Variable Compleja: 11 abril

Clase pasada: si $\exists\ F:\mathbb{C}\to\mathbb{C}$ tal que $F' = f \Rightarrow \int_\gamma f = 0$ para toda curva cerrada γ.

Hoy: si $f:\mathbb{C}\to\mathbb{C}$ holomorfa

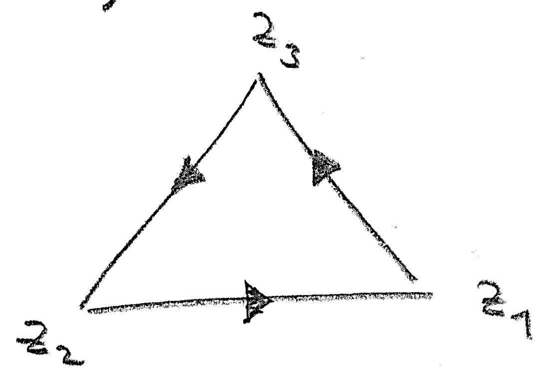
¿$\int_\gamma f = 0$? $\forall\ \gamma$ cerrada?

$$\langle z_1, z_2, z_3\rangle = \alpha := \alpha_1 \oplus \alpha_2 \oplus \alpha_3$$

$\alpha_1(t) = z_1 + (t-0)(z_2 - z_1)$ $\quad 0 \le t \le 1$

$\alpha_2(t) = z_2 + (t-1)(z_3 - z_2)$

$1 \le t \le 2.$



$\alpha_3(t) = z_3 + (t-2)(z_1 - z_3)$

$2 \le t \le 3.$

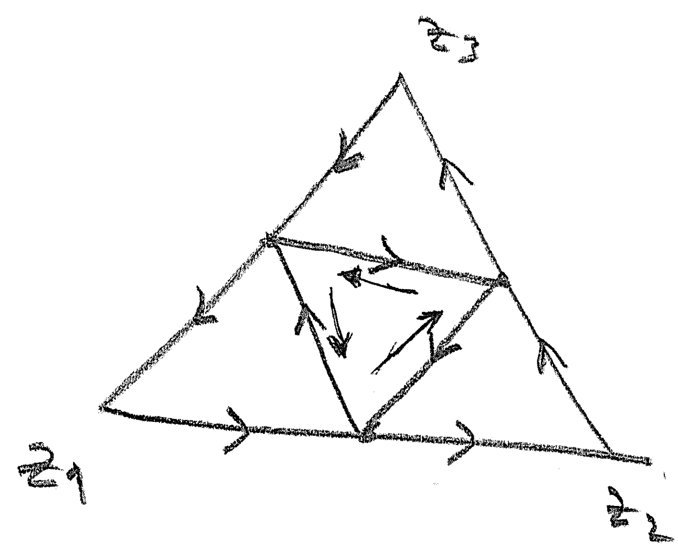
Teorema (Cauchy en Δ's)

$f: D \longrightarrow \mathbb{C}$ $\quad D \subseteq \mathbb{C}$ abierto, $\quad$ holomorfa en D.

$\Rightarrow$ si $\alpha = \langle z_1, z_2, z_3 \rangle \subset D$ entonces

$$\boxed{\int_\alpha f(\zeta)\, d\zeta = 0}$$

Demostración:



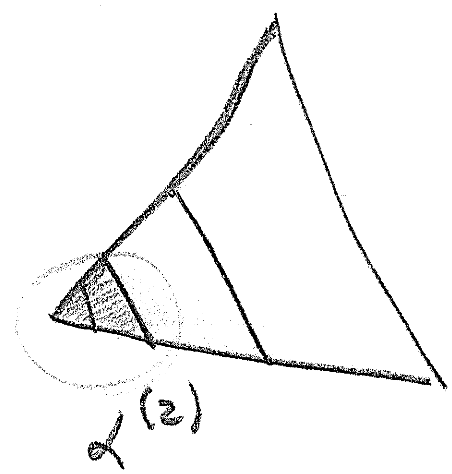
$$\int_\alpha = \int_{\alpha_1^{(1)}} + \int_{\alpha_2^{(1)}} + \int_{\alpha_3^{(1)}} + \int_{\alpha_4^{(1)}}$$

$$\left|\int_{\alpha^{(n)}} f\right| \le 4\left|\int_{\alpha^{(n+1)}} f\right|$$

elección

$$\Rightarrow \quad \left|\int_{\alpha} f(y)\,dy\right| \le 4^n \left|\int_{\alpha^{(n)}} f(y)\,dy\right|$$

Los triángulos generados por los $\alpha^{(n)}$'s están anidados: $\Delta^{(n)}$



$$\Delta^{(0)} \supset \underset{\underset{\alpha^{(1)}}{\uparrow}}{\Delta^{(1)}} \supset \underset{\underset{\alpha^{(2)}}{\uparrow}}{\Delta^{(2)}} \supset \dots$$

CANTOR: ¡$\exists\, z_0 \in D$ contenido en todos los Δ's!

entonces

conti

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + r(z)$$

con $\displaystyle\lim_{z \to z_0} \frac{r(z)}{z - z_0} = 0.$

Observar

$$\int_{\alpha^{(n)}} f(\gamma)\, d\gamma = \int_{\alpha^{(n)}} r(\gamma)\, d\gamma$$

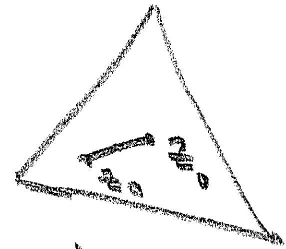
$$\Rightarrow \quad \left| \int_{\alpha} f(\gamma)\, d\gamma \right| \leq \left| 4^n \int_{\alpha^{(n)}} r(\gamma)\, d\gamma \right|$$

$r(\gamma)$ continuas el límite : $\forall \varepsilon > 0 \;\; \exists \delta > 0$

dice $|r(z)| < \varepsilon \cdot |z - z_0|$ si $|z - z_0| < \delta$

si $\Delta^{(n)} \subseteq B(z_0, \delta)$ entonces

$$|z - z_0| \leq l(\alpha^{(n)}) = \frac{1}{2^n}\, l(\alpha) \quad \forall\, z \in \Delta^{(n)}$$



$$\Rightarrow \quad \left| \int_\alpha f(\zeta)\, d\zeta \right| \leq \underbrace{4^n\, l(\alpha^{(n)}) \cdot \varepsilon\, l(\alpha^{(n)})}_{l(\alpha)^2 \cdot \varepsilon}$$

$$\Rightarrow \quad \int_\alpha f(\zeta)\, d\zeta = 0 \qquad \blacksquare$$

NO ejemplo $\qquad f(z) = |z|^2 = z \cdot \bar{z}$