,j) \times \{-1\}$. For $q \in S^*(i,j)$ we have paths $\alpha_{(q,\pm 1)} = \alpha_q \times \{\pm 1\}$. Theorem (1) gives that $F_{ij}^p, F_{ij}^{p+1}$ are the operators induced on $\ker(I - \rho(T))$ and $\operatorname{coker}(I - \rho(T))$ by

$$\sum_{q \in S^*(i,j)} I_q \rho(J_*(\alpha_q)).$$

If $l \leq n$ and $u_j = 1$, $W^u(z_j) \setminus \{z_j\}$ is the union of two trajectories of $\phi^*$, then $S^*(l,j) = W^u(z_j) \cap \partial\Sigma$ is a set with at most two points and $W^u(\gamma_j) \cap \partial N_l$ is the union of the Seifert orbits over these points. Since $W^s(y_l) \cap W^u(x_j) = \emptyset$ by transversality, we have $S_1(l,j) = S^s(y_l) \cap W^u(\gamma_j)$. For each $q \in S^*(l,j)$, $S_1(l,j)$ contains $\{b_1, \dots, b_{n_l}\}$ where $b_k = (z\exp(2ks_l\pi i/n_l), -1)$. For a path $\beta : [0,1] \to M$ we put $\beta^+ = \beta|[1/2,1]$. Theorem (1) gives

$$F^1_{lj} = \sum_{q\in S^*(l,j)} I_q \sum_{k=1}^{n_l} \rho(\alpha_{b_k}) = \sum_{q\in S^*(l,j)} I_q \rho(J_*(\alpha^+_q)) \sum_{k=0}^{n_l-1} \rho(\gamma_l)^k$$

$$F^0_{lj} = \sum_{q\in S^*(l,j)} I_q \rho(J_*(\alpha^+_q)).$$

We now consider the map $d_2$. If $l > n$, $W^s(y_l) \cap W^u(x_j) = \emptyset$ and therefore $G_{y_l,x_j} = 0$. If $l \leq n$, $S^s(y_l) \cap c_l = \bigcup_{u_j=2} S^s(y_l) \cap S^u(x_j)$.

Let $Q_l = \{j : u_j = 2, W^s(y_l) \cap W^u(x_j) \neq \emptyset\}$. For $j \in Q_l$, $S^s(y_l) \cap S^u(x_j) = \{(w_j \exp(2kr_l\pi i/m_l), -1) : 0 \leq k \leq p_j - 1\}$ where $a_j = (w_j, -1) \in S^s(y_l)$. We have $a_j = (q_j, 1) \in E_N$ with $q_j \in S^*(l,j)$ and

$$G_{y_l,x_j} = I_{q_j} \rho(J_*(\alpha_{q_j})) \sum_{k=0}^{p_j-1} \rho(\gamma_l)^k$$

## References


[1] C. Conley. *Isolated invariant sets and the Morse index* Conf. Series in Math **38** (1978).

[2] A. Floer. *Witten's complex and infinite dimensional Morse Theory.* J. Diff. Geom. **30**(1989) 207-221.

[3] J. Franks. *Morse-Smale flows and homotopy theory.* Topology. **18** (1979) 196-215.

[4] D. Fried. *Counting Circles.* Springer LNM **1342** (1988) 196-215.

[5] D. Fried. *R-torsion for nonunitary representations.* Notes.

[6] W. Wilson. *Smoothing derivatives of functions and applications.* Trans. Amer. Math. Soc. **139** (1969) 413-428.



Instituto de Matemáticas. Universidad Nacional Autónoma de México., Ciudad Universitaria C. P. 04510, México D. F., México

*E-mail address*: hector@matem.unam.mx