RATIONAL CURVES ON $K3$ SURFACES

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1. Basics

1.1. $K3$ **surfaces.** A Calabi-Yau (CY) manifold is a compact Kähler manifold that

- has trivial canonical bundle and
- is simply connected.

A $K3$ surface X is a CY manifold of (complex) dimension $\dim_{\mathbb{C}} X = 2$.

Theorem 1.1 (Siu)**.** *Let X be a simply-connected compact complex surface with the trivial canonical bundle. Then X is Kähler.*

A quick computation gives the Hodge numbers:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{1.1}$$

Here $H^0(X, \mathbb{Z}) = \mathbb{Z}$, $H^1(X, \mathbb{Z}) = 0$ and $H^{2,0}(X) = H^0(K_X) = \mathbb{C}$ follow directly from the definition, while $h^{1,1}(X) = 20$ follows from Noether's formula:

$$\chi_{top}(X) = 12(K_X^2 + \chi(\mathcal{O}_X)) = 24 \tag{1.2}$$

So $H^2(X) = \mathbb{C}^{22}$. A subtle point here is that $H^2(X, \mathbb{Z}) = \mathbb{Z}^{22}$ is torsion free. This follows from Lefschetz $(1, 1)$ theorem and Riemann-Roch.

Proposition 1.2. *$H^2(X, \mathbb{Z})$ is torsion free for a $K3$ surface X.*

Proof. By Lefschetz $(1, 1)$ theorem, every torsion element of $H^2(X, \mathbb{Z})$ lies in the image of $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. If $H^2(X, \mathbb{Z})$ is not torsion free, there exists a line bundle L such that $L \neq \mathcal{O}_X$ and $L^{\otimes m} = \mathcal{O}_X$ for some $m > 1$. By Riemann-Roch,

$$h^0(L) - h^1(L) + h^0(L^{-1}) = 2 \tag{1.3}$$

Therefore $h^0(L) + h^0(L^{-1}) \geq 2$. So at least one of L and L^{-1} is effective. WLOG, assume that $h^0(L) > 0$. Let $s \in H^0(L)$. Then $s^m \in H^0(\mathcal{O}_X)$.

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Consequent, s nowhere vanishes and hence $L = \mathcal{O}_X$. Contradiction. $\square$

There are (at least) two conclusions we can draw from $h^{1,1}(X) = 20$. First, since $H^1(X, \mathcal{O}_X) = 0$, we see that

Proposition 1.3. $\mathrm{Pic}(X)$ *is a lattice of rank at most* 20 *contained* $H^2(X, \mathbb{Z}) = \mathbb{Z}^{22}$.

1.2. **Deformations of $K3$ surfaces.** Let $\pi : M \to S$ be a smooth family of compact complex manifolds. Restricting the sequence

$$0 \to T_{M/S} \to T_M \to \pi^* T_S \to 0 \tag{1.4}$$

to a point $0 \in S$ and taking the long exact sequence, we obtain

$$\begin{tikzcd} H^0(\pi_* T_S\big|_0) \arrow[r] \arrow[d, equal] & H^1(T_{M/S}\big|_0) \arrow[d, equal] \\ T_{S,0} \arrow[r, "\mathrm{ks}"] & H^1(T_{M_0}) \end{tikzcd} \tag{1.5}$$

where $M_0 = \pi^{-1}(0)$. Here ks is call the Kodaira-Spencer map of the family M/S and $\mathrm{ks}(\partial/\partial t)$ is the Kodaira-Spencer class of M/S if S is the unit disk $\Delta = \{|t| < 1\}$.

If ks is an isomorphism, M/S is called a versal deformation space of M_0. A theorem of Kuranishi says that if $H^2(T_X) = 0$ for $X = M_0$, then the versal deformation space of X exists and is smooth of dimension $h^1(T_X)$.

For $K3$ surfaces, by Serre duality

$$H^{1,1}(X) = H^1(\Omega_X) = H^1(T_X)^\vee = \mathbb{C}^{20} \tag{1.6}$$

This, along with $H^2(T_X) = H^0(\Omega_X)^\vee = 0$, implies that the versal deformation space of X is smooth of dimension 20. That is, the moduli space of $K3$ surfaces, if exists, has dimension 20. In case that X is projective, a general deformation of X is, however, no longer algebraic.

1.3. **Gauss-Manin Connection.** Let M/S be a smooth family of $K3$ surfaces. We know that $R^2\pi_*\mathbb{C}$ is a flat complex vector bundle. Diffeomorphically and locally, $R^2\pi_*\mathbb{C} \cong H^2(M_0, \mathbb{C}) \times U$ for a polydisk U around 0. There is a natural flat connection call *Gauss-Manin* connection

$$\nabla : \mathcal{O}_S \otimes R^2\pi_*\mathcal{C} \to \Omega_S^1 \otimes R^2\pi_*\mathcal{C} \tag{1.7}$$

satisfying that $\nabla^2 = 0$ and the Griffiths transversality:

$$\nabla\big(\mathcal{O}_S \otimes F^p R^2\pi_*\mathcal{C}\big) \subset \Omega_S^1 \otimes F^{p-1} R^2\pi_*\mathcal{C}. \tag{1.8}$$

For a holomorpic tangent vector $u \in T_{S,0}$, we define

$$\nabla_u \omega = \langle \nabla\omega, u\rangle \in F^{p-1}H^2(X) \tag{1.9}$$

for $\omega \in H^0(\mathcal{O}_S \otimes F^p R^2\pi_*\mathcal{C})$ for S a polydisk around 0 and $M_0 = X$. Then we have well-defined maps

$$\mathbf{P}_{0,2} \circ \nabla_u : H^{1,1}(X) \to H^{0,2}(X) \tag{1.10}$$

and

$$\mathbf{P}_{1,1} \circ \nabla_u : H^{2,0}(X) \to H^{1,1}(X) \tag{1.11}$$

where $\mathbf{P}_{m,n}$ are the projections $H^{m+n}(\bullet) \to H^{m,n}(\bullet)$.

Take $p = 1, 2$ and the maps (1.10) and (1.11) are given by the pairings

$$H^1(T_X) \times H^{1,1}(X) \to H^{0,2}(X) \tag{1.12}$$

and

$$H^1(T_X) \times H^{2,0}(X) \to H^{1,1}(X), \tag{1.13}$$

via the natural maps $T_X \otimes \Omega_X \to \mathcal{O}_X$ and $T_X \otimes \wedge^2\Omega_X \to \Omega_X$, respectively. It turns out both (1.12) and (1.13) are nondegenerate pairings.

One consequence of the above discussion is that a general $K3$ surface is not projective. If M/S is a projective family of $K3$ surfaces, then there exists a nontrivial line bundle $L \in \mathrm{Pic}(M)$ over S. Since $c_1(L) \in H^0(R^2\pi_*\mathbb{Z})$,

$$\nabla c_1(L) = 0. \tag{1.14}$$

Consequently,

$$\langle \mathrm{ks}(u), c_1(L)\rangle = 0 \tag{1.15}$$

for all $u \in T_{S,0}$. By the nondegeneracy of the pairing (1.12), the image of the Kodaira-Spencer map has dimension at most 19. Therefore, M/S cannot be a versal deformation space of $X = M_0$. In other words, if M/S is a versal deformation space of X, a general fiber M_s of M/S is a $K3$ surface with Picard group $\mathrm{Pic}(M_s) = 0$.

1.4. **Examples.** The simplest examples of $K3$ surfaces are complete intersections in $\mathbb{P}^n$. Let $X \subset \mathbb{P}^n$ be a complete interesction cut out by hypersurfaces of degrees $d_1, d_2, ..., d_{n-2}$. By weak Lefschetz, $\pi_1(X) = 0$. By adjunction,

$$K_X = \mathcal{O}_X \Leftrightarrow d_1 + d_2 + ... + d_{n-2} = n + 1 \tag{1.16}$$

We also require that X be nondegerate, i.e., $d_i \geq 2$. Therefore, here are all the possibilities:

1. $X \subset \mathbb{P}^3$ a quartic surface;
2. $X \subset \mathbb{P}^4$ a complete intersection of a quadric and a cubic;

(3) $X \subset \mathbb{P}^5$ a complete intersection of three quadrics.

Theorem 1.4 (Noether-Lefschetz)**.** *For a very general surface* $X \subset \mathbb{P}^3$ *of degree* $d \geq 4$*,* $\operatorname{Pic}(X) \cong \mathbb{Z}$ *is generated by the hyperplane section* $\mathcal{O}_X(1)$*.*

Noether's original argument is seriously flawed. Lefschetz gave the first correct proof using the powerful tool of Hodge theory. Later Griffiths and Harris gave a more elementary proof using degeneration. Both proofs are quite novel in their own ways. We will give outlines of both proofs.

Lefschetz's proof of Noether-Lefschetz. Let $W \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a Lefschetz pencil of quartic surfaces, B be the finite set of points b over which the fiber W_b is singular and $S = \mathbb{P}^1 \backslash B$. Lefschetz's famous argument shows that the monodromy action of $\pi_1(S)$ on $H^2_{\text{prim}}(X, \mathbb{Q})$ is irreducible for some $0 \in S$ and $X = W_0$, where

$$H^2_{\text{prim}}(X, \mathbb{Q}) = \{\eta \in H^2(X, \mathbb{Q}) : \langle \eta, c_1(L) \rangle = 0\} \tag{1.17}$$

and $L = \mathcal{O}_X(1)$. For a line bundle $M_b \in \operatorname{Pic}(W_b)$ for $b \in S$ general, it is easy to see that $\gamma(c_1(M_b))$ is a Hodge $(1,1)$ class in $H^2(W_b)$ for every $\gamma \in \pi_1(S)$. If M_b is not a multiple of L, $\gamma(c_1(M_b))$ generates $H^2(X, \mathbb{Q})$ for $\gamma \in \pi_1(S)$ by the irreducibility of the action $\pi_1(S)$ on $H^2_{\text{prim}}(X, \mathbb{Q})$. This is impossible since $H^{2,0}(X) \neq 0$. Contradiction. □

Griffiths-Harris' Proof of Noether-Lefschetz. Let $W \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a pencil of quartic surfaces containing a member $W_0 = S_1 \cup S_2$, where S_1 and S_2 are quadric surfaces. Let $D = S_1 \cap S_2$. The 3-fold W has 16 rational double points $p_1, p_2, ..., p_{16}$ on D. Blowing up W along S_1 resolves these singularities. Let Z be the resulting 3-fold, $Z_0 = R_1 \cup R_2$ and $D = R_1 \cap R_2$. It is easy to see that R_1 is the blowup of S_1 at $p_1, p_2, ..., p_{16}$ and $R_2 \cong S_2$. We have the left exact sequence

$$0 \to \operatorname{Pic}(W_0) \to \operatorname{Pic}(R_1) \oplus \operatorname{Pic}(R_2) \to \operatorname{Pic}(D) \tag{1.18}$$

over $\mathbb{Z}$. We can show that for $L_1 \in \operatorname{Pic}(R_1)$ and $L_2 \in \operatorname{Pic}(R_2)$,

$$L_1\Big|_D = L_2\Big|_D \tag{1.19}$$

if and only if $L_1 = aL + bK_{R_1}$ and $L_2 = aL - bK_{R_2}$ for some $a, b \in \mathbb{Z}$. Then $\operatorname{Pic}(W_0) = \mathbb{Z} \oplus \mathbb{Z}$ and $\operatorname{Pic}(W_t) = \mathbb{Z}$ for $t \in \mathbb{P}^1$ general. □

The key fact in Lefschetz's proof is $H^{2,0}(X) \neq 0$ while Griffiths-Harris' proof relies on the fact that $g(D) > 0$ for the degenerated $K3$ surface $X = S_1 \cup S_2$; both are the consequence of $H^0(K_X) \neq 0$.

A consequence of this theorem is

Corollary 1.5. *Let X_1 and X_2 be two very general quartic surfaces and let $f : X_1 \to X_2$ be an isomorphism. Then f is induced by an action of $\mathbb{P}GL(4)$.*

Proof. f induces an isomorphism $\mathrm{Pic}(X_2) \to \mathrm{Pic}(X_1)$. Obviously, $f^*\mathcal{O}_{X_2}(1) = \mathcal{O}_{X_1}(1)$ and it also induces a linear map $|\mathcal{O}_{X_2}(1)| \to |\mathcal{O}_{X_1}(1)|$. Both $|\mathcal{O}_{X_i}(1)| \cong |\mathcal{O}_{\mathbb{P}^3}(1)| = \mathbb{P}^3$. Therefore, f induces an automorphism of $\mathbb{P}^3$, say $\sigma \in \mathbb{P}GL(4)$. In return, it is easy to see that σ induces f. □

Now we can compute the dimension of the moduli space of quartic surfaces, if it exists

$$\dim \mathcal{M} = \dim(|\mathcal{O}_{\mathbb{P}^3}(4)|/\sim) = \binom{7}{3} - 1 - \dim \mathbb{P}GL(4) = 19 \tag{1.20}$$

Similarly, we can compute the dimension of the space of the complete intersections $X = Q \cap C \subset \mathbb{P}^4$ of type $(2,3)$:

$$\begin{aligned} &\dim|\mathcal{O}_{\mathbb{P}^4}(2)| + \dim|\mathcal{O}_{\mathbb{P}^4}(3)| - \dim|I_X(2)| - \dim|I_X(3)| \\ &= \binom{6}{4} + \binom{7}{4} - 1 - (5+1) = 43 \end{aligned} \tag{1.21}$$

where I_X is the ideal sheaf of X and $h^0(I_X(d))$ can be computed via Kozul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \to I_X \to 0 \tag{1.22}$$

Again we can show that the isomorphism between two very general complete intersections of such type is induced by $\mathbb{P}GL(5)$. Hence the moduli space of $X = Q \cap C$ has dimension $43 - 24 = 19$.

Exercise 1.6. *Prove Noether-Lefschetz for a general complete intersection of type $(2,3)$ and $(2,2,2)$ in $\mathbb{P}^4$ and $\mathbb{P}^5$, respectively.*

Exercise 1.7. *Compute the dimension of the moduli space of the complete intersection $X \subset \mathbb{P}^5$ of type $(2,2,2)$.*

Here is another example. Let $\pi : X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ ramified over a smooth sextic curve $D \subset \mathbb{P}^2$. To see that X is a $K3$ surface, we first prove

Proposition 1.8. *Let X_0 be a smooth hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$ of type $(2,3)$. Then the projection $X_0 \to \mathbb{P}^2$ is a double cover ramified along a sextic curve.*

By weak Lefschetz and adjunction, X_0 is a $K3$ surface. Obviously, every double cover X of $\mathbb{P}^2$ ramified along a smooth sextic curve can be deformed to an $X_0 \subset \mathbb{P}^1 \times \mathbb{P}^2$ of type $(2,3)$. Hence X is a $K3$ surface.

The dimension of the moduli space of such X is the same as the dimension of the moduli space of sextic curves:

$$\dim |\mathcal{O}_{\mathbb{P}^2}(6)| - \dim \mathbb{P}GL(3) = 27 - 8 = 19. \tag{1.23}$$

Let $A = \mathbb{C}^2/\Lambda$ be a 2-dimensional complex torus. We have $\mathbb{Z}_2$ action on A by sending $(x, y) \to (-x, -y)$. Let $X = A/\mathbb{Z}_2$ and $\widetilde{X}$ be the minimal resolution of X. Then $\widetilde{X}$ is a special $K3$ surface called Kummer surface.

Proposition 1.9. *Let $G = \mathbb{Z}_2$ act on $\mathbb{A}^2_{xy}$ by sending $(x, y) \to (-x, -y)$. Then $\mathbb{A}^2/G$ is a hypersurface in $\mathbb{A}^3_{uvw}$ given by $uv = w^2$.*

Proof. $\mathbb{A}^2/G = \operatorname{Spec} k[x, y]^G = \operatorname{Spec} k[x^2, y^2, xy] = \operatorname{Spec} k[u, v, w]/(uv-w^2)$. □

The action $\mathbb{Z}_2$ on A has sixteen fixed points. Therefore, X has sixteen singularities where X is locally given by $\operatorname{Spec} \mathbb{C}[u, v, w]/(uv - w^2)$, i.e, X has sixteen rational double points.

Proposition 1.10. *Let $X = (xy = z^2) \subset Y = \mathbb{A}^3_{xyz}$ and let $\pi : \widetilde{Y} \to Y$ be the blowup of Y at the origin p. Let $\widetilde{X}$ be the proper transform of X under π and E_X be the exceptional divisor of $\pi : \widetilde{X} \to X$. Then $\widetilde{X}$ is smooth, E_X is a smooth rational curve with $E_X^2 = -2$ on $\widetilde{X}$ and $K_{\widetilde{X}} = \pi^* K_X$.*

Proof. $\widetilde{Y} \subset \mathbb{A}^3 \times \mathbb{P}^2$ is given by $x/X = y/Y = z/Z$ and $\widetilde{X}$ is by

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z} \text{ and } XY = Z^2 \tag{1.24}$$

It is straightforward to check that $\widetilde{X}$ is smooth. The exceptional divisor is a smooth conic curve $XY = Z^2$ in $\mathbb{P}^2$. Let E_Y be the exceptional divisor of $\widetilde{Y} \to Y$. By

$$\pi^* X = \widetilde{X} + 2E_Y \Rightarrow \pi^* X \cdot E_Y^2 = \widetilde{X} \cdot E_Y^2 + 2E_Y^3 \Rightarrow \widetilde{X} \cdot E_Y^2 = -2E_Y^3 \tag{1.25}$$

we see that $E_X^2 = \widetilde{X} \cdot E_Y^2 = -2$. Since

$$K_{\widetilde{Y}} = \pi^* K_Y + 2E_Y, \tag{1.26}$$

$$K_{\widetilde{X}} = (\pi^* K_Y + 2E_Y + \widetilde{X})|_{\widetilde{X}} = \pi^*(K_Y + X)|_{\widetilde{X}} = \pi^* K_X \tag{1.27}$$

□

So we see that $\widetilde{X}$ contains sixteen (-2)-curves. And since $\nu^* K_X = K_A$ and $\pi^* K_X = K_{\widetilde{X}}$, $K_{\widetilde{X}}$ is trivial. By classification of complex surfaces, $\widetilde{X}$ can be either a $K3$ surface or abelian surface. Since an abelian surface does not contain any rational curves, $\widetilde{X}$ must be a $K3$ surface.

1.5. **Deformation of $K3$ surfaces, A second look.** As we pointed out in the examples of $K3$ surfaces as complete intersections in $\mathbb{P}^n$ or double covers of $\mathbb{P}^2$, the corresponding moduli space of $K3$ surfaces has dimension 19, while the versal deformation of a $K3$ surface has dimension 20. So where does the extra dimension go? The answer, we have shown in 1.3, is that a general deformation of a $K3$ surface is not algebraic; 19 is the dimension of the deformation of a polarized $K3$. Let us take another look at this problem via a more elementary approach.

First let us illustrate this using the example of quartic surfaces. Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. We have the exact sequence

$$0 \to T_X \to T_{\mathbb{P}^3}|_X \to N_X \to 0 \tag{1.28}$$

where N_X is the normal bundle of X. The induced long exact sequence is

$$H^0(N_X) \to H^1(T_X) \xrightarrow{\beta} H^1(T_{\mathbb{P}^3}|_X) \tag{1.29}$$

By Euler sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus 4} \to T_{\mathbb{P}^3}|_X \to 0 \tag{1.30}$$

we see that $H^1(T_{\mathbb{P}^3}|_X) \cong H^2(\mathcal{O}_X) = \mathbb{C}$. Consider the dual map $\beta^\vee$ of β:

$$\begin{tikzcd} H^1(T_X) \arrow[r, "\beta"] & H^1(T_{\mathbb{P}^3}|_X) \\ \times & \times \\ H^1(\Omega_X) \arrow[d] & H^1(\Omega_{\mathbb{P}^3}|_X) \arrow[l, "\beta^\vee"'] \arrow[d] \\ \mathbb{C} & \mathbb{C} \end{tikzcd} \tag{1.31}$$

Since $H^1(\Omega_{\mathbb{P}^3}|_X) = H^1(\Omega_{\mathbb{P}^3}) = \mathbb{C}$ is generated by $c_1(L)$,

$$\operatorname{Im} \beta^\vee = \{\lambda c_1(L) : \lambda \in \mathbb{C}\} \tag{1.32}$$

where L is the hyperplane bundle. Consequently,

$$\begin{aligned} \operatorname{Im}(H^0(N_X) \to H^1(T_X)) &= \ker \beta \\ = \{\varepsilon \in H^1(T_X) : \langle \varepsilon, c_1(L)\rangle = 0\} &= c_1(L)^\perp \end{aligned} \tag{1.33}$$

By deformation theory, $H^0(N_X)$ classifies embedded deformations of $X \subset \mathbb{P}^3$ and $H^1(T_X)$ classifies the deformations of X as a complex manifold. So the image $\operatorname{Im}(H^0(N_X) \to H^1(T_X))$ classifies the deformations of the pair (X, L), i.e., a polarized $K3$ surface. The above argument actually applies to any polarized $K3$ surface, not only quartic surfaces. Therefore, the versal deformation space of a polarized $K3$ surface (X, L) is a hyperplane in $H^1(T_X)$ that is perpendicular to $c_1(L)$.

The above technique can be generalized to prove the following:

Proposition 1.11. *Let X/Δ be a family of smooth projective surfaces over disk Δ with central fiber $S = X_0$ and let $D \subset S$ be an effective divisor on S. Suppose that D can be extended to X, i.e., there exists a flat family Y/Δ with the commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ \Delta \arrow[r] & \Delta \end{tikzcd} \tag{1.34}$$

such that Y_0 embeds into X_0 with image D. For each $w \in H^0(K_S)$, let μ_w be the map

$$\mu_w : H^1(\Omega_S) \xrightarrow{\otimes w} H^1(\Omega_S(K_S)) \tag{1.35}$$

where K_S is the canonical class of S. Then the Kodaira-Spencer class $\mathrm{ks}(\partial/\partial t) \in H^1(T_S)$ of X lies in the subspace

$$\{v \in H^1(T_S) : \langle v, \mu_w(c_1(D))\rangle = 0 \text{ for all } w \in H^0(K_S)\} \tag{1.36}$$

where $\langle \cdot, \cdot \rangle$ is the pairing $H^1(T_S) \times H^1(\Omega_S(K_S)) \to \mathbb{C}$ given by Serre duality.

Using the above proposition, we can give yet another proof of Noether-Lefschetz for quartic surfaces.

Let $M = |\mathcal{O}_{\mathbb{P}^3}(4)|$ and $S = \{(p, X) : p \in X\} \subset \mathbb{P}^3 \times M$. Let $\mathcal{L} = \pi_1^* L$ be the pullback of the hyperplane bundle of $\mathbb{P}^3$. If for a very general X, there is a line bundle $D \in \mathrm{Pic}(X)$ such that D is not a multiple of L, then after a possible base change of M, there exists a line bundle $\mathcal{D}$ on S such that $\mathcal{D}_X$ is not a multiple of $\mathcal{L}_X$ when restricted to a very general point $[X] \in M$. Then by the above proposition, the image of

$$T_{M,[X]} \xrightarrow{\mathrm{ks}} H^0(N_X) \to H^1(T_X) \tag{1.37}$$

is contained in $c_1(D)^\perp$. Also $\mathrm{Im}(H^0(N_X) \to H^1(T_X))$ is $c_1(L)^\perp$ and ks is obviously surjective. Therefore, we necessarily have

$$c_1(L)^\perp = c_1(D)^\perp \tag{1.38}$$

That is, $c_1(L)$ and $c_1(D)$ are linearly dependent over $\mathbb{Q}$. Therefore, $\mathrm{Pic}(X) = \mathbb{Z}$. Let J be a generator of $\mathrm{Pic}(X)$. Then $L = mJ$ for some $m \in \mathbb{Z}$. WLOG, assume that $m > 0$. Since $L^2 = 4$, $m = 1$ or $m = 2$. We are done if $m = 1$. If $m = 2$, $J^2 = 1$ and $(K + J)J = 1$. This is impossible by Riemann-Roch.

Exercise 1.12. *Let M be the moduli space of the tuple $(X, L_1, L_2, ..., L_m)$, where X is a $K3$ surface, $\{L_k\}$ are m linearly independent line bundles on X and L_1 is ample. Then* $\dim M \leq 20 - m$.

So far we reach the conclusion that the moduli space of polarized $K3$ surfaces (X, L) has dimension at most 19. Then how many components does this moduli space have?

We call a polarized $K3$ surface (X, L) a primitive $K3$ surface if there does not exist $D \in \mathrm{Pic}(X)$ and $m > 1$ such that $L = mD$ and in this case, we say L is a primitive line bundle over X. Let $C \in |L|$. Then $2p_a(C) - 2 = L^2$. This number $g = p_a(C)$ is called the genus of X. By a $K3$ surface of genus g, we mean a polarized $K3$ surface (X, L) with a primitive line bundle L and $L^2 = 2g - 2$.

Theorem 1.13. *For each $g \geq 2$, there exists a moduli space $\mathcal{M}_g$ parameterizing genus g $K3$ surfaces; $\mathcal{M}_g$ is quasi-projective, smooth and irreducible of dimension 19.*

Genus 2 $K3$ surfaces are double covers of $\mathbb{P}^2$ ramified along a smooth sextic curves. Genus 3 $K3$ surfaces are quartic surfaces in $\mathbb{P}^3$. Genus 4 $K3$ surfaces are complete intersections in $\mathbb{P}^4$ of type $(2, 3)$. Genus 5 $K3$ surfaces are complete intersections in $\mathbb{P}^5$ of type $(2, 2, 2)$. Here I will give an elementary proof of existence of $K3$ of any genus g.

Proposition 1.14. *For every $g \geq 2$, there exists a $K3$ surface X of genus g and* $\mathrm{Pic}(X) = \mathbb{Z}$.

Proof. Let X be a smooth surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of type $(2, 3)$. We embed $X \hookrightarrow \mathbb{P}^{3k+2}$ by the very ample linear series $|\pi_1^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(1)|$ with $k > 0$. In the exact sequence

$$H^1(T_X) \to H^1(T_{\mathbb{P}^g}|_X) \to H^1(N_X) \to H^2(T_X) \tag{1.39}$$

we have already seen that $H^1(T_X) \to H^1(T_{\mathbb{P}^g}|_X)$ is surjective, where $g = 3k + 2$. And since $H^2(T_X) = 0$, $H^2(N_X) = 0$ and the embedded deformations of $X \subset \mathbb{P}^g$ are unobstructed. Therefore, there exists a flat family $Y \subset \mathbb{P}^g \times \Delta^m$ such that $Y_0 = X \subset \mathbb{P}^g$ and the Kodaira-Spencer map $T_{\Delta^m,0} \to H^0(N_X)$ is an isomorphism. We have proved that $\mathrm{Im}(H^0(N_X) \to H^1(T_X))$ is $c_1(L)^\perp$, where $L = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2}(1)$. For a general fiber Y_t of $Y \to \Delta^m$, $\mathrm{Pic}(Y_t) = \mathbb{Z}$ is generated by L.

So this proves the proposition when $g = 3k + 2$. For $g = 3k, 3k + 1$, see the following exercise. $\square$

Exercise 1.15. *Let $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$ be a rank three vector bundle over $\mathbb{P}^1$ and $Y = \mathbb{P}E$. Let $X \in |-K_Y|$ be a smooth anti-canonical surface in $\mathbb{P}E$. Show that the complete linear series $|\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(k)|$ $(k \geq 1)$ embeds X into $\mathbb{P}^{3k}$.*

Change E to $E = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ and do the same thing.

1.6. **Degeneration of $K3$ surfaces by Ciliberto-Lopez-Miranda.** Let S_1 and S_2 be two Del Pezzo surfaces. We "glue" S_1 and S_2 along a smooth anti-canonical curve $D \subset S_1$ and $D \subset S_2$. More precisely, we glue S_1 and S_2 transversely via two immersions $i_k : D \hookrightarrow S_k$ for $k = 1, 2$. The union $S = S_1 \cup S_2$ is not necessarily projective. It is projective if and only there are ample line bundles $L_1 \in \operatorname{Pic}(S_1)$ and $L_2 \in \operatorname{Pic}(S_2)$ such that

$$i_1^* L_1 = i_2^* L_2. \tag{1.40}$$

This pair (L_1, L_2) defines an ample line bundle $L \in \operatorname{Pic}(S)$, which polarizes S.

Since $\pi_1(S_1) = \pi_1(S_2) = 0$ and D is connected, S is simply connected. The dualizing sheaf ω_S is trivial since

$$\omega_S \Big|_{S_k} = \omega_{S_k} + D = 0 \tag{1.41}$$

for $k = 1, 2$. So S is a "degenerated" $K3$ surface.

We let

- if g is odd, we let $S_i \cong \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and

$$L_i = L\Big|_{S_i} = C_i + \frac{g-1}{2} F_i \tag{1.42}$$

  where C_i and F_i are the generators of $\operatorname{Pic}(S_i)$ with $C_i^2 = F_i^2 = 0$ and $C_i F_i = 1$ for $i = 1, 2$;
- if $g \geq 4$ is even, we let $S_i \cong \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and

$$L_i = L\Big|_{S_i} = C_i + \frac{g}{2} F_i \tag{1.43}$$

  where C_i and F_i are the generators of $\operatorname{Pic}(S_i)$ with $C_i^2 = -1$, $F_i^2 = 0$ and $C_i F_i = 1$ for $i = 1, 2$.

We can embed S to $\mathbb{P}^g$ by $|L|$. Let N_S be the normal bundle of $S \subset \mathbb{P}^g$:

$$N_S = \mathcal{H}om(I_S / I_S^2, \mathcal{O}_S). \tag{1.44}$$

Theorem 1.16 (Ciliberto-Lopez-Miranda)**.** [CLM] *For $S \subset \mathbb{P}^g$ given as above, $H^1(N_S) = 0$ and $H^0(N_S) = g^2 + 2g + 19$.*

So a general embedded deformation of $S \subset \mathbb{P}^g$ is a $K3$ surface of genus g. Namely, there exists $W \subset \mathbb{P}^g \times \Delta$ such that $W_0 = S_1 \cup S_2$ and W_t is a smooth $K3$ surface of genus g.

It is worthwhile to take a closer look at W. Since the T^1

$$T^1(W_0) = \mathcal{E}\mathrm{xt}(\Omega_{W_0}, \mathcal{O}_{W_0})) = H^0(\mathcal{O}_D(-K_{S_1} - K_{S_2})) \tag{1.45}$$

W has sixteen rational double points $p_1, p_2, .., p_{16}$. Let X be the blowup of W along S_1. As in Griffiths-Harris' proof of Noether-Lefschetz, the central fiber X_0 of X/Δ is a union $R_1 \cup R_2$, where R_1 is the blowup of S_1 at $p_1, p_2, ..., p_{16}$ and $R_2 = S_2$. So one may use Griffiths-Harris' argument to establish Noether-Lefschetz for $K3$ surfaces of all genera g.

2. Rational curves on $K3$

2.1. **Existence.** First of all, we have

Proposition 2.1. *There are at most countably many rational curves on a $K3$ surface X.*

Proof. Otherwise, X is covered by rational curves, i.e., X is uniruled. There exists a dominant rational map $\mathbb{P}^1 \times \Gamma \to X$, where Γ is a smooth projective curve. This rational map can resolved by a sequence of blowups. Let $f : Y \to \mathbb{P}^1 \times \Gamma \to X$ be such a resolution. So Y is a fiberation over Γ whose general fibers are $\mathbb{P}^1$. Since f is surjective, we have the injection

$$f^* : H^0(K_X) \hookrightarrow H^0(K_Y). \tag{2.1}$$

So K_Y is effective since K_X is. Let $Y_p = C$ be a general fiber of $Y \to \Gamma$. Then

$$K_Y|_C = K_C = -2 \tag{2.2}$$

Yet $K_Y|_C = K_Y \cdot C \geq 0$ since K_Y is effective. Contradiction. □

Exercise 2.2. *Let X be a smooth projective variety satisfying that mK_X is effective for some $m > 0$. Show that X is not uniruled.*

More generally, according to Kodaira's classification of compact complex surfaces, for a smooth projecitve surface X over $\mathbb{C}$,

- If $\kappa(X) < 0$, X is covered by rational curves.
- If $\kappa(X) = 0$, there are at most countably many rational curves on X.
- If $\kappa(X) > 0$, there are only finitely many rational curves on X conjecturally (Lang conjecture).

Yet the existence of rational curves are more subtle. The existence of rational curves on $K3$ surfaces was established by S. Mori and S. Mukai. I made it more precise:

Theorem 2.3 (Chen). *For any integers $n \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general $K3$ surface S in $\mathbb{P}^n$ contains an irreducible nodal rational curve.*

The idea for the proof is to degenerate a $K3$ surface to a union of rational surfaces given in 1.6. It is best illustrated by quartic surfaces.

Let us consider $X_0 = Q_1 \cup Q_2 \subset \mathbb{P}^3$ be a union of two quadrics. This is a "special" quartic surface and any smooth quartics can be degenerated to it. It is a common knowledge that $Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ is embedded into $\mathbb{P}^3$ by $|H_1 + H_2|$, where H_i are two rulings of Q_i. Let $E = Q_1 \cap Q_2$. Then E is an elliptic curve in the linear series $|2H_1 + 2H_2|$. In addition we have

$$\mathcal{O}_{Q_1}(H_1 + H_2)|_E = \mathcal{O}_{Q_2}(H_1 + H_2)|_E \tag{2.3}$$

Let $X \subset \mathbb{P}^3 \times \Delta$ be a pencil of quartics whose central fiber is X_0. So the defining equation of X looks like

$$F_{Q_1} F_{Q_2} + tF = 0 \tag{2.4}$$

where F_{Q_i} are the defining equations of Q_i. We choose X to be general enough. The idea is to find a curve $Y_0 \in |\mathcal{O}_{X_0}(d)|$ and show that Y_0 can be deformed to a nodal rational curve $Y_t \in |\mathcal{O}_{X_t}(d)|$.

For example, let us work out the case $d = 1$. Obviously, there exists $r \in E$ such that

$$\mathcal{O}_E(H_1 + H_2) = \mathcal{O}_E(4r) \tag{2.5}$$

Actually there are exactly 16 such points. There exists a unique curve $C_i \in |\mathcal{O}_{Q_i}(H_1 + H_2)|$ such that $C_i \cdot E = 4r$. This is due to the fact

$$H^0(\mathcal{O}_{Q_i}(H_1 + H_2)) = H^0(\mathcal{O}_E(H_1 + H_2)) \tag{2.6}$$

Also for E general, C_i is irreducible and smooth.

Let $U^{d,\delta}(S)$ be the subset of $|\mathcal{O}_S(d)|$ consisting of irreducible nodal curves with δ nodes on a quartic surface S. Let

$$W_{d,\delta} = \bigcup_{t \neq 0} U^{d,\delta}(X_t) \subset |\mathcal{O}_X(d)| \tag{2.7}$$

and let $\overline{W}_{d,\delta}$ be the closure of $W_{d,\delta}$ in $|\mathcal{O}_X(d)|$

A theorem of Caporaso-Harris-Ran shows that

Proposition 2.4. *The following are true:*

(1) $[C_1 \cup C_2] \in \overline{W}_{1,3}$*;*
(2) $\overline{W}_{1,3}$ *has an ordinary singularity of multiplicity* 4 *at* $[C_1 \cup C_2]$*;*
(3) *for any open neighborhood* O_r *of* $r \in \mathbb{P}^3$*, there exists an open neighborhood* $V_{[C_1 \cup C_2]}$ *of* $[C_1 \cup C_2] \in \overline{W}_{1,3}$ *such that for any* $[C] \in V_{[C_1 \cup C_2]}$*, the nodes of* C *lies in* O_r*.*

From the above proposition, we see that $U^{1,3}(X_t)$ is nonempty for $t \neq 0$. So there exists an irreducible curve Y_t with 3 nodes in $|\mathcal{O}_{X_t}(1)|$. This curve Y_t is obviously a rational curve.

For $d \geq 2$, a slight different construction is needed but the basic idea is the same. For example, let us work out the case $d = 2$.

The threefold X has sixteen rational double points lying on E. Let p be one of them. We let $Y_0 = C_{11} \cup C_{12} \cup C_{21} \cup C_{22}$ with

(1) $C_{i1} \in |\mathcal{O}_{Q_i}(H_1)|$ and $C_{i2} \in |\mathcal{O}_{Q_i}(H_1 + 2H_2)|$;
(2) $C_{i1} \cdot E = p + q_i$ and $C_{i2} \cdot E = q_{3-i} + 5r$

where q_1, q_2, r are determined by p up to 25 different choices.

Again we can show there is a flat family $Y \subset X$ of nodal curves after a base change such that $Y_t \in |\mathcal{O}_{X_t}(2)|$ has 9 nodes, with 4 of them approaching r, 1 of them approaching p, 2 of them approaching $C_{11} \cap C_{12}$ and 2 of them approaching $C_{21} \cap C_{22}$ as $t \to 0$. Obviously, Y_t is a rational curve. For details, please see [C1].

2.2. **Density of rational curves on $K3$ surfaces.** There has been a revival of interest in the existence of rational curves on $K3$ surfaces. There are two main results.

Theorem 2.5 (Bogomolov-Hassett-Tschinkel, Li-Liedtke). *There are infinitely many rational curves on every projective $K3$ surface X with $\mathrm{Pic}(X) = \mathbb{Z}$.*

Their proof goes like the following: it suffices to prove it for such a $K3$ surface over a number field. For a family X of $K3$ surfaces over $\operatorname{Spec}\mathbb{Z}$, the reduction X_p at p is a $K3$ surface of even Picard rank. Using this fact, they can construct rational curves on X_p of arbitrarily high degree that can be "lifted" to the generic fiber of X.

In another direction, we proved that

Theorem 2.6 (Chen-Lewis). *For all $g \geq 2$, the set*

$$\bigcup_{n=1}^{\infty} \mathcal{C}_{g,n}$$

is dense in $\mathcal{S}_g$ under the analytic topology, where

$$\mathcal{C}_{g,n} \subset \mathcal{S}_g = \{(X, L, p) : (X, L) \in \mathcal{K}_g, p \in X\}$$

is the closed subscheme of $\mathcal{S}_g$ whose fiber over a general point $(X, L) \in \mathcal{K}_g$ is the union of all irreducible rational curves in the linear series

$|nL|$. *Here*

$$\begin{aligned}\mathcal{K}_g = \big\{(X, L) : X &\textit{ is a } K3 \textit{ surface,}\\ L \in \operatorname{Pic}(X) &\textit{ is ample primitive}\\ &\textit{and } L^2 = 2g - 2\big\}\end{aligned} \tag{2.8}$$

is the moduli space of $K3$ surfaces of genus g.

Our proof is based on the following facts:

- Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3 surface. Fixing a divisor L on X, let $\phi : X \dashrightarrow X$ be the rational map defined by sending
$$\phi(p) = L - (d - 1)p \tag{2.9}$$
for $p \in X_b = \pi^{-1}(b)$, where $d = L \cdot X_b$. Equivalently, let X_η be the generic point of π. Then ϕ is induced by an endomorphism of the elliptic curve X_η over $\mathbb{C}(t)$.
- For every rational curve $C \subset X$, $\phi(C)$ is also a rational curve on X.
- Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3 surface and $C \subset X$ be a rational curve not containing in a fiber of π. If $\phi^n(p)$ is dense on the fiber of $X/\mathbb{P}^1$ containing p for a general point $p \in C$, $\phi^n(C)$ is dense on X.
- There exists rational curves $C \subset X$ such that $\phi^n(p)$ is dense for a general point $p \in C$.
- Elliptic $K3$ surfaces are dense in the moduli space of K3 surfaces.
- For every rational curve $C' = \phi^n(C)$, there exists a rational curve C'' on X such that $C' \cup C'' \in |mL|$.

Exercise 2.7. *Let $T = (\mathbb{R}/\mathbb{Z})^n$. For which points p, is $\{mp : m \in \mathbb{Z}\}$ dense in T?*

2.3. **Counting rational curves.** The next natural question following the existence problem is how many irreducible rational curves there are in $|\mathcal{O}(d)|$ on a general $K3$ surface in $\mathbb{P}^n$. The number for $d = 1$ has been successfully calculated in [Y-Z]. They give the following remarkable formula

$$\sum_{g=1}^{\infty} n(g) q^g = \frac{q}{\Delta(q)} \tag{2.10}$$

where $\Delta(q) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$ is the well-known modular form of weight 12 and $n(g)$ is the nominated number of rational curves in $|\mathcal{O}(1)|$ on a general $K3$ surface in $\mathbb{P}^g$ for $g \geq 3$. More precisely, $n(g)$ is the sum of the Euler characteristics of the compactified Jacobians of all rational curves in $|\mathcal{O}(1)|$. Since the compactified Jacobian of a rational curve with singularities other than nodes is not very well understood, we only know this sum equals the number of rational curves in $|\mathcal{O}(1)|$ on a $K3$ surface in the case that all these rational curves are nodal.

Later J. Bryan and N.C. Leung redid and generalized Yau-Zaslow's counting via a different approach. Basically, they used a degeneration argument by degenerating a general $K3$ surface to a $K3$ surface S of Picard lattice

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

Let C and F be the generators of $\mathrm{Pic}(S)$ with $C^2 = -2$, $C \cdot F = 1$ and $F^2 = 0$. We also assume that F is effective.

The good thing about S is that $(S, C + gF)$ is a $K3$ surface of genus g and every member of the linear series $|C + gF|$.

Exercise 2.8. *Show that $h^0(C) = 1$, $h^0(F) = 2$ and the map $\pi : S \to \mathbb{P}^1$ given by $|F|$ realizes S as an elliptic fiberation. For S general, there are exactly 24 singular fibers of π.*

Exercise 2.9. *Show that every curve in $D \in |C + gF|$ is a union $C \cup F_1 \cup ... \cup F_g$ with $F_i \in |F|$.*

Using this degeneration, I proved the following theorem:

Theorem 2.10. *All rational curves in the primitive class of a general $K3$ surface of genus $g \geq 2$ are nodal.*

This justifies the number obtained by Yau-Zaslow is the number of rational curves.

Question 2.11. *Compute the number of rational curves in $|\mathcal{O}_S(d)|$.*

2.4. **Hodge-$\mathcal{D}$-Conjecture.** As another application of rational curves on $K3$, J. Lewis and I proved the following theorem, originally a conjecture of Beilinson:

Theorem 2.12 (Chen, Lewis)**.** *Hodge-$\mathcal{D}$ conjecture holds for a general $K3$ surface X (general under the real analytic topology). That is, the regulator map*

$$r_{2,1} : \mathrm{CH}^{2,1}(X) \to H^{1,1}(X, \mathbb{R}) \tag{2.12}$$

is surjective.

Here the rational curves are used to construct nontrivial classes in $\mathrm{CH}^{2,1}(X)$. By definition,

$$\mathrm{CH}^k(X,1) = \frac{\left\{\sum_j (f_j, Z_j) \ : \ \begin{array}{c} \mathrm{cd}_X Z_j = k-1,\ f_j \in \mathbb{C}(Z_j)^\times \\ \sum_j \mathrm{div}(f_j) = 0 \end{array}\right\}}{\mathrm{Image(Tame\ symbol)}}. \tag{2.13}$$

Choose two rational curves C_1 and $C_2 \subset X$. Suppose that there are two points $p, q \in C_1 \cap C_2$. Then there exists two rational functions f_1 and f_2 on C_1 and C_2, respectively, such that $(f_1) = p - q$ and $(f_2) = q - p$. Then $(f_1, C_1) + (f_2, C_2)$ is a class in $\mathrm{CH}^{2,1}(X)$.

On the other hand, there exist $K3$ surfaces X with maximal Picard rank $h^{1,1}(X) = 20$. For such surfaces,

$$\mathrm{Im}(r_{2,1}) = H^{1,1}(X, \mathbb{Q}) \otimes \mathbb{R} = H^{1,1}(X, \mathbb{R}). \tag{2.14}$$

A natural approach is via degeneration: Let W/Δ be a one-parameter family of K3 surfaces with $\mathrm{Pic}(W_0) = \mathbb{Z}^{20}$. Show that

$$\lim_{t\to 0} r_{2,1}(\mathrm{CH}^2(W_t, 1)) = r_{2,1}(\mathrm{CH}^2(W_0, 1)). \tag{2.15}$$

To show (2.15), we need to construct higher Chow cycles on W_t using rational curves:

$$\xi_t = (f_t, \mathcal{C}_t) + (g_t, \mathcal{D}_t) \tag{2.16}$$

where $\mathcal{C}_t$ and $\mathcal{D}_t$ are rational curves on W_t and f_t and g_t are rational functions with zero and pole at two intersections of $\mathcal{C}_t$ and $\mathcal{D}_t$. Here great care has to be taken such that the limit ξ_0 of ξ_t exists as a higher Chow cycle in $\mathrm{CH}^2(W_0, 1)$; unlike regular Chow cycles, higher Chow cycles on the generic fiber do not necessarily extend over a one-parameter family. So it is important to choose the right W_0: we used a special elliptic $K3$ with singular fibers of $W_0 \to \mathbb{P}^1$ looking like

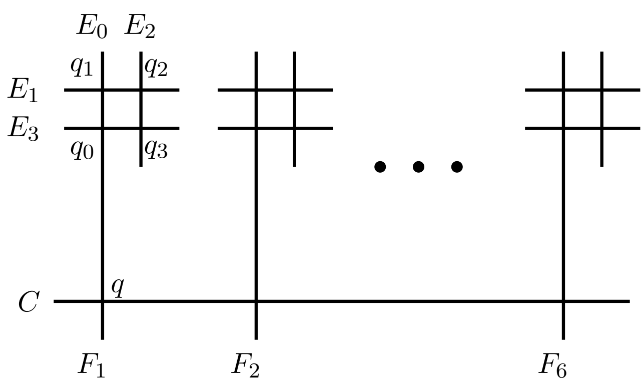

FIGURE 1. The BL $K3$ surface we will use

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