The Motivic Zeta Function

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February 2013

Arithmetic, cycles, motives and algebraic geometry

Zeta functions over finite fields

$$k = \mathbb{F}_q$$

$$k_r = \mathbb{F}_{q^r}$$

$\overline{k}$ an algebraic closure of k

X an algebraic variety over k

$$\overline{X} = X \times_k \overline{k}$$

Weil zeta function

The Weil zeta function is given by

$$Z(X;t) = \exp\left(\sum_{r=1}^{\infty} \#\overline{X}(k_r)\frac{t^r}{r}\right) .$$

Weil conjectures for $Z(X,t)$

- **Rationality.** $Z(X,t)$ is a rational function of t.
- **Functional equation.** Let $E=\Delta_X.\Delta_X$. Then

$$Z\left(X,\frac{1}{q^n t}\right)=\pm q^{nE/2}t^E Z(X,t)$$

- **Riemann hypothesis.**

$$Z(X,t)=\frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

with $P_0(t)=1-t$, $P_{2n}(t)=1-q^n t$ and for $1\leq i\leq 2n-1$ we have

$$P_i(t)=\prod_j(1-\alpha_{i,j}t)$$

for some algebraic integers with $|\alpha_{i,j}|=q^{i/2}$.

The Weil zeta function can be rewritten as

$$Z(X,t) = \sum_{n=0}^{\infty} \# \mathrm{Sym}^n(X)(k)\, t^n$$

The ring of varieties

For an arbitrary field k, $K_0(Var(k))$ is given by:

Generators:	$[X]$ isomorphism class fo the variety X.
Relations generated by:	$[X \setminus Y] = [X] - [Y]$, $Y \subseteq X$ closed $[\mathrm{X} \times Y] = [X][Y]$

Remark

Denote $\mathbb{L} := [\mathbb{A}^1_k]$ and $1 := [pt]$. Then

$$[\mathbb{P}^1_k] = \mathbb{L} + 1$$

A multiplicative Euler characteristic with compact support is a function

$$\mu : K_0(Var(k)) \to R$$

such that

- $\mu[X \times Y] = \mu[X] \cdot \mu[Y]$,
- $\mu[X \setminus Y] = \mu[X] - \mu[Y]$ for $Y \subseteq X$ closed.

Kapranov's zeta function

$$Z_\mu(X,t) = \sum_{n=0}^{\infty} \mu[\mathrm{Sym}^n(X)]\, t^n \in R[[t]].$$

If $\mu = id_{K_0(Var(k))}$ then $Z_\mu(X,t)$ is called the universal Kapranov zeta function.

Properties

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- If $k = \mathbb{C}$, $\mu = \chi_C$ then $Z_\mu(X, t) = (1 - t)^{-\chi_C(X)}$.

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- If $k=\mathbb{C}$, $\mu=\chi_C$ then $Z_\mu(X,t) = (1-t)^{-\chi_C(X)}$.
- If $k=\mathbb{Q}$ and $X \in \mathrm{Obj}(Var(k))$ then $X(\mathbb{F}_p)$ is well defined for all but a finite number of primes. Therefore for $p>>0$ the universal Kapranov zeta function interpolates the Weil zeta functions of the reductions of X mod p.

Analogues of Weil's conjectures

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- $Z_\mu(X,t) = \dfrac{P_\mu(X,t)}{(1-t)(1-\mathbb{L}t)}$ for some degree 2 polynomial P_μ.

Nevertheless..

Theorem (Larsen,Lunts)

If X is a product of two curves of genus $g > 1$ then the universal Kapranov zeta function is not rational.

$\mathsf{K}_0(M(k))$

The category $M(k)$ of Chow motives over k is a monoidal category under sums:

$$(X,p)\oplus(Y,q)=(X\coprod Y,p+q)\ .$$

Therefore, we can construct the Grothendieck group $K_0(M(k))$ over the monoid of isomorphism classes of motives. We will denote the class of the motive (X,p) by $[X,p]$.
$K_0(M(k))$ can be endowed with a ring structure with multiplication induced by the tensor product of motives.

Moreover, we have a ring morphism:

$$\begin{array}{rcl} \eta : K_0(Var(k)) & \to & K_0(M(k)) \\ [X] & \mapsto & [X, \Delta_X] \end{array}$$

The motivic zeta function

The motivic zeta function is the Kapranov zeta function given by

$$Z_{mot}(X,t) := Z_\eta(X,t) = \sum_{n=0}^{\infty} [Sym^n(X)]t^n \in K_0(M(k))[[t]].$$

Remark. $Z_{mot}(M \oplus M', t) = Z_{mot}(M,t) \cdot Z_{mot}(M',t)$.

Finite dimensionality

We say a motive $M \in M(k)$ is finitely dimensional if it can be decomposed:

$$M \cong M_+ \oplus M_-$$

and there is a positive integer N such that

$$\Lambda^N M_+ = 0$$

$$\mathrm{Sym}^N M_- = 0 \ .$$

Kimura-O'Sullivan conjecture

Every motive with $\mathbb{Q}$ coefficients is finitely dimensional.

Consequences

- (Andre) Z_{mot} is rational.

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- (Kahn) We have a functional equation

$$Z_{mot}(M^{\vee}, t^{-1}) = (-1)^{\chi_+(M)} \cdot \det(M) \cdot t^{\chi_-(M)} \cdot Z_{mot}(M, t) \ .$$

where $\det(M) = \Lambda^{\chi_+} M_+ \otimes (\mathrm{Sym}^{-\chi_-} M_-)^{-1}$.

Chow-Künneth decomposition

Definition

Let $X \in \mathrm{Obj}(\mathit{Var}(k))$ with $\dim X = d$. We say that X has a Chow-Künneth decomposition if we can find cycle classes $\pi_0(X), \ldots, \pi_{2d}(X) \in CH^d(X \times X, \mathbb{Q})$ such that

a) $\pi_i(X) \circ \pi_j(X) = \delta_{i,j} \pi_i(X)$.

b) $\Delta_X = \sum_{i=0}^{2d} \pi_i(X)$.

c) (over $\bar{k}$) π_i modulo (co)homological equivalence (for example, in étale cohomology) is the usual Künneth component $\Delta_X(2d-i, i)$.

If we define $h^i(X) := (X, \pi_i(X))$, then we will say that

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)$$

(or equivalently, the collection $\pi_0(X), \ldots, \pi_{2d}(X)$) is a Chow-Künneth (CK) decomposition for X.

Murre's conjectures.

A) Every smooth projective d dimensional variety X has a Chow-Künneth decomposition:

$$h(X) \cong \bigoplus_{i=0}^{2d} (X, \pi_i(X))$$

B) For each j, $\pi_0(X), \ldots, \pi_{j-1}(X), \pi_{2j+1}(X), \ldots, \pi_{2d}(X)$ act as zero on $CH^j(X, \mathbb{Q})$.

C) If $F^\nu CH^j(X) = \bigcap_{i=0}^{\nu-1} \ker(\pi_{2j-i}) \subseteq CH^j(X)$ then this descending filtration is independent of the choice of the $\pi_i' s$.

D) $F^1 CH^j(X) = CH^j(X)_{hom}$.

Theorem (Jannsen)

Murre's conjectures are equivalent to Bloch-Beilinson conjecture on a filtration for Chow groups. Moreover both filtrations coincide.

Some results about Murre's conjectures

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- For abelian varieties A) is true and part of B) is true. If B) is true then C) is also valid.
- For uniruled threefolds A) and B) are true.
- If $\pi : Y \to X$ is a fibration locally trivial in the Zariski topology, and with fibers having a Bruhat decomposition then A) and B) are true.

Relations between all of these concepts

$$
\begin{tikzcd}
 & Kimura-O'Sullivan \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
Murre\ conj.\ \mathrm{A)} & & Z_{mot}\ rational
\end{tikzcd}
$$

$$Murre's\ A) - D)\ for\ powers\ of\ X$$

$$\Downarrow$$

$$finite - dimensionality\ for\ X$$

$$\Downarrow$$

$$rationallity\ of\ Z_{mot}(X,t)$$

Thank You