LECTURE NOTES: CLUSTERS, POSITIVITY, SCATTERING

LARA BOSSINGER

Universidad Nacional Autónoma de México, Instituto de Matemáticas, Unidad Oaxaca

Abstract. These lectures notes cover the following topics: (1) Motivation: total positivity (2) Introduction to cluster algebras (3) cluster structures on partial flag varieties (4) realization of configuration spaces in quantum field theory as partial flag varieties (5) applications of cluster structures to scattering amplitudes.

THIS IS A PRELIMINARY VERSION, COMMENTS ARE WELCOME.

Contents

1. Total positivity

For $n \in \mathbb{Z}_{>0}$ let $[n] := \{1, \dots, n\}$ and $\binom{[n]}{k}$ be the set of k-element subsets of $[n]$, $k \leq n$.

Let $M = (m_{ij})_{i,j\in[n]} \in \mathbb{R}^{n\times n}$ be a matrix and let $I, J \in \binom{[n]}{k}$. Then

$$\Delta_{I,J}(M) := \det((m_{i,j})_{i\in I, j\in J})$$

E-mail address: lara@im.unam.mx.

Date: October 2025.

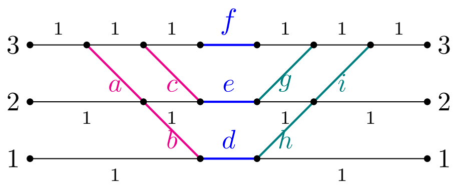


FIGURE 1. A planar network of order 3.

is called a *minor* of M. If all its minors are positive (resp. non negative) real numbers M is called *totally positive (resp. non negative)* or *TP (resp. TNN)* for short.

Example 1.1. Take $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then M is totally positive if and only if a, b, c, d and $\Delta = ad - bc \in \mathbb{R}_{>0}$. Observe that $d = \frac{1}{a}(\Delta + bc)$. Hence, it suffices to verify that $a, b, c, \Delta \in \mathbb{R}_{>0}$. The set $\{a, b, c, \Delta\}$ is a *positivity test.*

Question:

(i) How can we efficiently test for total positivity?
(ii) How can we characterize totally positive matrices?

1.1. **Planar networks.** A *planar network* (G, ω) *of order* n consists of a planar directed cycle free graph G with $2n$ vertices of valency one, of which n are sources and n are sinks; and a vector ω assigning scalar weights $\omega(e)$ to edges e of G.

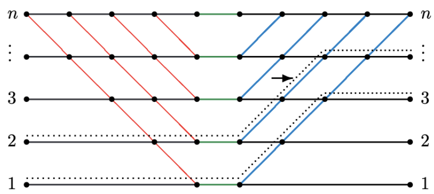
The *weight matrix* of a planar network (G, ω) of order n is an $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is the sum of weights of all paths from i to j in G.

Exercise 1. Complete the weight matrix of the diagram depicted in Figure 1.1 $\begin{pmatrix} d & dh & dhi \\ * & * & * \\ * & * & * \end{pmatrix}$

Lemma 1.2 (Lindström)**.** Let (G, ω) be a planar network of order n and M its weight matrix. The minor $\Delta_{I,J}(M)$ equals the sum of weights of collections of vertex disjoint paths connect source vertices in I with sink vertices in J.

Example 1.3. In Figure 1.1 we have $\Delta_{23,23}(M) = bcdegh + bdfh + fe$.

The following graph is denoted G_0:



Its colored edges are called *essential.* A weightning ω of G_0 is called *essential* if $\omega(e) \neq 0$ for at least one essential edge e and $\omega(e) = 1$ for all non essential edges. It is called *positive* if all edge weights are positive.

Theorem 1.4 (Whitney, Fomin–Zelevinsky)**.** There is a bijection between the set of essential positive weightnings of G_0 and the set of all totally positive $n \times n$ matrices.

1.2. **Grassmannians.** The *Grassmannian* as a set is denfined for $d \leq n$ integers as

$$\mathrm{Gr}_{d,n} := \{V \subset \mathbb{K}^n : \dim_{\mathbb{K}} V = d\}.$$

its elements can be represented by matrices:

(i) fix a basis $e_1, \dots, e_n$ for $\mathbb{K}^n$;
(ii) fix a basis $v_1, \dots, v_d \in \mathbb{K}^n$ for $V \in \mathrm{Gr}_{d,n}$;
(iii) express each $v_i = \sum_{j=i}^n m_{ij} e_j$ and define $M_V = (m_{ij})_{i \in [n], j \in [d]}$

Observe, M_V is unique up to base change.

Given $V \in \mathrm{Gr}_{d,n}$ and $I \in \binom{[n]}{d}$ define the *Plücker coordinate*

$$p_I(V) := \Delta_{[d],I}(M_V).$$

If $w_1, \dots, w_d$ is another basis for V yielding a coefficient matrix M'_V then $\exists\ A \in GL_d(\mathbb{K})$ such that $AM_V = M'_V$ and

$$\Delta_{[d],I}(M'_V) = \Delta_{[d],I}(AM_V) = \det(A)\Delta_{[d],I}(M_V).$$

In particular, we deduce that Plücker coordinates are unique up to common rescaling. This observation motivates the following result:

Theorem 1.5. The map $V \mapsto [p_I(V) : I \in \binom{[n]}{d}]$ defines an embedding $\mathrm{Gr}_{d,n} \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$ known the *Plücker embedding*. This way $\mathrm{Gr}_{d,n}$ obtains the structure of a projective variety.

The proof may be found, for example in [LB15, Theorem 5.2.1].

Observe that by definition of the Plücker coordinates for $\sigma \in S_n$ we have

$$p_{\sigma(i_1),\dots,\sigma(i_d)} = (-1)^{\ell(\sigma)} p_{i_1,\dots,i_d}.$$

The Grassmannian in $\mathbb{P}^{\binom{n}{d}-1}$ is cut out by *Plücker relations*. They are defined for $I = \{i_1, \dots, i_{d-1}\}$ and $J = \{j_1, \dots, j_{d+1}\}$ in $[n]$ as

$$R_{I,J} := p_{i_1,\dots,i_{d-1},j_1} p_{j_2,\dots,j_{d+1}} - \sum_{2 \le k \le d+1} p_{i_1,\dots,i_{d-1},j_k} p_{j_1,\dots,\hat{j_k},\dots,j_{d+1}}.$$

The *Plücker ideal* is the ideal generated by all Plücker relations, that is $\mathcal{I}_{d,n} := \langle R_{I,J} : I \in \binom{[n]}{d-1}, J \in \binom{[n]}{d+1} \rangle$. As a consequence of the proof of Theorem 1.5 the homogeneous coordinate ring of the Grassmannian with respect to the Plücker embedding, denoted $\mathbb{K}[\mathrm{Gr}_{d,n}]$, is of form:

$$\mathbb{K}[\mathrm{Gr}_{d,n}] = \mathbb{K}[p_I : I \in \tbinom{[n]}{d}]/\mathcal{I}_{d,n}.$$

Exercise 2. Verify that for $d = 2$ all Plücker relations are of form $p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}$ where $1 \le i < j < k < l \le n$

The Grassmannian admitds a stratification determined by combinatorial objects called matroids.

Definition 1.6. A *matroid* is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B} \neq \varnothing$ a collection of subsets of E called *bases* such that for all distinct $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$ there exists $b_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ is a basis.

A point $V \in \mathrm{Gr}_{d,n}$ determines a matroid, denoted by $\mathcal{M}(V)$, on $[n]$ with bases $\left\{ I \in \binom{[n]}{d} : p_I(V) \neq 0 \right\}$. Matroids of this form are called *realizable (over* $\mathbb{K}$*)*.

Example 1.7. Consider $M = \begin{pmatrix} 1 & 0 & -3 & -6 \\ 0 & 1 & 1 & 2 \end{pmatrix} \in \mathrm{Gr}_{d,n}$ then $p_{12}(M) = p_{13}(M) = 1$, $p_{14}(M) = 2$, $p_{23}(M) = 3$, $p_{24}(M) = 6$, and $p_{34} = 0$. Hence,

$$\mathcal{M}(M) = ([4], \{12, 13, 14, 23, 24\}).$$

This way we obtain the *matroid (or Gelfand–Serganova) stratification* of the Grassmannian

$$\mathrm{Gr}_{d,n} = \bigcup_{\mathcal{M} \subseteq \binom{[n]}{d}} \{V \in \mathrm{Gr}_{d,n} : \mathcal{M}(V) = \mathcal{M}\},$$

see [GGMS87]. The topology of strata can be as complicated as any projective variety. Surprisingly, this changes drastically if instead of focussing on the entire complex Grassmannian we focus our attention on the positive part of the real Grassmannian. The study of this object was pioneered by Postnikov.

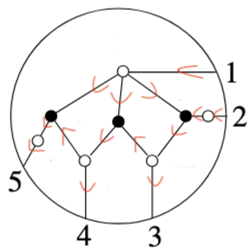


FIGURE 2. A plabic graph with perfect orientation and labelling of its faces.

1.3. **Totally positive Grassmannians and positroids.** For this section let $\mathbb{K} = \mathbb{R}$. Define the *totally positive* (resp. *totally non negative*) *Grassmannian* as

$$\begin{aligned} \mathrm{Gr}_{d,n}^{>0} &:= \{V \in \mathrm{Gr}_{d,n} : p_I(V) > 0, I \in \tbinom{[n]}{d}\}, \\ \mathrm{Gr}_{d,n}^{\geq 0} &:= \{V \in \mathrm{Gr}_{d,n} : p_I(V) \geq 0, I \in \tbinom{[n]}{d}\}. \end{aligned}$$

Recall that Plücker coordinates are defined up to overall scaling. In particular, the condition $p_I(V) > 0$ for $V \in \mathrm{Gr}_{d,n}$ has to be understood as there *exists* a matrix M_V representing V such that $p_I(M_V) > 0$. From now on we will use this abuse of notation.

Example 1.8. $M = \left(\begin{smallmatrix} 1 & 0 & -3 & -6 \\ 0 & 1 & 1 & 2 \end{smallmatrix}\right) \in \mathrm{Gr}_{d,n}^{\geq 0}$ but $A \notin \mathrm{Gr}_{d,n}^{>0}$ as $p_{34}(A) = 0$.

Theorem 1.9 (Postnikov)**.** For $\mathcal{M} \subseteq \binom{[n]}{d}$ let $S_\mathcal{M} := \{V \in \mathrm{Gr}_{d,n}^{\geq 0} : p_I(V) > 0 \Leftrightarrow I \in \mathcal{M}\}$. Then

$$\mathrm{Gr}_{d,n}^{\geq 0} = \bigcup S_\mathcal{M}$$

is a cell decomposition (in fact, a regular CW decomposition), i.e. each $S_\mathcal{M}$ is an open ball.

If $S_\mathcal{M} \neq 0$ then $\mathcal{M}$ is called a *positroid* and $S_\mathcal{M}$ a *positroid cell.*

1.4. **Plabic graphs.**

Definition 1.10. A *plabic graph G* is a *pla*nar *bi*colored graph embedded in a disk with (non colored) vertices in the boundary of the disk labelled $1, \dots, n$ in clockwise order, such that:
(i) each boundary vertex is incident to a single edge
(ii) each internal vertex is colored black or white
(iii) all vertices are connected to the boundary by some path
(iv) G has no internal leaves

We denote by $F(G)$ the set of faces of G. Faces adjacent to the boundary are called *boundary faces*, all other faces are called *internal.*

Definition 1.11. A *perfect orientation* of a plabic graph G is an orientation of its edges satisfying that every black vertex has a unique outgoing arrow and every white vertex has a unique incoming arrow.

A plabic graph G is called *of type (k,n)* if it has n boundary vertices and

$$k - (n-k) = \sum_{v \text{ vertex}} \mathrm{col}(v)(\deg(v) - 2),$$

where $\mathrm{col}(v) = 1$ if v is black and -1 if v is white. Denote by $I_\mathcal{O}$ the set of boundary vertices that are sources of a given perfect orientation $\mathcal{O}$.

Exercise 3. Let G be perfect orientable of type (k, n). Show that $|I_\mathcal{O}| = k$ for all perfect orientations $\mathcal{O}$.

We define the *matroid of G* as

$$\mathcal{M}(G) := \{I_\mathcal{O} \subset [n] : \mathcal{O} \text{ perfect orientation of } G\}$$

Example 1.12. The source set of the perfect orientation depicted in Figure 1.4 is $I_\mathcal{O} = \{1, 2\}$.

Let G be a plabic graph of type (k, n) with perfect orientation $\mathcal{O}$ and let $y = (y_f)_{f \in F(G)}$ be an assignment of weights $y_f \in \mathbb{R}_{>0}^{F(G)}$ to faces $f \in F(G)$ such that $\prod_{f \in F(G)} y_f = 1$. The triple $(G, \mathcal{O}, y)$ is called a *plabic network*.

For every pair of boundary vertices (i, j) we define the following number counting weighted paths from i to j in $(G, \mathcal{O}, y)$:

$$M_{ij} := \sum_{P : b_i \to b_j} (-1)^{\mathrm{wind}(P)} \prod_{f \text{ to the left of } P} y_f \tag{1}$$

with wind(P) counting the number of 360° turns of P.

Exercise 4. Consider the plabic network depicted in Figure 1.4. We have $M_{14} = y_1 y_2 y_3 + y_1 y_2 y_3 y_6$. Compute the missing M_{ij} for $i, j \in [5]$.

Theorem 1.13 (Prop. 11.7, Cor. 16.7 in Postnikov)**.** The assignment

$$(G, \mathcal{O}, y) \mapsto (M_{ij})_{i \in [k], j \in [n]}$$

defines the *boundary measurement map*

$$Meas : \{\text{plabic networks of type } (k, n)\} \to \mathrm{Gr}_{k,n}^{\geq 0}.$$

Restricting to the set of all plabic networks with fixed underlying plabic graph G yields $Meas_G : \mathbb{R}_{>0}^{F(G)-1} \to \mathrm{Gr}_{k,n}^{\geq 0}$ with

$$Meas_G(\mathbb{R}_{>0}^{F(G)-1}) = S_{\mathcal{M}(G)}.$$

Plabic graphs admit certain combinatorial operations called moves. They are determined as follows.

Figure 5: Square move (M1), and merging vertices of same colour (M2)

Figure 6: Insert/remove degree two vertex (M3), and reducing parallel edges (R)

Definition 1.14. Two plabic graphs G, G' are called *(move) equivalent* if they are related by a sequence of moves (M1),(M2),(M3). If the reduction (R) can be applied to G', it is called *reducible*. A plabic graph is called *reduced* if there are no reducible plabic graphs in its move equivalence class.

Exercise 5. Show that using (M1)-(M3) a plabic graph G can be transformed to either a trivalent or a bipartite plabic graph G'.

Theorem 1.15 (Theorem 12.7 in [Pos06])**.** Let G be a reduced plabic graph. Then G is perfectly orientable and the boundary measurement map $Meas_G : \mathbb{R}_{>0}^{F(G)-1} \to S_{\mathcal{M}(G)}$ gives a subtraction-free rational parametrization of the corresponding totally nonnegative Grassmann cell $S_{\mathcal{M}(G)}$. Moreover,

(i) $\dim S_{\mathcal{M}(G)} = F(G) - 1$
(ii) for any positroid cell S there exists G such that $S = S_{\mathcal{M}(G)}$
(iii) for any two different parametrizations $Meas_G$ and $Meas_{G'}$ of the same cell, the plabic graphs G and G' are related by moves (M1) - (M3).

2. Cluster algebras

2.1. Motivation: total positivity. We start with a motivational example of the totally positive Grassmannian $\mathrm{Gr}_{2,n}$. Let $M = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{smallmatrix}\right)$ be a $2 \times n$ matrix with real entries representing a point in $\mathrm{Gr}_{2,n}$. Then M is contained in the totally positive Grassmannian $\mathrm{Gr}_{2,n}^{>0}$ if all its maximal minors (Plücker coordinates) satisfy

$$p_{ij}(M) := \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = a_i b_j - a_j b_i > 0 \quad \text{for all } i < j.$$

Example 2.1. Consider $M = \left(\begin{smallmatrix} 1 & 0 & -3 & -5 \\ 0 & 2 & 1 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 4}$. We compute its maximal minors

$$\begin{array}{rrr} p_{12} = \det \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) = 2, & p_{13} = \det \left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right) = 1, & p_{14} = \det \left(\begin{smallmatrix} 1 & -5 \\ 0 & 1 \end{smallmatrix}\right) = 1, \\ p_{23} = \det \left(\begin{smallmatrix} 0 & -3 \\ 2 & 1 \end{smallmatrix}\right) = 6, & p_{24} = \det \left(\begin{smallmatrix} 0 & -5 \\ 2 & 1 \end{smallmatrix}\right) = 10, & p_{34} = \det \left(\begin{smallmatrix} -3 & -5 \\ 1 & 1 \end{smallmatrix}\right) = 2. \end{array}$$

In particular, $M \in \mathrm{Gr}_{2,4}^{>0}$. We observe that the Plücker relation holds $p_{12}p_{34} + p_{14}p_{23} = 2\cdot 2 + 1\cdot 6 = 10 = p_{13}p_{24}$.

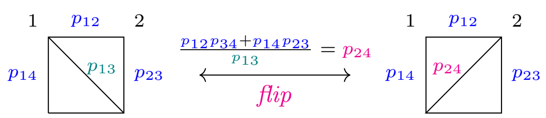
Definition 2.2. A set of Plücker coordinates Δ for $\mathrm{Gr}_{2,n}$ is called a *positivity test* if given any matrix $M \in \mathrm{Gr}_{2,n}$ we have that $p_{ij}(M) > 0$ for all $p_{ij} \in \Delta$ implies $M \in \mathrm{Gr}_{2,4}^{>0}$.

A positivity test is called *efficient* if it is of minimal cardinality.

Example 2.3. In the case of $\mathrm{Gr}_{2,4}$ if $p_{12}, p_{13}, p_{14}, p_{23}, p_{34} > 0$ then

$$p_{24} = \frac{p_{12}p_{34} + p_{14}p_{23}}{p_{13}} > 0.$$

Hence, the set $\{p_{13}, p_{12}, p_{14}, p_{23}, p_{34}\}$ is a positivity test. The same is true for $\{p_{24}, p_{12}, p_{14}, p_{23}, p_{34}\}$ and we can visualize the two as the arcs and boundary edges of a triangulation of a quadrilateral

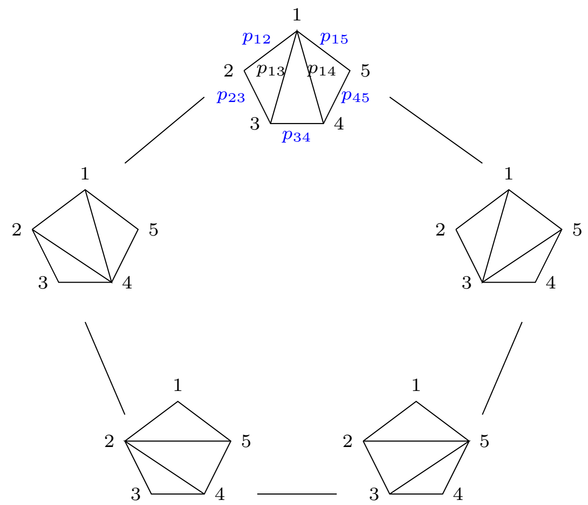
(2) 

The operation that exchanges one triangulation for another by changing the diagonal is called a *flip*.

The above example can be extended to $\mathrm{Gr}_{2,n}$: there is a bijection between efficient positivity test and triangulations of an n-gon. Hence, to determine whether a given matrix represents a point in $\mathrm{Gr}_{2,n}^{>0}$ or not it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_{2,n} + 1$ Plücker coordinates for positivity.

The triangulations of an n-gon are organized in the *associahedron*, also called *Stasheff polytope*: the vertices of this polytope are in correspondence with triangulations of the n-gon, two vertices are connected by an edge if the two triangulations are related by a flip.

Example 2.4. For $n = 5$ the associahedron is depicted below:



The positivity tests associated to each triangulation can be read off the labels of its edges (diagonals and boundary edges). For example, in case of the triangulation at the top the positivity test is $\{p_{13}, p_{14}, p_{12}, p_{15}, p_{23}, p_{34}, p_{45}\}$.

2.2. **Quivers and mutation.** A *quiver* Q is a directed graph, consisting of a finite set of vertices and arrows between them. Technical assumption: for us, a quiver Q does not contain any loops (arrows starting and ending at the same vertex) or 2-cycles (directed cycles consisting of two arrows).

We label the vertices $1, \dots, m$ and divide the vertex set $\{1, \dots, m+n\}$ into two subsets: $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$ for some $n \leq m$. The vertices $1, \dots, n$ are called *mutable* while the vertices $n+1, \dots, n+m$ are called *frozen*. When visualizing a quiver, frozen vertices are depicted in a box, e.g. $1 \rightrightarrows 2 \to \boxed{3}$.

Definition 2.5 (Quiver mutation)**.** Given a quiver Q and a mutable vertex k, the *mutation in direction* k $\mu_k(Q)$ is a quiver obtained from Q in three steps:

(i) for every path $i \to k \to j$ add an arrow $i \to j$;
(ii) invert every arrow incident to k;
(iii) remove a maximal set of disjoint 2-cycles and all arrows between frozen vertices that have been created as a result of steps (i) and (ii).

Example 2.6. The mutation in direction 2 of the quiver $1 \rightrightarrows 2 \to \boxed{3}$ is

$$\begin{tikzcd} & 2 \arrow[dl, shift left] \arrow[dl, shift right] & \\ 1 \arrow[rr, shift left] \arrow[rr, shift right] & & \boxed{3} \arrow[ul] \end{tikzcd}$$

Exercise 6. (i) Show that quiver mutation is an *involution*, that is $\mu_k(\mu_k(Q)) = Q$.

(ii) Go to Bernhard Keller's website [1] and explore the mutation of the quivers

$$Q_1 : \begin{tikzcd} \boxed{4} & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[dr] & \boxed{5} \\ 1 \arrow[u] \arrow[ur] \arrow[rr] & & 3 \arrow[u] \end{tikzcd} \quad \text{and} \quad Q_2 : \begin{tikzcd} & 2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

What do you observe about the quivers you obtain as results of iterated mutations?

A quiver Q with n mutable vertices and all the quivers obtained from Q by iterated mutation are in bijection with the vertices of the n-regular tree $\mathbb{T}_n$: it is an infinite graph whose vertices are all adjacent to exactly n edges labelled $1, \dots, n$ at every vertex. The bijection between the mutation class of Q and the vertices of $\mathbb{T}_n$ is obtained as follows: place Q at a vertex of $\mathbb{T}_n$, then for every $k \in \{1, \dots, n\}$ there is a unique vertex connected to Q via an edge labelled k.

[1] https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/

Associate the quiver $\mu_k(Q)$ to this vertex. Now iterate this process with each of the quivers $\mu_1(Q), \dots, \mu_n(Q)$. Notice that this is well defined as quiver mutation is an involution.

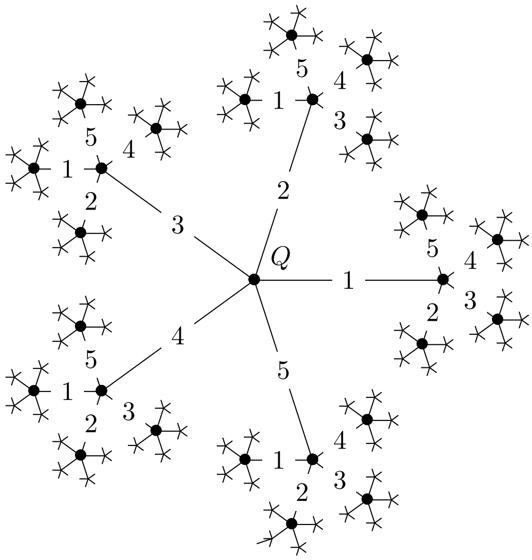


FIGURE 3. The 5-regular tree depicted until depth four.

2.3. **Seeds and mutation.** From now on, we fix $\mathcal{F}$ to be a field of rational functions over $\mathbb{Q}$ of transcendence degree m. It is called the *ambient field* and we will later define cluster algebras as subalgebras of this field.

Definition 2.7. A *seed* s is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ is a collection of variables satisfying $\mathbb{Q}(x_1, \dots, x_m) \cong \mathcal{F}$ called a *cluster* and Q is a quiver with n mutable and m frozen vertices.

Definition 2.8 (Seed mutation)**.** Given a seed $s = (\mathbf{x}, Q)$ and a mutable vertex k of Q, the *mutation in direction k of s*, denoted $\mu_k(s)$, is the pair $(\mu_k(\mathbf{x}), \mu_k(Q))$, where $\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$ with x'_k determined by the *exchange relation*

$$x_k x'_k := \prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j. \tag{3}$$

If the set of vertices $\{i \in Q : \exists i \to k \in Q\}$ is empty, the product is set to 1 (similar for vertices j with arrows $k \to j$).

Two seeds are called *mutation equivalent* is there exists a finite sequence of mutations from one to the other. We denote this as $(Q, \mathbf{x}) \sim (Q', \mathbf{x}')$.

Notice that the products in (3) are indexed by arrows in Q. In particular, if there are multiple arrows say from a vertex i to the mutation vertex k then the variable x_i appears with the exponent equal to the number of arrows $i \to k$ on the right hand side.

Exercise 7. Verify that seed mutation is an *involution*: $\mu_k(\mu_k(s)) = s$.

Example 2.9. Consider the quiver $1 \to 2$ without frozen vertices. To simplify the notation of a seed $(\{x_1, x_2\}, 1 \to 2)$ we write $x_1 \to x_2$. We explore the mutation class of the seed $x_1 \to x_2$ iterating mutations at the vertices 1 and 2, starting with mutation at 1:

$$\begin{tikzcd}
& x_1 \to x_2 \arrow[dl, dash, "1"'] \arrow[dr, dash, "\cong"] & \\
\frac{1+x_2}{x_1} \leftarrow x_2 \arrow[d, dash, "2"'] & & x_2 \leftarrow x_1 \arrow[d, dash, "1"] \\
\frac{1+x_2}{x_1} \to \frac{1+x_1+x_2}{x_1x_2} \arrow[dr, dash, "1"'] & & \frac{1+x_1}{x_2} \to x_1 \arrow[dl, dash, "2"] \\
& \frac{1+x_1}{x_2} \leftarrow \frac{1+x_1+x_2}{x_1x_2} &
\end{tikzcd}$$

For example, after performing the mutation in direction 1 at $\frac{1+x_2}{x_1} \to \frac{1+x_1+x_2}{x_1x_2}$ the new cluster variable is obtained from the exchange relation (3) by

$$\left(1+\frac{1+x_1+x_2}{x_1x_2}\right) \div \frac{1+x_2}{x_1} = \frac{x_1x_2+1+x_1+x_2}{x_2(1+x_2)} = \frac{1+x_1}{x_2}$$

After five mutations we arrive at the seed $x_2 \leftarrow x_1$. Up to a permutation of the vertices this seed coincides with $x_1 \to x_2$ and the mutation pattern will repeat itself after this point. In particular, we have discovered a periodic mutation pattern.

2.4. **Cluster algebra.** If $s = (\mathbf{x}, Q)$ is a seed with $\mathbf{x} = (x_1, \ldots, x_{n+m})$ and let $s' = (\mathbf{x}', Q')$ be a seed obtained from s by a sequence of mutations, then the cluster $\mathbf{x}' = (x_1', \ldots, x_n', x_{n+1}, \ldots, x_{n+m})$ satisfies

$$\mathbb{Q}(x_1', \ldots, x_n', x_{n+1}, \ldots, x_{n+m}) = \mathcal{F}.$$

Definition 2.10. The *cluster algebra* defined by the initial quiver Q is the $\mathcal{F}$-subalgebra

$$\mathcal{A}_Q := \langle \bigcup_{(\mathbf{x}', Q') \sim (\mathbf{x}, Q)} \mathbf{x}' \rangle \subset \mathcal{F}.$$

A first fundamental result is the following.

Theorem 2.11 ([FZ02])**.** The cluster algebra $\mathcal{A}_Q$ only depends on the mutation class of Q.

Example 2.12. Continuing Example 2.9, we find the associated cluster algebra $\mathcal{A}_Q = \left\langle x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\rangle \subset \mathbb{Q}(x_1, x_2)$.

To every (mutable) cluster variable Fomin and Zelevisnky associate its $\hat{y}$*-variable* defined as

$$\hat{y}_k = \frac{\prod_{i \to k \in Q} x_i}{\prod_{k \to j \in Q} x_k}. \tag{4}$$

Notice that if $x_k' = \mu_k(x_k)$ then the $\hat{y}$-variable associated with x_k' satisfies

$$\hat{y}_k' = \hat{y}_k^{-1}. \tag{5}$$

More generally, the tuple $(\hat{y}_1, \ldots, \hat{y}_n)$ associated with a seed $(\mathbf{x}, Q)$ is an example of a Y-seed, (also called coefficients) introduced in [FZ07]. We revisit this notion in Example 2.15. It also plays an important role in the application of cluter algebras to scattering amplitudes in section 3.3.

We close this subsection with the definition of a cluster subalgebra: a subalgebra that is also a cluster algebra which has a compatible cluster structure. The following definition may be found in the preliminary version of a text book authored by Fomin, Williams and Zelevinsky [SF].

Definition 2.13. Let $(Q, (x_i : i \in Q_0))$ be a seed, and let $I \cup J$ be a partition of the vertex set of Q such that there are no arrows between mutable vertices in I and vertices in J. Let Q' be the quiver obtained from Q by deleting all vertices in J (*i.e.* the vertex set of Q' is I). Then the seed $(Q', (x_i : i \in I))$ is called a *restricted seed* of $(Q, (x_i : i \in Q_0))$.

Passing to a restricted seed commutes with mutation, hence yields a *seed subpattern* and induces a *cluster subalgebra*.

2.5. **From triangulations to quivers.** Recall the Grassmannian $\mathrm{Gr}_{2,n}$ with its $\binom{n}{2}$ Plücker coordinates p_{ij} and the correspondence

$$\left\{\begin{matrix}\text{efficient positivity}\\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{triangulations}\\ \text{of an } n\text{-gon}\end{matrix}\right\}.$$

Definition 2.14. Given a triangulation T of the n-gon we define the associated quiver Q_T as follows:

(i) introduce a mutable vertex of Q_T for every diagonal of T;
(ii) introduce a frozen vertex of Q_T for every boundary edge;
(iii) add arrows between the vertices corresponding to each triangle inside T in clockwise order, see Figure 2.5;
(iv) eliminate arrows between frozen vertices.

Further we define the *seed associated with* T as the pair $(\mathbf{x}_T, Q_T)$ where

$$\mathbf{x}_T = (p_{ij} : \overline{ij} \in T),$$

where $\overline{ij} \in T$ runs over all diagonal and boundary edges of T.

Step (ii) of the rule for adding arrows to Q_T is depicted in Figure 2.5.

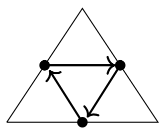


FIGURE 4. How to add arrows to Q_T in between vertices corresponding to a single triangle in T.

Example 2.15. Consider a triangulation T containing a quadrilateral with vertices $i < j < k < l$ and diagonal $\overline{jl}$. So $\mathbf{x}_T$ contains the Plücker coordinates $p_{ij}, p_{jk}, p_{kl}, p_{il}$ and p_{ik}. Then the $\hat{y}$-coordinate associated to p_{ik} is

$$\hat{y}_{ik} = \frac{p_{ij}p_{kl}}{p_{jk}p_{il}}.$$

Example 2.16. In Figure 2.5 we depict a triangulation T of the pentagon and its associated quiver as well as the triangulation T' obtained by performing a flip at the diagonal 13. The mutation at the vertex of Q_T corresponding to the diagonal 13 results in the quiver associated to the triangulation T', so that quiver mutation and flip are compatible.

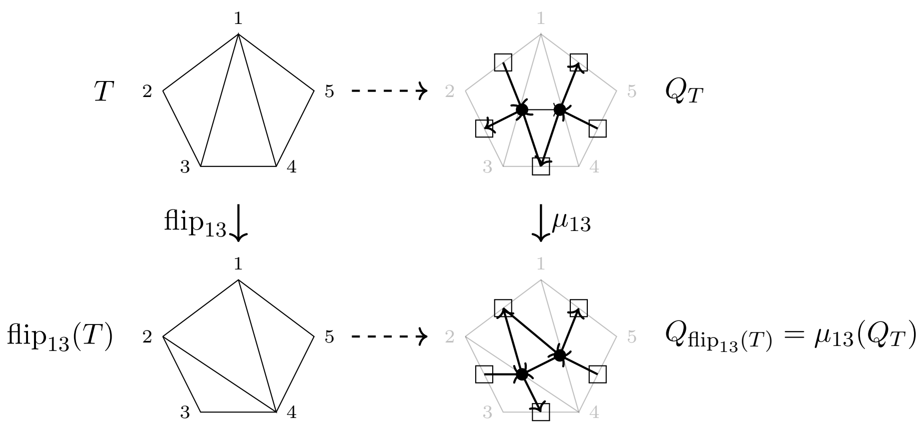


FIGURE 5. A triangulation of the pentagon, its associated quiver, a flip and the mutated quiver.

The above example hints at the following bijection

$$\left\{\begin{matrix}\text{triangulations } T\\ \text{of an } n\text{-gon}\end{matrix}, \mathit{flip}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{quivers of the cluster}\\ \text{algebra } \mathcal{A}_{Q_T}\end{matrix}, \mathit{mutation}\right\}.$$

Moreover, efficient positivity tests for $\mathrm{Gr}_{2,n}$ are in bijection with clusters of the cluster algebra $\mathcal{A}_{Q_T}$. The corresponding cluster algebra is

$$\mathcal{A}_{Q_T} = \mathbb{Q}[p_{ij} : 1 \leq i < j \leq n]/\text{Plücker relations.}$$

2.6. **Laurent phenomenon and finite type classification.** We very briefly summarize some fundamental results about cluster algebras. In the following let $x_1, \dots, x_n$ denote the mutable variables of an initial seed and let $x_{n+1}, \dots, x_m$ denote the frozen variables.

Theorem 2.17 (Laurent Phenomenon [FZ02])**.** All cluster variables are Laurent polynomials in the cluster variables of the initial seed with integer coefficients. More precisely, they are contained in

$$\mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}, x_{n+1}, \dots, x_{n+m}].$$

This result is extremely powerful and opens up the way to a geometric viewpoint on cluster algebras that we will see more about in the following section. However, the Laurent phenomenon is insufficient for the applications in total positivity, as those need more control about the signs of the coefficients appearing in the Laurent expressions. In fact, Fomin and Zelevisnky conjectured:

Theorem 2.18 (Positivity conjecture in [FZ02])**.** All cluster variables are contained in $\mathbb{N}[x_1^{\pm}, \dots, x_n^{\pm}, x_{n+1}, \dots, x_{n+m}]$.

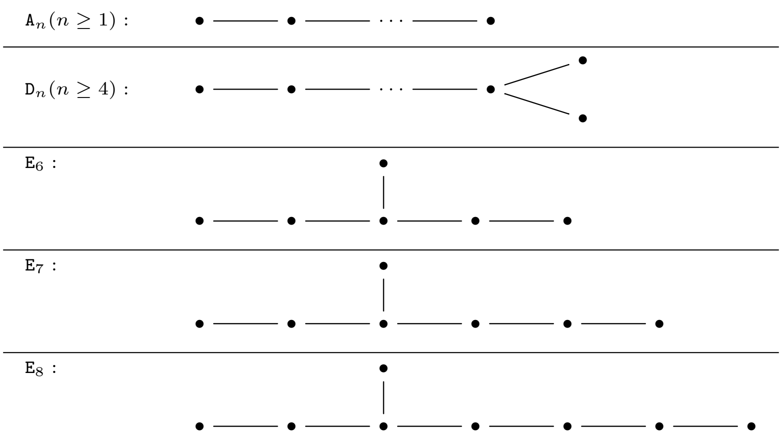
The positivity conjecture has gained a lot of interest due to its importance in total positivity and was approached successfully in varying generality by numerous mathematicians using representation theoretic techniques. In our setting (where a cluster algebra is associated to a quiver) it was proven by Lee and Schiffler.

Theorem 2.19 ([?])**.** The positivity conjecture is true.

There is a more general notion of cluster algebra associated with a skew-symmetrizable matrix. The proof in all generality however was obtained using techniques from birational geometry, more precisely log Calabi–Yau varieties, inspired by mirror symmetry, [GHKK18].

In Example 2.9 we observed a periodicity in the mutation pattern. As a result this cluster algebra only has a finite number of clusters, namely five, see Example 2.12. More generally, we call a cluster algebra of *finite type* if its set of cluster variables is finite, or equivalently, if its set of seeds is finite. Fomin and Zelevinsky classified cluster algebras of finite type as follows.

Theorem 2.20 (Fomin–Zelevinsky 2003)**.** A cluster algebra $\mathcal{A}_Q$ is of finite type if and only if (the mutable part of) Q is mutation equivalent to an orientation of a type `ADE` Dynkin diagram:



2.7. **Quivers from plabic graphs.** In section 1 we have seen the totally positive Grassmannian and how reduced plabic graphs yield parametrizations of its positroid cells. In the case of the Grassmannian $\mathrm{Gr}_{2,n}$ we have seen how triangulations of the n-gon and bijectively cluster of the associated cluster algebra provide positivity tests. It is therefore a natual question to ask whether the observations made for $\mathrm{Gr}_{2,n}$ extend to arbitrary Grassmannians. In this section we summarize some of the results based on [Sco06] and [Pos06].

Recall from Exercise 5 that every reduced plabic graph may be transformed into either a trivalent or a bipartite plabic graph.

Definition 2.21. Let G be a bipartite reduced plabic graph. We define its associated quiver Q_G as follows:

- mutable vertices of Q_G correspond to internal faces of G
- frozen vertices of Q_G correspond to boundary faces of G
- arrows of Q_G are perpendicular to internal edges of G and oriented so that the white vertex is on its left

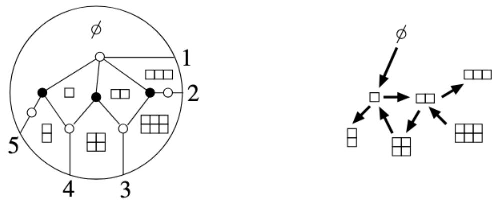


FIGURE 6. A reduced plabic graph of type $(3, 5)$ and its associated quiver.

The aim of this section is to associate a cluster algebra to a plabic graph. To define the cluster we need another combinatorial tool introduced by Postnikov, called trips, building on previous work of Kenyon who called it *zig-zag paths* [Ken04].

Definition 2.22. Let G be a reduced plabic graph of type (d, n). For each $i \in [n]$ we define the *trip* T_i as the oriented path starting at the boundary vertex i and following the *rules of the road*

- turning maximally right at a black vertex
- turning maximally left at a white vertex

until it arrives at another boundary vertex $j =: \sigma_G(i)$. The resulting map $\sigma_G : [n] \to [n]$ is in fact a permutation called the *trip permutation of G*.

The trip permutation is an invariant of the move equivalence class of a plabic graph [Pos06, Theorem 13.4] Notice that each trip divides the disk into two parts: left and right with respect to the orientation of T_i.

Lemma 1 ([Pos06])**.** Placing an i in every face of G to the left of T_i for every $1 \leq i \leq n$ yields a face labelling of G so that every face contains a d-element subset of $[n]$.

Define the permutation $\sigma_{d,n} := [n-d+1, \ldots, n, 1, \ldots, n-d]$. Given a reduced plabic graph of type (d, n) with trip permutation $\sigma_G = \sigma_{d,n}$ we define a cluster

$$\mathbf{x}_G := (p_I : I \text{ is the face label of some face in } G).$$

Theorem 2.23 ([Sco06])**.** Given a reduced plabic graph G with trip permutation $\sigma_{d,n}$ and associated quiver Q_G and cluster $\mathbf{x}_G$. Then $(Q_G, \mathbf{x}_G)$ is a seed for the cluster algebra $\mathcal{A}_{d,n}$ which is isomorphic to $\mathbb{Q}[\mathrm{Gr}_{d,n}]$.

A seed $(\mathbf{x}, Q)$ satisfying that $\mathbf{x}$ consists only of Plücker coordinates is called a *Plücker seed.* There is a bijection

$$\left\{\begin{matrix}\text{reduced plabic graphs}\\ G \text{ with } \sigma_G = \sigma_{k,n},\\ \textit{squaremoves}\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{Plücker seeds } (Q, \mathbf{x}) \text{ in } \mathcal{A}_{k,n},\\ \textit{mutations at 4-valent vertices}\end{matrix}\right\}$$

Scott further established the following finite type classification for Grassmannian cluster algebras.

Theorem 1 ([Sco06])**.** The only finite type Grassmannian cluster algebras (assuming $k \leq n$) are the following:

(i) $\mathrm{Gr}_{2,n}$ is of cluster type $\mathtt{A}_{n-3}$,

(ii) $\mathrm{Gr}_{3,6}$ is of cluster type $\mathtt{D}_4$,
(iii) $\mathrm{Gr}_{3,7}$ is of cluster type $\mathtt{E}_6$,
(iv) $\mathrm{Gr}_{3,8}$ is of cluster type $\mathtt{E}_8$.

All other Grassmannian cluster algebras are of infinite cluster type.

2.8. **Cluster varieties.** Shortly after cluster algebras were introduced by Fomin and Zelevinsky, Fock and Goncharov introduced the geometric analogue, cluster varieties, algebraic schemes that generalize algebraic tori [FG06, FG09]. Much of the geometric theory of cluster varieties is similar to toric geometry, for example cluster varieties come in dual pairs, and can be compactified by certain generliazations of polytopes and fans [GHKK18, CMNC22]. In this section we very briefly summarise the definition of type $\mathcal{A}$ cluster varieties and their tropical geometry.

Definition 2.24 (§1.2 in [FG09])**.** Given a seed $s=(Q,\mathbf{x})$ define the *seed torus* $T_s:=(\mathbb{C}^*)^{n+m}_{\mathbf{x}}=((\mathbb{C}^*)^{n+m},\mathbf{x})$ as the algebraic torus $(\mathbb{C}^*)^{n+m}$ endowed with coordinates $\mathbf{x}=(x_1,\dots,x_{n+m})$. The $\mathcal{A}$-cluster variety associated to given seed $s_0=(Q_0,\mathbf{x}_0)$ is defined as the scheme glued from seed tori

$$\mathcal{A}_{s_0}:=\bigcup_{s\text{ mutation equivalent to }s_0} T_s$$

subject to the transition functions induced by the mutation rule (3) , that is, for s,s' two seeds related by mutation in direction k we have $\mu_{s,s'}:T_s\dashrightarrow T_{s'}$

$$(t_1,\dots,t_{n+1})\mapsto\left(t_1,\dots,\frac{1}{t_k}\left(\prod_{i\to k\text{ in }Q}t_i+\prod_{k\to j\text{ in }Q}t_j\right),\dots,t_{n+m}\right). \tag{6}$$

The seed tori are glued along the biggest open subset where the transition functions are well defined.

Example 2.25. The affine cone of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$, denoted by $\widetilde{\mathrm{Gr}_2}(\mathbb{C}^4)$ contains the cluster variety

$$(\mathbb{C}^*)^5_{p_{13},p_{12},p_{23},p_{34},p_{14}}\cup(\mathbb{C}^*)^5_{p_{24},p_{12},p_{23},p_{34},p_{14}}$$

with gluing given by

$$(t_{13},t_{12},t_{23},t_{34},t_{14})\mapsto\left(\frac{t_{12}t_{34}+t_{14}t_{23}}{t_{13}},t_{12},t_{23},t_{34},t_{14}\right).$$

2.9. **Tropicalization.** Observe that the transition fundtions (6) does not involve any substraction. In particular, the transition functions are well defined over a *semifield*: a set P equipped with the operations of addition and multiplication, so that addition is commutative and associative, multiplication makes P an abelian group, and they are compatible in the sense that $(a+b)c=ac+bc$ for $a,b,c\in P$. Examples include $\mathbb{R}_{>0}$ or $\mathbb{Z}^t=(\mathbb{Z},\min,+)$. Denote by $T_s(P)$ the P-points of T_s for any semifield P. For example, we have an identification as sets

$$T_s(\mathbb{Z}^T)\equiv N_s\otimes_{\mathbb{Z}}(\mathbb{Z}^t)^\times,$$

with N_s denoting the cocharacter lattice of T_s and $(\mathbb{Z}^T)^\times$ denoting the multiplicative group of $\mathbb{Z}^t$, see *e.g.* [GHKK18, §2]. Let s,s' be two seeds related by mutation in direction k. Then the transition functions (6) over $\mathbb{Z}^T$ are piecewise linear maps $\mu^T_{s,s'}:N_s\to N_{s'}$

$$(a_1,\dots,a_n)\mapsto\left(a_1,\dots,-a_k+\min\left(\sum_{i\to k}a_i,\sum_{k\to j}a_j\right),\dots,a_n\right) \tag{7}$$

Definition 2.26. The *Fock-Goncharov tropicalization* (or *FG tropicalization*, for short) of the cluster variety $\mathcal{A}_s$ is $\mathcal{A}_s(\mathbb{Z}^T):=\bigcup_{s'\sim s}N_{s'}$ glued along the tropical transition functions of form $\mu^T_{s,s'}$ as given in (7).

Given a mutation sequence $s\overset{k_1}{\to}s_1\to\cdots\to s_{r-1}\overset{k_r}{\to}s'$ denote by $\boldsymbol{\mu}^T_{s,s'}$ the composition of tropical transition functions $\mu^T_{s,s_1}\circ\cdots\circ\mu^T_{s_{r-1},s'}$. A point in the FG tropicalization $\mathcal{A}_s(\mathbb{Z}^T)$ is therefore an equivalence class of points $(a_{s'} \in N_{s'})_{s \sim s'}$, one for each seed s' mutation equivalent to s such that the maps $\mu^T_{s,s'}$ map one representative to another.

The following construction may be found in [GHKK18, §2]: denote by N_s^+ the positive orthant in $N_s \otimes_{\mathbb{Z}} \mathbb{R}$. Each $\boldsymbol{\mu}^T_{s,s'}$ yields a fan (of linear domains) in $N_{s'} \otimes_{\mathbb{Z}} \mathbb{R}$ and the pullback of $N_{s'}^+$ is a cone in this fan.

Definition 2.27 (Definition 2.9 in [GHKK18])**.** For a given seed s define the fan $\Sigma_s \subset N_s \otimes \mathbb{R}$ as the union of the full dimensional cones $(\boldsymbol{\mu}^T_{s,s'})^{-1}(N_{s'}^+)$ for all s' mutation equivalent t s. This fan is called the *(Fock-Goncharov) cluster complex.*

Moreover, the cluster complex is contained in the fan obtained as the common refinement of all the fans that are domains of linearity of all the tropical transition functions. Notice that, if s does not determine a cluster algebra of finite type, then the cluster complex is infinite. In fact, Σ_s is a complete fan if and only if s is of finte type.

Example 2.28. In Example 2.25 we have seen the cluster variety inside $\widetilde{\mathrm{Gr}_2}(\mathbb{C}^4)$, namely two copies of $(\mathbb{C}^*)^5$ with two sets of coordinates given by the two seeds as depicted in (2). The cluster complex is a union of two cones determined by the domains of linearity of the tropical transition function

$$(a_{13}, a_{12}, a_{14}, a_{23}, a_{34}) \mapsto (-a_{13} + \min(a_{12} + a_{34}, a_{14} + a_{23}), a_{12}, a_{14}, a_{23}, a_{34}).$$

So the cluster complex consists of the two cones

$$\sigma_s = \{(u_{13}, u_{12}, u_{14}, u_{23}, u_{34}) \in \mathbb{R}^5 : u_{12} + u_{34} \geq u_{14} + u_{23}\}$$

and

$$\sigma_{s'} = \{(u_{13}, u_{12}, u_{14}, u_{23}, u_{34}) \in \mathbb{R}^5 : u_{12} + u_{34} \leq u_{14} + u_{23}\}$$

Let A be a finite type cluster algebra of rank d and $x_1, \ldots, x_N$ all its cluster variables, so that

$$A \cong \mathbb{Z}[x_1, \ldots, x_N]/I \tag{8}$$

There is an interesting connection between the positive part of the tropicalization of the ideal I and the FG tropicalization of the cluster variety $\mathcal{A}$. Before stating the theorem it is necessary to recall some notions regarding the tropicalization of an ideal. For more details consider [MS15].

Definition 2.29. Let $f = \sum_{m \in \mathbb{Z}^N_{\geq 0}} a_m \mathbf{x}^m \in k[x_1, \ldots, x_N]$ be a polynomial. The for $w \in \mathbb{R}^N$ we define its *initial form with respect to* w as

$$\mathrm{in}_w(f) = \sum_{n \in \mathbb{Z}^N_{\geq 0} : n = \min(m \cdot w : m \in in \mathbb{Z}^N_{\geq 0} \text{ with } a_m \neq 0)} a_n \mathbf{x}^n.$$

For an ideal $I \subset k[x_1, \ldots, x_N]$ its *initial ideal with respect to* w is defined as $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$.

Then the *Gröbner fan* of I, defined in [MR88], is a full dimensional fan in $\mathbb{R}^N$ whose cones are determined by inital ideals of I, that is by the equivalence relation

$$w \sim v \text{ if and only if } \mathrm{in}_w(I) = \mathrm{in}_v(I).$$

Maximal cones in the Gröbner fan have monomial initial ideals associated to them. Monomial initial ideals can equivalently (and more classically) be obtained by monomial orders instead of weight vectors. For a generic weight vector w the initial ideal is monomial. There is an interesting non generic locus in the Gröbner fan, it is called the *tropicalization of* I, denoted by $\mathcal{T}(I)$, and it is the subfan consisting of only those cones with initial ideals that do not contain any monomials.

If the ideal I is defined over the real numbers positivity questions can also be addressed in the tropicalization.

Definition 2.30 ([SW05])**.** An ideal $I \subset \mathbb{R}[x_1, \ldots, x_N]$ is called totally positive if it does not contain any nonzero polynomial in $\mathbb{R}_{\geq 0}[x_1, \ldots, x_N]$. The *totally positive part of* $\mathcal{T}(I)$ is the subfan of $\mathcal{T}(I)$ consisting only of those cones which have totally positive initial ideals.

The definition of totally positive ideal is motivated by the fact that an ideal is totally positive if and only if there exists a weight vector $w \in \mathbb{R}^N$ such that $V(\mathrm{in}_w(I)) \cap \mathbb{R}^N_{>0} \neq \varnothing$ [ET01].

Coming back to the setup of (8) where the ideal I presents a rank d cluster algebra of finite type and $x_1, \dots, x_N$ are all its cluster variables, then $V(I) \subset \mathbb{A}^N$ contains the corresponding cluster variety $\mathcal{A}$. Fixing an initial seed $s = (Q, (x_1, \dots, x_d))$ determines an embedding

$$\iota_s : \mathcal{A} \hookrightarrow V(I) \quad (a_1, \dots, a_d) \mapsto (a_1, \dots, a_d, X_{d+1}(\mathbf{a}), \dots, X_N(\mathbf{a}))$$

where $X_j \in \mathbb{N}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ is the Laurent polynomial expression of the cluster variable x_j in the variables ob the initial seed s. Moreover, denote by B the *exchange matrix* of the quiver Q associated to s, that is the entry b_{ij} of B is given by the number of arrows between i and j in Q. Precisely, $b_{ij} = \#\{i \to j\} - \#\{j \to i\}$.

Theorem 2.31 ([Bos22])**.** Assuming B is of full rank, the induced map between tropical spaces $\iota_s^T : \mathbb{R}^d \to \mathbb{R}^N$ yields an isomorphism of fans

$$\iota^T(\Sigma_s) = \mathcal{T}^+(I).$$

In particular, there is a bijection between the rays of $\mathcal{T}^+(I)$ and cluster variables and between maximal cones in $\mathcal{T}^+(I)$ and seeds.

Example 2.32. In Example 2.28 we have seen the FG tropicalization of the cluster variety inside $\widetilde{\mathrm{Gr}}_2(\mathbb{C}^4)$. The ideal presenting the corresponding cluster algebra is generated by a single Plücker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$$

The tropicalization of this ideal consists of three maximal cones that are in correspondence with the three monomials of the relation, namely

$$\begin{aligned}
\tau_{12} &= \{w \in \mathbb{R}^6 : w_{12} + w_{34} \leq w_{13} + w_{24} = w_{14} + w_{23}\},\\
\tau_{13} &= \{w \in \mathbb{R}^6 : w_{12} + w_{34} = w_{14} + w_{23} \geq w_{13} + w_{24}\},\\
\tau_{14} &= \{w \in \mathbb{R}^6 : w_{12} + w_{34} = w_{13} + w_{24} \geq w_{14} + w_{23}\}.
\end{aligned}$$

It is not hard to verify that only τ_{12} and τ_{14} belong to $\mathcal{T}^+(I)$. The tropicalization of the map ι is $\iota^T : \mathbb{R}^5 \to \mathbb{R}^6$ given by

$$\begin{pmatrix} u_{13} \\ u_{12} \\ u_{23} \\ u_{34} \\ u_{14} \end{pmatrix} \longmapsto \begin{pmatrix} u_{13} \\ \max\{u_{12}+u_{34}, u_{14}+u_{23}\} - u_{13} \\ u_{12} \\ u_{23} \\ u_{34} \\ u_{14} \end{pmatrix}$$

Observe that in fact $\iota^T(\sigma_s) = \tau_{14}$ and $\iota^T(\sigma_| s') = \tau_{12}$, so that $\iota(\Sigma) = \mathrm{Trop}^+(I_{2,4})$, as predicted by Theorem 2.31.

2.10. **Partial flag varieties.** Consider $1 \leq d_1 < \cdots < d_k < n \in \mathbb{N}$ and define the *partial flag variety*

$$\mathcal{F}_{d_1,\dots,d_k;n} := \{0 \in V_1 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{K}^n : \dim_{\mathbb{K}} V_i = d_i\}.$$

Partial flag varieties admit a natural embedding into a product of Grassmannians given by

$$\mathcal{F}_{d_1,\dots,d_k;n} \hookrightarrow \mathrm{Gr}_{d_1;n} \times \cdots \times \mathrm{Gr}_{d_k;n}, \quad \mathcal{V} \mapsto (V_1, \dots, V_k).$$

Also, when $k = 1$ this is a Grassmannian and similar as in the case of Grassmannians partial flag varieties admit a parametrization induced by minors of matrices representing flags. The *Plücker embedding* of a partial flag variety is given by concatenating the above embedding with the Plücker embedding of each Grassmannian. More precisely,

$$\mathcal{F}_{d_1,\dots,d_k;n} \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{d_k}-1}$$

Let $\mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}]$ denote the (multi-)homogeneous coordinate ring. Plücker coordinates can also be contructed directly as follows. Associate to $\mathcal{V} \in \mathcal{F}_{d_1,\dots,d_k;n}$ a matrix $M_{\mathcal{V}} = (m_{ij}) \in \mathbb{K}^{d_k \times n}$ such that V_i is generated by the first d_i rows of M for all $1 \leq i \leq k$. Then Plücker coordinates are defined as before: Let $1 \leq j \leq k$ and $\{i_1, \dots, i_{d_j}\} \subset \{1, \dots, n\}$, define the *Plücker coordinate*

$$P_{i_1,\dots,i_{d_j}}(\mathcal{V}) := \det(m_{ab})_{1 \leq a \leq d_j,\ b \in \{i_1,\dots,i_{d_\ell}\}}.$$

Example 2.33. Let $M_{\mathcal{V}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -3 \end{bmatrix}$ represent a point in $\mathcal{F}_{1,2,3;4}$. Then, for example, $P_{134}(\mathcal{V}) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}\right) = -6$.

Geiss, Leclerc and Schröer generalized Scott's result about the cluster structure on Grassmannians to partial flag varieties. The following is a summary of the construction of an initial seed for the cluster algebra. For more details on this construction see [BL24]. The combinatorial gadgets used in the construction are pseudoline arrangements (sometimes also called wiring diagrams). For $\mathcal{F}_{d_1,\dots,d_k;n}$ the *pseudoline arrangement* $\mathcal{P}_{d_1,\dots,d_k;n}$ is a pictorial presentation of the permutation $\sigma \in S_n$ whose one-line presentation is

$$\sigma = [d_k + 1, d_k + 2, \dots, n, d_{k-1} + 1, d_{k-1} + 2 \dots, d_k, \dots, 1, \dots, d_1].$$

Notice that σ is the permutation corresponding to the minimal representative of the coset of the longest word in $S_n/(s_i : i \notin \{d_1, \dots, d_k\})$.

Algorithm 1: Pseudoline arrangement

We draw $\mathcal{P}_{d_1,\dots,d_k;n}$ inside a two-dimeinsional positive orthant.

(i) Label the x- and y-axes by $1, \dots, n$
(ii) For each i draw a line segment from $(i, 0)$ to $(i, \sigma(i))$ and another line segment from $(0, \sigma(i))$ to $(i, \sigma(i))$. The union of the two line segments is called the *pseudoline* (or *wire*) ℓ_i.

An example of the resulting pseudoline arrangement $\mathcal{P}_{2,5;7}$ is given in Figure 7.

We associate a quiver $Q_{d_1,\dots,d_k;n}$ to $\mathcal{P}_{d_1,\dots,d_k;n}$ that determines a seed in the cluster structure of the multi-homogeneous coordinate ring $\mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}]$ with respect to the Plücker embedding, compare to [**?**, §9.3.2].

Algorithm 2: From pseudoline arrangement to quiver

(i) **Vertices of $Q_{d_1,\dots,d_k;n}$:**
  (a) mutable vertices of $Q_{d_1,\dots,d_k;n}$ correspond to bounded faces of $\mathcal{P}_{d_1,\dots,d_k;n}$;
  (b) there are two types of frozen vertices: $n-1$ of them correspond to the unbounded faces along the y-axis; additionally there are k frozen vertices, we denote them by $v_{d_1}, \dots, v_{d_k}$.
(ii) **Arrows of $Q_{d_1,\dots,d_k;n}$:** There are four types of arrows:
  (a) from left to right perpendicular to a vertical straight lines segment connecting adjacent faces of $\mathcal{P}_{d_1,\dots,d_k;n}$;
  (b) from top to bottom perpendicular to a horizontal straight line segment connecting adjacent faces of $\mathcal{P}_{d_1,\dots,d_k;n}$;
  (c) diagonally from bottom right to top left through a crossing of two straight line segments connecting faces of $\mathcal{P}_{d_1,\dots,d_k;n}$ that share a vertex;
  (d) arrows to and from the extra frozen vertices $v_{d_1}, \dots, v_{d_k}$: there is an arrow from the face bounded by ℓ_{d_i-1}, ℓ_{d_i} vertically and by $\ell_{d_i+1}, \ell_{d_i+2}$ horizontally to the vertex v_{d_i}, and an arrow from v_{d_i} to the face bounded by ℓ_{d_i} on the left, by ℓ_{d_i+1} on the top and right (this is where ℓ_{d_i+1} bends) and by $\ell_{d_{i+2}}$ on the bottom[2].

The quiver $Q_{2,5;7}$ is depicted in Figure 7. The frozen vertices v_2, v_5 are labelled ω_2, ω_5, respectively.

Every face of the pseudoline arrangement $\mathcal{P}_{d_1,\dots,d_k;n}$ can be associated with a minor of an $n \times n$-matrix. The minors are of form form $\Delta_{I,J}$ with $I, J \subset [n]$ of the same size. In our case, the column index set J is always of form $\{n - |I| - 1, \dots, n\}$. We associate index sets I_F to faces F of $\mathcal{P}_{d_1,\dots,d_k;n}$:

$$I_F := \{i : \ell_i \text{ passes north-east of } F\} \quad \text{and} \quad \Delta_{I_F} := \Delta_{I_F,\{n-|I_F|+1,\dots,n\}}. \tag{9}$$

Observe that all index sets associated to $\mathcal{P}_{d_1,\dots,d_k;n}$ are all of form $[i_j, d_j] \cup [i_{j+1}, d_{j+1}]$. Plücker coordinates are top bound minors, that is for I an index set of cardinality d we have $P_I = \Delta_{[d],I}$. Using Laplace expansion minors associated to $\mathcal{P}_{d_1,\dots,d_k;n}$ are translated to Plücker coordinates

[2] In case that $\ell_{d_{i+2}}$ does not exists the latter arrow also does not exist.

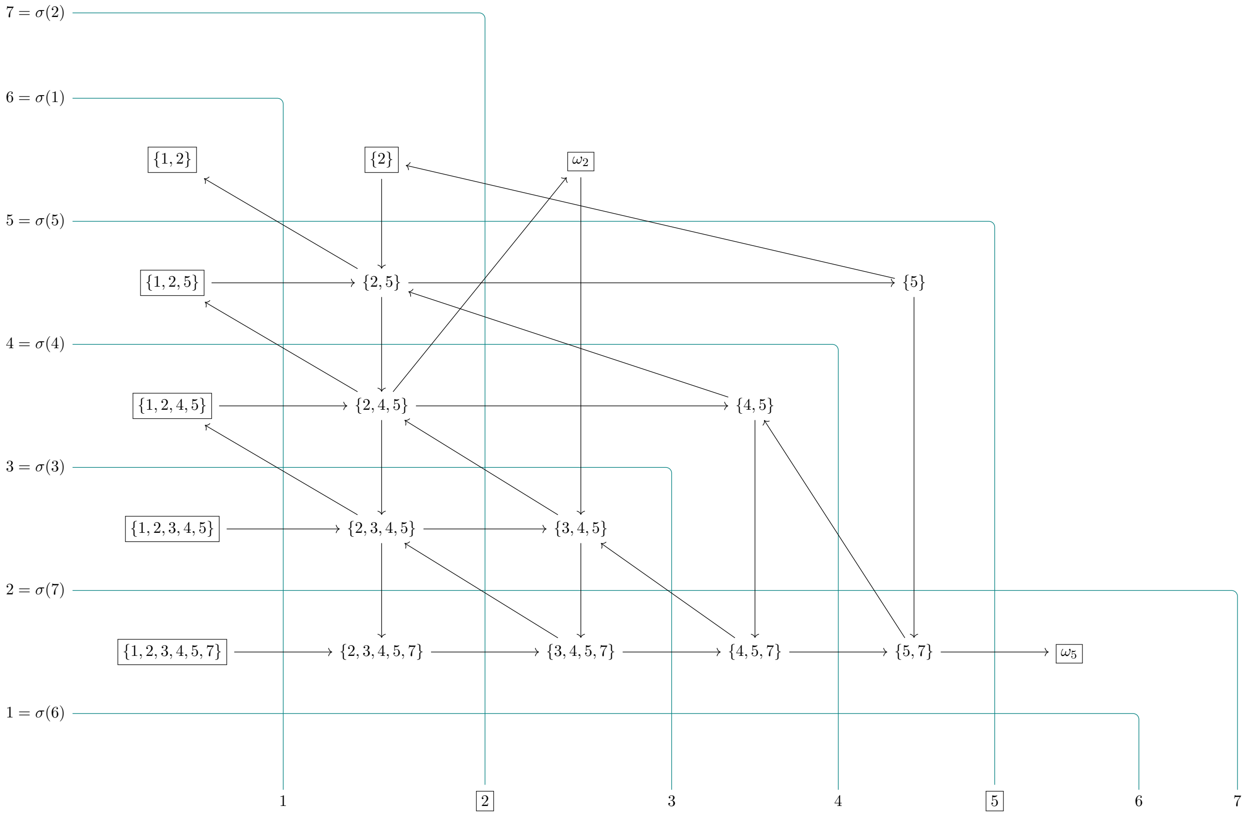


FIGURE 7. The pseudoline arrangement $\mathcal{P}_{2,5;7}$ and its quiver $Q_{2,5;7}$ together with the index sets I_F of the initial minors and the additional frozen vertices corresponding to $d_1 = 2, d_2 = 5$ labelled by ω_2 and ω_5.

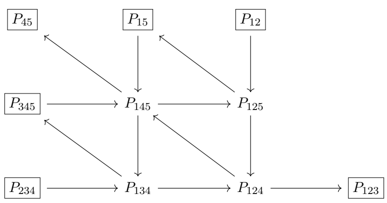
Proposition 2.34 (Proposition 3.4 in [BL24])**.** Consider an arbitrary flag variety $\mathcal{F}_{d_1,\dots,d_k;n}$ and an arbitrary initial minor $\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]}$ with $1 \le i_j \le d_j < i_{j+1} \le d_{j+1} \le n$ and $0 \le j < k$ (recall, that $d_0 := 0, d_{k+1} := n$). Set $\ell = n - d_j - d_{j+1} + i_j + i_{j+1} - 1$. Then

$$\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} = \sum_{J\in\binom{[\ell,n]}{d_j-i_j+1},\ J'=[\ell,n]\setminus J} (-1)^{\Sigma(i_j,d_j,J)} P_{[i_j-1]\cup J} P_{[i_{j+1}-1]\cup J'} \tag{10}$$

where $\Sigma(i_j,d_j,J) := \sum_{q=i_j}^{d_j} q + \sum_{j\in J} j$.

In particular, the initial minors are well defined elements in $\mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}]$.

Exercise 8. For the partial flag variety $\mathcal{F}_{2,3;5}$ we obtain the following initial seed:



Show that the mutable part of the quiver is of type $\mathtt{D}_4$.

Theorem 2 ([GLS08])**.** The quiver $Q_{d_1,\dots,d_k;n}$ together with the cluster $(\Delta_{I_F} : F \text{ face of } \mathcal{P}_{d_1,\dots,d_k;n})$ form an initial seed for the cluster algebra $A_{d_1,\dots,d_k;n}$ which has the property that

$$A_{d_1,\dots,d_k;n} \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}].$$

The expression of initial minors in terms of Plücker coordinates (10) in fact also reveals the tableau associated to these cluster variables. Cluster variables are elements of Lusztig's dual

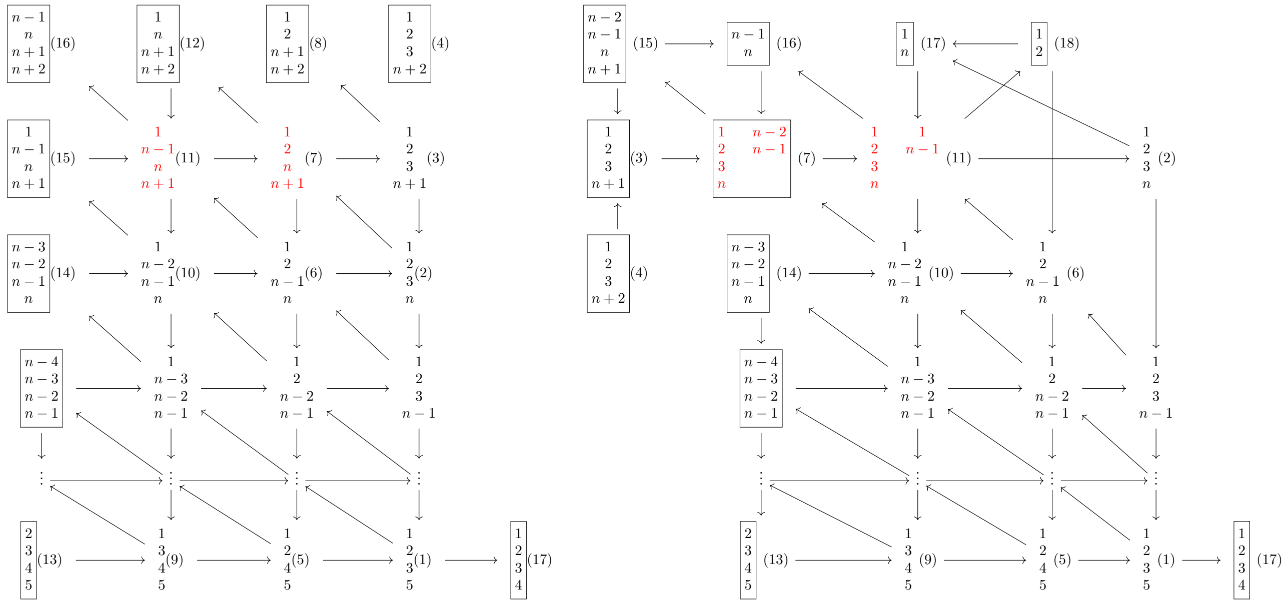


FIGURE 8. LHS: The initial seed for Grassmannian $\mathrm{Gr}_{4;n+2}$. RHS: a seed obtained by mutating (11), (7), (11) of the seed on LHS and freeze (7). The full subquiver on RHS on all vertices but (3), (4), (15) coincides with the initial seed for $\mathcal{F}_{2,4;n}$.

canonical basis which is parametrized by Young tableaux of appropriate shape [Lus90, CP96, HL21, Li24]

Recall the notion of cluster subalgebras from Definition 2.13.

Theorem 3 ([BL24])**.** Consider the partial flag variety $\mathcal{F}_{d_1,\dots,d_k;n}$ with Plücker coordinates $\{P_I : I \in \binom{[n]}{d_i}, i \in [k]\}$, and the Grassmannian $\mathrm{Gr}_{d_k;N}$ with $N = n + d_k - d_1$ with Plücker coordinates $\{p_J : J \in \binom{[N]}{d_k}\}$. Then the natural map

$$\varphi : P_I \mapsto p_{I\cup\{n+1,\dots,n+d_k-|I|\}}$$

extends to an embedding of cluster algebras $\mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}] \hookrightarrow \mathbb{C}[\mathrm{Gr}_{d_k;N}]$.

The proof follows the simple idea of constructing a seed of $\mathrm{Gr}_{d_k;N}$ containing the initial seed of $\mathcal{F}_{d_1,\dots,d_k;n}$ as a restricted seed (up to applying the above map φ). This is done by exhibiting an explicit mutation sequence starting from the initial seed [BL24, §3]. Figure 8 demonstrates this for the partial flag varieties $\mathcal{F}_{2,4;n}$ where the mutation sequence is of length three. In the figure, cluster variables are represented by their tableaux which may be thought of as the leading term of their expression in Plücker coordinates (the number of columns indicating the degree of the expression, a one column tableau correspond to Plücker coordinates with index set the filling of the tableau, see [BL24, §4]).

The result is unexpected from a mathematical point of view and was in fact inspired by the application of cluster algebras in particle physics which is the topic of the last section.

3. Scattering amplitudes

A scattering amplitude is a function used in particle physics to calculate the likelihood of a specific particle interaction. It is proportional to the scattering cross-section, a measurable quantity at particle accelerators which makes it relevant for experiments. Feynman systematized the calculation of scattering amplitudes expressing them in terms of Feynman integrals: they are infinite sums of such integrals indexed by Feynman graphs.

3.1. **Symbol calculus.** In this section I mostly follow the reference [DGR12], however I use the definition of the symbol map as originally given in [GSVV10]. As a function a scattering amplitude is a *multiple polylogarithm* defined recursively by

$$G(a_1,\ldots,a_n;x)=\int_0^x \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t),$$

with $G(x)=G(;x)=1$ unless $x=0$ then $G(0)=0$, $n\geq 0$ and $a_i\in\mathbb{C}$. The vector $(a_1,\ldots,a_n)$ is called the *vector of singularities of* G, its length n is called the *weight* of G. These functions were studied already by Poincar´e and of Lappo- Danilevsky [LD35] who called them *hyperlogarithms* and by Chen [Che77] in his study of iterated integrals. In the physics literature they are often called *Goncharov polylogarithms* due to Goncharov's extensive work on the subject [Gon98, Gon99, GSVV10, Gon13]. Alternatively, they are also known as *multiple polylogarithms*. Multiple polylogarithms form a shuffle algebra with shuffle product defined as

$$G(a_1,\ldots,a_{n_1};x)G(a_{n_1+1},\ldots,a_{n_1+n_2};x)=\sum_{\sigma\in\Sigma(n_1,n_2)}G(a_{\sigma(1)},\ldots,a_{\sigma(n_1+n_2)};x),$$

where $\Sigma(n_1,n_2)\subset S_{n_1+n_2}$ denotes all possible shuffles. More generally, multiple polylogarithms form a Hopf algebra over $\mathbb{Q}$ [**?**, §6].

Example 3.1. The following well known functions have expressions as multiple polylogarithms. Let $\mathbf{a}_n:=(a,\ldots,a)\in\mathbb{C}^n$ for $a\in\mathbb{C}$.

$$\begin{aligned}
G(\mathbf{0}_n;x) &= \frac{1}{n!}\log^n(x),\\
G(\mathbf{a}_n;x) &= \frac{1}{n!}\log^n\left(1-\frac{x}{a}\right),\\
G(\mathbf{0}_{n-1},a;x) &= \mathrm{Li}_n\left(\frac{x}{a}\right),\\
G(\mathbf{0}_n,\mathbf{a}_p;x) &= (-1)^pS_{n,p}\left(\frac{x}{a}\right),
\end{aligned}$$

where $S_{n,p}$ denoted the Nielsen polylogarithm [Nie09].

The recursive definition of a Goncharov polylogarithm is inherit in its differential structure as well. In the generic case (all a_i are distinct and non zero) we obtain

$$dG(a_{n-1},\ldots,a_1;a_n)=\textstyle\sum_{i=1}^{n-1}G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z)d\log\left(\frac{a_i-a_{i+1}}{a_i-a_{i-1}}\right).$$

This motivates the definition of the *symbol map*: a linear map that associates an element in the n-fold tensor product of a certain vector space of 1-forms with a multiple polylogarithm.

Definition 3.2 (Equation 12 in [GSVV10])**.** Given a multiple polylogarithm $G(a_{n-1},\ldots,a_1;a_n)$ its *symbol* is defined as

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n))=\sum_{i=1}^{n-1}\mathcal{S}(G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z))\otimes\frac{a_i-a_{i+1}}{a_i-a_{i-1}}. \tag{11}$$

The simple tensors in $\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n))$ are called the *(symbol) words* of $G(a_{n-1},\ldots,a_1;a_n)$, the tensor factors appearing are called *(symbol) letters* and a complete set of (a choice of) letters is called the *symbol alphabet of* G.

When G is a multiple polylogarithm and a function on a manifold M, then its symbol $\mathcal{S}(G)$ lies in the vector space $\bigotimes_{i=1}^n\mathcal{O}^*_{M,\mathrm{an}}$, where $\mathcal{O}^*_{M,\mathrm{an}}$ denotes the multiplicative group of the invertible analytic functions on M. Then the *symbol map* given by (11) is a linear map.

Example 3.3. We have seen in Example 3.1 that the polylogarithmic fuction $\mathrm{Li}_k(z)=\sum_{j=1}^\infty\frac{z^j}{j^k}$ is a multiple polylogarithm. It satisfies $\mathrm{Li}_1(z)=-\log(1-z)$ and

$$\mathrm{Li}_k(z)=\int_0^z\mathrm{Li}_{k-1}(t)d\log(t).$$

From this presentation we obtain the symbols $\mathcal{S}(\mathrm{Li}_1(z))=-(1-z)$ and $\mathcal{S}(\mathrm{Li}_k(z))=-(1-z)\otimes z\otimes\cdots\otimes z$, where the factor z appears $(k-1)$-many times.

Each tensor factor is understood as a logarithmic one form, that is $d\log(a_i) \equiv \frac{da_i}{a_i}$. This yields the following identities (obtained from multilinearity and shuffle product) with the only caveat that we are switching from multiplicative to additive notation

Proposition 3.4. The symbol of a multiple polylogarithm satisfies the following identities.

(i) **Distributivity.** We have $C \otimes (ab) \otimes D = C \otimes a \otimes D + C \otimes b \otimes D$ and hence, $C \otimes a^n \otimes D = n\,(C \otimes a \otimes D)$ for $n \in \mathbb{Z}$.

(ii) **Neglecting torsion.** For $n \in \mathbb{Z}$ and ξ an n^{th} root of unity we set $C \otimes \xi \otimes D = 0$ which corresponds to working up to torsion.

(iii) **Shuffle product** Recall the *shuffle product* of two simple tensors is defined as

$$(a_1 \otimes \cdots \otimes a_{n_1}) \amalg (a_{n_1+1} \otimes \cdots \otimes a_{n_1+n_2}) = \sum_{\sigma \in \Sigma(n_1,n_2)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n_1+n_2)},$$

then the symbol satisfies

$$\mathcal{S}\left(G(a_1, \dots, a_{n_1}; x)G(b_1, \dots, b_{n_2}); x\right) = \mathcal{S}(G(a_1, \dots, a_{n_1}; x)) \amalg \mathcal{S}(G(b_1, \dots, b_{n_2}; x)).$$

Exercise 9. Using the following identity (valid for a, b distinct and non zero)

$$G(a,b;x) = \mathrm{Li}_2\left(\frac{b-x}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1-\frac{x}{a}\right)\log\left(\frac{x-a}{b-a}\right)$$

compute the symbol $\mathcal{S}(G(a,b;x))$.

Example 3.5. This is an example of a function relevant for scattering amplitudes: in [GSVV10, Equation (3)] Goncharov, Spradlin, Vergu and Volovich present a compact expression of the remainder function for a six particle scattering amplitude in $\mathcal{N} = 4$ super Yang–Mills theory (computed up to loop level two). The innovative idea of this paper was to use a change of coordinates following twistor theory which lead to an expression much simpler than the ones known before which mainly relied on Feynman integral. We get back at this example in the following subsection, but for now want to compute its symbol alphabet. The function is defined as

$$R_6^{(2)} = \sum_{i=1}^{3}\left(L_i - \frac{1}{2}\mathrm{Li}_4(-v_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(-v_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}, \tag{12}$$

in terms of

$$\begin{aligned} L_i &= \frac{1}{384}P_i^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}P_i^m\left((\ell_{4-m}(x_i^+) + \ell_{4-m}(x_i^-)\right), \\ P_i &= 2\mathrm{Li}_1(-v_i) - \sum_{j=1}^{3}\mathrm{Li}_1(-v_j), \\ J &= \sum_{i=1}^{3}\ell_1(x_i^+) - \ell_1(x_i^-), \\ \ell_n(x) &= \frac{1}{2}\left(\mathrm{Li}_n(-x) - (-1)^n\mathrm{Li}_n\left(\frac{1}{x}\right)\right). \end{aligned}$$

Notice that all functions are combinations of Li_n with $n \leq 4$. Therefore, with Example 3.3 we deduce that the symbol alphabet of $R_6^{(2)}$ is given by

$$\{v_i, x_i^+, x_i^- : i = 1,2,3\}.$$

In the next subsection we explain what there variables are and where they are defined.

3.2. **Twistor theory.** Twistor theory was introduced by Penrose in his seminal 1967 paper [Pen67]. Essentially it refers to clever changes of variables from 4-dimensional real Minkowski space to twistor space, a projective 3-space. Points in Minkowski space are transformed to Riemann spheres while rays of lights translate to point in twistor space. Among other things, this change of variables revealed symmetries, such as conformal invariance, that are hard to detect in Minkowski. Problems such as solving certain differential equations on the real side are translated to problems in terms of complex and algebraic geometry that can be attacked using tools such as sheaf cohomology [AW77] (Einstein's field equations), [PR84]. And most interesting to us, twistor theory was successfully applied to find new compact expressions of scattering amplitudes [Wit04]. Example 3.5 is an illustration of this phenomenon.

In what follows I will present two parametrizations for spaces of particle configurations (*i.e.* configuration spaces) that rely on twistor theory. More precisely, these are parametrizations in terms of momentum twistors and helicity spinors.

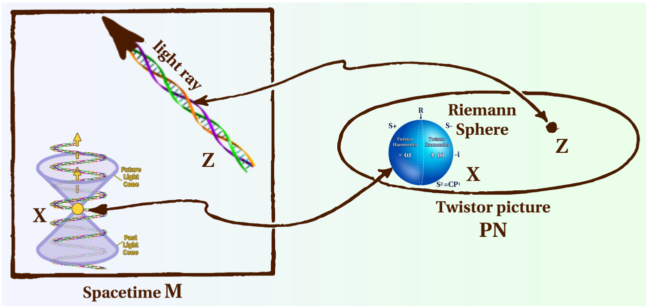


FIGURE 9. A cartoon depicting twistor theory ©`https://twistor.li/`

3.3. **Momentum twistors.** A popular approach to twistor theory was initiated by Hodges in [Hod13] using so called *momentum twistors*. For details on the precise coordinate transformation we refer to *loc.cit.*.

In $\mathcal{N} = 4$ super Yang–Mills theory a massless particle $\mathbf{p}$ in Minkowski space $\mathbb{R}^4$ as an input and yields a point Z in twistor space $\mathbb{P}^3$ called the momentum twistor of $\mathbf{p}$. So, a particle configuration $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^4$ is transformed to a configuration of n points $Z_1, \dots, Z_n$ in $\mathbb{P}^3$. In this theory dual conformal invariance holds which translates to a planar symmetry on the point configuration: the cycle action $c_6 = (123456) \in S_6$ given by $Z_i \mapsto Z_{i+1}$ (indices are taken modulo n) leaves the point configuration invariant.

Choose representatives $\bar{Z}_i \in \mathbb{C}^4$ for each Z_i and consider these as columns of a $4 \times n$-matrix. The configuration space is then parametrized by 4×4-minors p_{ijkl} for $1 \leq i < j < k < l \leq n$

$$p_{ijkl} := \det(\bar{Z}_i \bar{Z}_j \bar{Z}_k \bar{Z}_l), \tag{13}$$

satisfying determinantal identities, *i.e.* Plücker relations. Up to column rescaling, which does not change the point configuration a matrix represents, the configuration space is the Grassmannian $\mathrm{Gr}_{4,n}$. In fact, the configuration space is a quotient of the affine cone $\widetilde{\mathrm{Gr}_4}(\mathbb{C}^n)$.

Example 3.6. Recall the remainder function of the two loop $\mathcal{N} = 4$ super Yang–Mills scattering amplitude for six particles from Equation 12. The variables v_i, x_i^+, x_i^- are coordinates on the configuration space, more precisely we have

$$v_1 = \frac{p_{1245}p_{1346}}{p_{1245}p_{1346}}, \quad x_1^+ = \frac{p_{1456}p_{2356}}{p_{1256}p_{3456}}, \quad x_1^- = \frac{p_{2356}p_{1234}}{p_{1236}p_{2345}}. \tag{14}$$

The missing variables v_i, x_i^+, x_i^- for $i \neq 0$ are obtained from the above by applying the cycle c_6 to the indices of Plücker coordinates. There is an isomorphism $\mathrm{Gr}_2(\mathbb{C}^6) \cong \mathrm{Gr}_4(\mathbb{C}^6)$ induced on Plücker coordinates by $p_{ijkl} \mapsto p_{pq}$ where p, q are such that $\{i, j, k, l, p, q\} = \{1, \dots, 6\}$. Rewriting the variables (14) with this isomorphism we have the following expressions for the symbol alphabet of $R_6^{(2)}$:

$$v_1 = \frac{p_{35}p_{26}}{p_{23}p_{56}}, \quad v_2 = \frac{p_{13}p_{46}}{p_{16}p_{34}}, \quad v_3 = \frac{p_{15}p_{24}}{p_{45}p_{12}},$$

$$x_1^+ = \frac{p_{14}p_{23}}{p_{12}p_{34}}, \quad x_2^+ = \frac{p_{25}p_{16}}{p_{56}p_{12}}, \quad x_3^+ = \frac{p_{36}p_{45}}{p_{34}p_{56}},$$
$$x_1^- = \frac{p_{14}p_{56}}{p_{45}p_{16}}, \quad x_2^- = \frac{p_{25}p_{34}}{p_{23}p_{45}}, \quad x_3^- = \frac{p_{36}p_{12}}{p_{16}p_{23}}.$$

These coordinates are in fact nine out of the twelve $\hat{y}$-variables associated with $\mathrm{Gr}_2(\mathbb{C}^6)$ up to taking inverse. See (5) and Example 2.15. Notice that Plücker relations induce relations among $\hat{y}$-variables. The nine $\hat{y}$-variables represent an minimal set in the sense that it is not possible to eliminate one of the variables using the relations that hold among all twelve $\hat{y}$-variables. The symbol alphabet may equivalently be expressed as the complete set of twelve $\hat{y}$-variables, a different set of nine $\hat{y}$-variables or even as the set of all Plücker coordinates. These reformulations are possible due to the identities that hold for symbols, see Proposition 3.4.

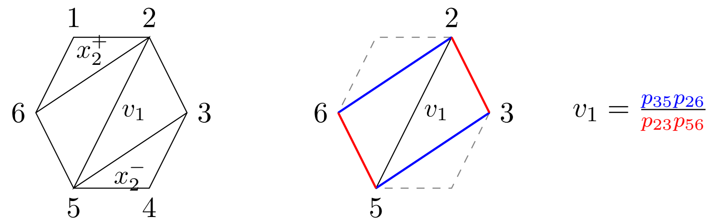


FIGURE 10. Identifying letters in the symbol alphabet of $R_6^{(2)}$ with quadrilaterals, *i.e.* $\hat{y}$-variables. See Example 3.6.

Example 3.6 generalizes to the case of $n = 7$ and yields the following result.

Theorem 4 ([GSVV10]). In planar $\mathcal{N} = 4$ super Yang–Mills with $n = 6$ the symbol alphabet for the remainder function of the scattering amplitude (up to loop level two)[3] consists of the cluster variables of the Grassmannian $\mathrm{Gr}_{4,6}$.

In fact for $n = 7$ the analogous result has been verified in [DDH+17]. Higher loop orders have been studied in [DDH11, DDvHP13] for $\ell = 3$, in [DDDP14] for $\ell = 4$, in [CHDMvH16] for $\ell = 5$, and in [CHDD+19] for $\ell = 6, 7$.

We have seen the finite type classification of Grassmannians in Theorem 1. The only finite type Grassmannians $\mathrm{Gr}_{4,n}$ are those for $n = 6$ and $n = 7$. So it is natural to expect that the above Theorem cannot generalize as is for $n \geq 8$: the set of cluster variables is infinite while the symbol alphabet remains a finite set (at any given loop level). However, the symbol alphabet may still be recovered from the cluster structure. Drummond, Foster, Gürdoğan and Kalousios show the following

Theorem 5 ([DFGK21]). The 272 letters of the symbol alphabet for $\mathrm{Gr}_{4,8}$ are obtained by

(i) a finite set of cluster variables, and
(ii) square root expressions obtained as limits of cluster variables along infinite mutation sequences of Kronecker type.

In section 3.5 we describe expressions obtained in (ii) with more detail. It is worth mentioning that the main tool used in[DFGK21] is the positive part of the tropicalization of $\mathrm{Gr}_{4,8}$ in its Plücker embedding. A generalization of Theorem 2.31 explains the relationship between rays and cluster variables.

Remark 3.7. These findings have gauged wide interested in symbol calculus which is now called the *(cluster) boostrap* programme, initiated by Lance Dixon and several collaborators. The programme addresses the question whether a multiple polylogarithm on a given configuration space is uniquely determined by its symbol. It has been verified in numerous cases that this is in fact true for scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills. Besides the works already cited above important contributions are [DvH14, DvHM16] (studying the NMHV for $\ell = 3$ resp. $\ell = 4$), [DPS15] addressing uniqueness of the bootstrapped amplitude, [DFGP19, DFG18] recovering the rules for how to form word from symbol letters– called *adjacencies* from exchange relations in the cluster algebras.

[3]The notion of loop level is explained at the beginning of section 3.6.

3.4. **Beyond dual conformal symmetry.** The success of the bootstrap programme (Remark 3.7) opened up ways to new research questions, including questions if this could also work outside of the supersymmetric toy model $\mathcal{N} = 4$ super Yang–Mills. This model–while very popular in theoretical physics due to its mathematical beauty incarnated, for example, in additional symmetries– is often critiqued for being 'too far' from the real world modelled by the standard model and QCD. The real challenge–so critics– is taking symbol calculus to QCD.

Mathematically speaking, this translates to dropping the dual conformal invariance which is present in the momentum twistor parametrization as the cyclic symmetry or planarity. This is solved by introducing two more points $Z_{n+1}, Z_{n+2} \in \mathbb{P}^3$ to the configuration of n points $Z_1, \ldots, Z_n \in \mathbb{P}^3$. These dummy points are present to break the cyclic symmetry. Physically speaking the line spanned by Z_{n+1}, Z_{n+2} represents an *infinity twistor*. A configurations of n lightlike particles $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^4$ (without assuming dual conformal invariance) translates in momentum twistors to a a configuration of n points $Z_1, \ldots, Z_n$ and a line $\overline{Z_{n+1}Z_{n+2}}$ in complex projective space $\mathbb{P}^3$.

The corresponding configuration space also admits a parametrization in terms of minors. Similar to (13) consider choose representatives $\bar{Z}_i \in \mathbb{C}^4$ for each $Z_i \in \mathbb{P}^3$ and set

$$p_{ijkl} := \det(\bar{Z}_i\bar{Z}_j\bar{Z}_k\bar{Z}_l), \quad \text{and additionally} \quad p_{ab} := \det(\bar{Z}_a\bar{Z}_b\bar{Z}_{n+1}\bar{Z}_{n+2}),$$

for $1 \leq i < j < k < l \leq n$ and $1 \leq a < b \leq n$. The relations satisfied by the p_{ijkl} and p_{ij} are exactly the Plücker relations satisfied by the partial flag variety $\mathcal{F}_{2,4;n+2}$. This motivates the following definition.

Definition 3.8. The *momentum twitsor variety* for configurations of n lightlike particles without assuming dual conformal invariance is the partial flag variety $\mathcal{F}_{2,4;n+2}$

3.5. **Spinor helicity variety.** The spinor helicity formalism is an example of a change of coordinates in twistor theory. It has proven itself highly useful in computations of scattering amplitudes of massless particles. It relies on minimal assumptions for the model of particle configurations (only the *on-shell* condition) and describes all helicities (gluons, fermions, scalars) of massless particles. For more details see [HP14]. In particular, it applies in quantum chromo dynamics, which is part of the standard model modelling real world phenomena.

Recall that a particle p in Minkowski space is presented as a vector $(p_0, p_1, p_2, p_3) \in \mathbb{R}^4$. Define a linear map

$$\alpha : \mathbb{R}^4 \to \mathbb{C}^{2\times 2}, \quad \mathbf{p} = (p_0, p_1, p_2, p_3) \mapsto \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}. \tag{15}$$

Minkowski space has an inner product with sign $(+, -, -, -)$. The product of a particle vector with itself reveals the mass m of the particle

$$\mathbf{p} \cdot \mathbf{p} = m^2;$$

this is known as the mass-shell condition in the physics literature. Using (15) we obtain

$$\det(\alpha(\mathbf{p})) = p_0^2 - p_3^2 - p_1^2 - p_2^2 = \mathbf{p} \cdot \mathbf{p} = m^2.$$

In particular, when p is a *lightlike* particle, that is $m = 0$, then $\alpha(\mathbf{p})$ has rank less or equal one and hence, may be written as $\lambda\tilde{\lambda}^T$ for two vectors $\tilde{\lambda}, \lambda \in \mathbb{C}^2 = \mathbb{C}^{2\times 1}$. More precisely,

$$\lambda = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 \\ p_1 + ip_2 \end{pmatrix}, \quad \tilde{\lambda} = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} p_0 + p_3 \\ p_1 - ip_2 \end{pmatrix}.$$

The vectors $\lambda, \tilde{\lambda}$ associated with a particle $\mathbf{p}$ are called its *helicity spinors*.

Exercise 10. Verify that *momentum conservation* holds, that is $\tilde{\lambda}^T\lambda = 0$.

Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^4$ be a configuration of n lightlike particles. Then its helicity spinors are n pairs $(\lambda_i, \tilde{\lambda}_i) \in \mathbb{C}^{2\times 2}$ satisfying $\tilde{\lambda}_i^T\lambda_i = 0$. Define $\Lambda = (\lambda_1, \ldots, \lambda_n), \tilde{\Lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in \mathbb{C}^{2\times n}$. As Λ and $\tilde{\Lambda}$ are (generically) of full rank we may interpret them as points in $\mathrm{Gr}_2(\mathbb{C}^n)$. This motivates the following definition.

Definition 3.9 ([MPS24])**.** The *spinor helicity variety* is

$$\mathcal{SH}_n := \{(\Lambda, \tilde{\Lambda}) \in \mathrm{Gr}_{2;n} \times \mathrm{Gr}_{2;n} : \Lambda\tilde{\Lambda}^T = 0\}$$

The Plücker coordinates for $\mathrm{Gr}_2(\mathbb{C}^n)$ induce a parametrization of $\mathcal{SH}_n$. For $1 \leq i < j \leq n$ set

$$P_{ij} := \det(\lambda_i \lambda_j) \quad \text{y} \quad \tilde{P}_{ij} := \det(\tilde{\lambda}_i \tilde{\lambda}_j).$$

These satisfy the Plücker relations of $\mathrm{Gr}_2(\mathbb{C}^n)$ (also known as *Schouten identities* in the physics literature)

$$0 = P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} = \tilde{P}_{ij}\tilde{P}_{kl} - \tilde{P}_{ik}\tilde{P}_{jl} + \tilde{P}_{il}\tilde{P}_{jk} \tag{16}$$

and *momentum conservation*

$$0 = \sum_{s=1}^{n} P_{is}\tilde{P}_{sj} \quad (\Leftrightarrow \Lambda\tilde{\Lambda}^T = 0). \tag{17}$$

These are in fact the generators of the vanishing ideal of $\mathcal{SH}_n \subset \mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{2}-1}$. Recall the partial flag variety $\mathcal{F}_{2,n-2;n}$ from §2.10.

Proposition 3.10. The map $\mathcal{F}_{2,n-2;n} \to \mathcal{SH}_n$ defined via pull-back by

$$P_{ij} \mapsto P_{ij} \quad \text{and} \quad \tilde{P}_{ij} \mapsto (-1)^{i+j-1} P_{[n]-ij},$$

where $[n] - ij := \{1, \ldots, n\} - \{i, j\}$ is an isomorphism.

The relations (16) and (17) translate to the following Plücker relations of $\mathcal{F}_{2,n-2;n}$:

$$\begin{aligned} P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} &= 0 \\ P_{[n]-ij}P_{[n]-kl} - P_{[n]-ik}P_{[n]-jl} + P_{[n]-il}P_{[n]-jk} &= 0 \\ \sum_{s=1}^{n}(-1)^{s+j-1} P_{is}P_{[n]-js} &= 0 \end{aligned}$$

Moreover, that $\dim \mathcal{SH}_n = 4(n-3) = \dim \mathcal{F}_{2,n-2;n}$ holds is shown in [MPS24], so that we have an isomorphism of the defining ideals.

3.6. **Symbol alphabet for scattering in QCD.** We start with the case of five particles. The configuration space is the spinor helicity variety $\mathcal{SH}_5 \cong \mathcal{F}_{2,3;5}$.

Analytic computations have found a symbol alphabet of 31 letters [GHLP16, CHM18]. These computations are done by approximating the scattering amplitude by a finite sum of Feynman integrals. Recall that the occuring Feynman integrals are indexed by Feynman graphs. The approximation (of the scattering amplitude) of *loop level* ℓ is the sum of all relevant Feynman integrals that are indexed by Feynman graphs with at most ℓ loops. This leads to a phenomenon called *spurious letters*: letters appearing in the symbol alphabet of the approximation of loop level ℓ that are not letters in the symbol alphabet of the approximation of loop level $\ell + 1$. One of the 31 letters in the alphabet in this case is expected to be spurious.

Reformulated in spinor helicity variables the symbol alphabet is given by

$$\begin{aligned} W_1 &= P_{12}\tilde{P}_{12}, \\ W_6 &= P_{12}\tilde{P}_{12} - P_{35}\tilde{P}_{35}, \\ W_{11} &= P_{12}\tilde{P}_{12} - P_{45}\tilde{P}_{45}, \\ W_{16} &= P_{13}\tilde{P}_{13}, \\ W_{21} &= P_{13}\tilde{P}_{13} + P_{34}\tilde{P}_{34}, \\ W_{26} &= \frac{P_{45}\tilde{P}_{15}P_{12}\tilde{P}_{24}}{\tilde{P}_{45}P_{15}\tilde{P}_{12}P_{24}}, \end{aligned}$$

with their cyclic copies induced by the action of $c_5 = (12345) \in S_5$ on the indices, for example $W_2 = c_5(W_1)$, and

$$W_{31} = \tilde{P}_{45}P_{15}\tilde{P}_{12}P_{24} - P_{45}\tilde{P}_{15}P_{12}\tilde{P}_{24}.$$

We have seen in Proposition 3.10 that $\mathcal{SH}_5$ is isomorphic to $\mathcal{F}_{2,3;5}$ which is of finite cluster type $\mathtt{D}_4$, see Exercise 8. In this case we have 16 cluster variables and 6 frozen variables. Among the 16 mutable cluster variables, 14 are Plücker coordinates and two are quadratic binomials in Plücker coordinates.

Recall that in $\mathcal{N} = 4$ super Yang–Mills dual conformal symmetry manifests as cyclic symmetry in the momentum twistor parametrization. This is also manifest in the corresponding cluster algebra for $\mathrm{Gr}_{4,n}$: the set of cluster variables is closed under the action of the cycle $c_n = (12 \cdots n) \in S_n$. This is no longer true for partial flag varieties: for example in Exercise 8 we see that P_{12}, P_{15}, P_{45} are frozen variables, while P_{23}, P_{34} are not. In order to recover the symbol alphabet we have to consider the orbit of cluster variables with respect to the cyclic action. The main result is summarized below

Theorem 6 ([BDG23])**.** The symbol alphabet of the scattering amplitude for 5 light like particles can be recovered from the cluster algebra structure on the spinor helicity variety $\mathcal{SH}_5 \cong \mathcal{F}_{2,3;5}$. More precisely,

- the letters $W_1, \dots, W_5, W_{16}, \dots, W_{20}, W_{26}, \dots, W_{30}$ are multiplicative combinations of Plücker coordinates (cluster variables of degree one and their cyclic copies)
- the letters $W_6, \dots, W_{10}, W_{11}, \dots, W_{15}, W_{21}, \dots, W_{25}$ are cluster variables of degree two and their cyclic copies
- the letter W_{31} is not recovered by the cluster algebra.

Six particles. The relevant cluster structure of the momentum twistor variety $\mathcal{F}_{2,4;6}$ if affine type $\mathtt{D}_6^{(1)}$, hence the cluster type is inifite (also for the spinor helicity variety). However using methods similar to those used in [DFGK21] some results have been achieved simultaneously but independently in [PSVW25] and [BDG$^+$25].

As for $n = 5$ these rely on previous analytic computations that have found a symbol alphabet of 289 letters that split into 38 permutation classes [HMM$^+$25]. A direct analysis reveals that there are 54 cluster variables that are letters and give 11 permutation classes. The embedding $\mathbb{C}[\mathcal{F}_{2,4;6}] \hookrightarrow \mathbb{C}[\mathrm{Gr}_{4,8}]$ from Theorem 3 combined with the known results and methods that worked for $\mathrm{Gr}_{4,8}$ are the main tools. Letters that are sqare roots (and therefor cannot be cluster variables) are recovered as follows: a seed for $\mathrm{Gr}_{4,8}$ (without frozen variables depicted) that restricts to a seed for $\mathcal{F}_{2,4;6}$:

$$\begin{tikzcd}
 & & & & z_0 \arrow[dll] \arrow[dl] \arrow[dr] & & & \\
a_0 \arrow[r] & a_2 \arrow[r] & b_3 \arrow[drr] & b_2 \arrow[dr] & & b_1 \arrow[dl] \arrow[r] & a_4 \arrow[r] & a_3 \\
 & & & & w_0 \arrow[uu, shift left] \arrow[uu, shift right] & & &
\end{tikzcd}$$

where the cluster variables are

$$\begin{aligned}
z_0 &= p_{1236}p_{1578} - p_{1235}p_{1678} & w_0 &= p_{1356} \\
b_1 &= p_{1236}p_{3578} - p_{1235}p_{3678} & a_1 &= p_{1345} \\
b_2 &= p_{1256} & a_2 &= p_{1346} \\
b_3 &= p_{1346}p_{1578} - p_{1345}p_{1678} & a_3 &= p_{1237} \\
a_4 &= p_{1236}p_{4578} - p_{1235}p_{4678} + p_{1234}p_{5678} & &
\end{aligned}$$

Mutating consecutively at w_0 and z_0 yields a sequencen of cluster variables z_i that satisfy

$$z_{i+2}z_i = b_1b_2b_3F + z_{i+1}^2,$$

where F is a monomial in frozen variables. Solving the equation for z_i we get an expression containing the squareroot of

$$\Delta = P_{12}^2P_{3456}^2 - 2P_{1256}P_{12}P_{3456}P_{34} + P_{1256}^2P_{34}^2 - 2P_{1234}P_{12}P_{3456}P_{56} - 2P_{1234}P_{1256}P_{3478}P_{56} + P_{1234}^2P_{56}^2$$

which is a letter in the alphabet. In summary, in [BDG$^+$25] we recover a total of 32 out of the 38 permutation classes

References

[AW77] M. F. Atiyah and R. S. Ward. Instantons and algebraic geometry. *Comm. Math. Phys.*, 55(2):117–124, 1977.

[BDG23] L. Bossinger, J. M. Drummond, and R. Glew. Adjacency for scattering amplitudes from the Gröbner fan. *JHEP*, 11:002, 2023.

[BDG+25] L. Bossinger, J. Drummond, R. Glew, Ö. Gürdoğan, and R. Wright. Singularities of massless scattering and cluster algebras. *arXiv:2507.01015 [hep-th]*, 2025.

[BL24] Lara Bossinger and Jian-Rong Li. Cluster structures on spinor helicity and momentum twistor varieties. *arXiv:2408.14956 [math.AG]*, 2024.

[Bos22] L. Bossinger. Tropical totally positive cluster varieties. *arxiv preprint arXiv:2208.01723 [math.AG]*, 2022.

[CHDD+19] Simon Caron-Huot, Lance J. Dixon, Falko Dulat, Matt von Hippel, Andrew J. McLeod, and Georgios Papathanasiou. Six-Gluon amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory at six and seven loops. *JHEP*, 08:016, 2019.

[CHDMvH16] Simon Caron-Huot, Lance J. Dixon, Andrew McLeod, and Matt von Hippel. Bootstrapping a Five-Loop Amplitude Using Steinmann Relations. *Phys. Rev. Lett.*, 117(24):241601, 2016.

[Che77] Kuo Tsai Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[CHM18] Dmitry Chicherin, Johannes Henn, and Vladimir Mitev. Bootstrapping pentagon functions. *JHEP*, 05:164, 2018.

[CMNC22] Man-Wai Cheung, Timothy Magee, and Alfredo Nájera Chávez. Compactifications of cluster varieties and convexity. *Int. Math. Res. Not. IMRN*, (14):1–54 [10858–10911 on table of contents], 2022.

[CP96] Vyjayanthi Chari and Andrew Pressley. Minimal affinizations of representations of quantum groups: the simply laced case. *J. Algebra*, 184(1):1–30, 1996.

[DDDP14] Lance J. Dixon, James M. Drummond, Claude Duhr, and Jeffrey Pennington. The four-loop remainder function and multi-Regge behavior at NNLLA in planar N = 4 super-Yang-Mills theory. *JHEP*, 06:116, 2014.

[DDH11] Lance J. Dixon, James M. Drummond, and Johannes M. Henn. Bootstrapping the three-loop hexagon. *JHEP*, 11:023, 2011.

[DDH+17] Lance J. Dixon, James Drummond, Thomas Harrington, Andrew J. McLeod, Georgios Papathanasiou, and Marcus Spradlin. Heptagons from the Steinmann Cluster Bootstrap. *JHEP*, 02:137, 2017.

[DDvHP13] Lance J. Dixon, James M. Drummond, Matt von Hippel, and Jeffrey Pennington. Hexagon functions and the three-loop remainder function. *JHEP*, 12:049, 2013.

[DFG18] James Drummond, Jack Foster, and Ömer Gürdoğan. Cluster Adjacency Properties of Scattering Amplitudes in $N = 4$ Supersymmetric Yang-Mills Theory. *Phys. Rev. Lett.*, 120(16):161601, 2018.

[DFGK21] James Drummond, Jack Foster, Ömer Gürdoğan, and Chrysostomos Kalousios. Tropical fans, scattering equations and amplitudes. *JHEP*, 11:071, 2021.

[DFGP19] James Drummond, Jack Foster, Ömer Gürdoğan, and Georgios Papathanasiou. Cluster adjacency and the four-loop NMHV heptagon. *JHEP*, 03:087, 2019.

[DGR12] Claude Duhr, Herbert Gangl, and John R. Rhodes. From polygons and symbols to polylogarithmic functions. *J. High Energy Phys.*, (10):075, front matter + 77, 2012.

[DPS15] James M. Drummond, Georgios Papathanasiou, and Marcus Spradlin. A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon. *JHEP*, 03:072, 2015.

[DvH14] Lance J. Dixon and Matt von Hippel. Bootstrapping an NMHV amplitude through three loops. *JHEP*, 10:065, 2014.

[DvHM16] Lance J. Dixon, Matt von Hippel, and Andrew J. McLeod. The four-loop six-gluon NMHV ratio function. *JHEP*, 01:053, 2016.

[ET01] Manfred Einsiedler and Selim Tuncel. When does a polynomial ideal contain a positive polynomial? volume 164, pages 149–152. 2001. Effective methods in algebraic geometry (Bath, 2000).

[FG06] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[FG09] V. V. Fock and A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[FZ07] S. Fomin and A. Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

[GGMS87] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987.

[GHKK18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608, 2018.

[GHLP16] T. Gehrmann, J. M. Henn, and N. A. Lo Presti. Analytic form of the two-loop planar five-gluon all-plus-helicity amplitude in QCD. *Phys. Rev. Lett.*, 116(6):062001, 2016. [Erratum: Phys.Rev.Lett. 116, 189903 (2016)].

[GLS08] Christof Geiss, Bernard Leclerc, and Jan Schröer. Preprojective algebras and cluster algebras. In *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep., pages 253–283. Eur. Math. Soc., Zürich, 2008.

[Gon98] A. B. Goncharov. Multiple polylogarithms, cyclotomy and modular complexes. *Math. Res. Lett.*, 5(4):497–516, 1998.

[Gon99] Alexander Goncharov. Volumes of hyperbolic manifolds and mixed Tate motives. *J. Amer. Math. Soc.*, 12(2):569–618, 1999.

[Gon13] A. B. Goncharov. A simple construction of Grassmannian polylogarithms. *Adv. Math.*, 241:79–102, 2013.

[GSVV10] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich. Classical polylogarithms for amplitudes and Wilson loops. *Phys. Rev. Lett.*, 105(15):151605, 4, 2010.

[HL21] David Hernandez and Bernard Leclerc. Quantum affine algebras and cluster algebras. In *Interactions of quantum affine algebras with cluster algebras, current algebras and categorification—in honor of Vyjayanthi Chari on the occasion of her 60th birthday*, volume 337 of *Progr. Math.*, pages 37–65. Birkhäuser/Springer, Cham, [2021] ©2021.

[HMM+25] Johannes Henn, Antonela Matijašić, Julian Miczajka, Tiziano Peraro, Yingxuan Xu, and Yang Zhang. Complete Function Space for Planar Two-Loop Six-Particle Scattering Amplitudes. *Phys. Rev. Lett.*, 135(3):031601, 2025.

[Hod13] Andrew Hodges. Eliminating spurious poles from gauge-theoretic amplitudes. *J. High Energy Phys.*, (5):135, front matter+22, 2013.

[HP14] Johannes M. Henn and Jan C. Plefka. *Scattering Amplitudes in Gauge Theories*, volume 883. Springer, Berlin, 2014.

[Ken04] Richard Kenyon. An introduction to the dimer model. In *School and Conference on Probability Theory*, volume XVII of *ICTP Lect. Notes*, pages 267–304. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[LB15] V. Lakshmibai and J. Brown. *The Grassmannian variety*, volume 42 of *Developments in Mathematics*. Springer, New York, 2015. Geometric and representation-theoretic aspects.

[LD35] J.A. Lappo-Danilevskij. Mémoires sur la théorie des systémes des équations différentielles linéaires. vol. ii. *Travaux Inst. Physico-Math. Stekloff*, 7:5–210, 1935.

[Li24] Jian-Rong Li. Dual canonical bases for unipotent groups and base affine spaces. *Journal of Algebra and its Applications, https://doi.org/10.1142/S0219498825503542*, 2024.

[Lus90] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[MPS24] Yassine El Maazouz, Anaëlle Pfister, and Bernd Sturmfels. Spinor-Helicity Varieties. 6 2024.

[MR88] Teo Mora and Lorenzo Robbiano. The Gröbner fan of an ideal. volume 6, pages 183–208. 1988. Computational aspects of commutative algebra.

[MS15] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[Nie09] N. Nielsen. Der eulersche dilogarithmus und seine verallgemeinerungen. *Nova Acta Leopoldina*, 90:123, 1909.

[Pen67] R. Penrose. Twistor algebra. *J. Math. Phys.*, 8:345, 1967.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. *arXiv preprint arXiv:math/0609764*, 2006.

[PR84] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984. Two-spinor calculus and relativistic fields.

[PSVW25] Andrzej Pokraka, Marcus Spradlin, Anastasia Volovich, and He-Chen Weng. Symbol Alphabets in QCD and Flag Cluster Algebras. 6 2025.

[Sco06] J. S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)*, 92(2):345–380, 2006.

[SF] Andrei Zelevinsky Sergey Fomin, Lauren Williams. Introduction to cluster algebras. chapters 4-5. *arXiv preprint arXiv:1707.07190 [math.CO]*.

[SW05] David Speyer and Lauren Williams. The tropical totally positive Grassmannian. *J. Algebraic Combin.*, 22(2):189–210, 2005.

[Wit04] Edward Witten. Perturbative gauge theory as a string theory in twistor space. *Comm. Math. Phys.*, 252(1-3):189–258, 2004.