Álgebra Moderna III: 21 NOV

Clase pasada: Campos conjugados y subgrupos normales

Hoy: Subgrupos normales & sus campos fijados

Recordar $H \trianglelefteq G$ subgrupo normal $\Rightarrow$

G/H es grupo.

si tenemos $G = Gal\left(\begin{array}{c} L \\ | \\ F \end{array}\right)$ y $H \trianglelefteq G$

entonces $F < L_H \subseteq L$ son extensiones

tal que
$$H = Gal(L \setminus L_H)$$

¿ Quien es $Gal(L_H \setminus F)$?

en términos de G & H

Teorema:

$$\begin{array}{l} L \\ | \\ K \\ | \\ F \end{array} \quad \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\rangle \text{Galois} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\rangle \text{Galois} \end{array}$$

Entonces

$$\mathrm{Gal}(L\backslash K) \triangleleft \mathrm{Gal}(L\backslash F) \quad \text{y}$$

$$\mathrm{Gal}(L\backslash F) \big/ \mathrm{Gal}(L\backslash K) \xrightarrow{\cong} \mathrm{Gal}(K\backslash F)$$

Demostración: El teorema de la clase pasada nos dice que $\mathrm{Gal}(L\backslash K) \triangleleft \mathrm{Gal}(L\backslash F)$.

Nos resta relacionar

$$\mathrm{Gal}(L\backslash F) \big/ \mathrm{Gal}(L\backslash K) \quad \text{y} \quad \mathrm{Gal}(K\backslash F).$$

Tomemos $\sigma \in \mathrm{Gal}(L\backslash F)$ y consideremos

$$\begin{tikzcd} \sigma|_K : K \arrow[rr] & & \sigma K = K \\ & F \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$\boxed{\text{F<K es Galois \& clase pasada}}$

III

$\Rightarrow \quad \sigma|_K \in Gal(K \backslash F)$. Por tanto

$$\Phi : Gal(L\backslash F) \longrightarrow Gal(K\backslash F)$$
$$\sigma \longmapsto \sigma|_K$$

Observar

$$\sigma\cdot\tau|_K = (\sigma\circ\tau)|_K = \sigma|_K \circ \tau|_K$$

$$= \sigma|_K \cdot \tau|_K$$

Por tanto Φ es un homomorfismo de grupos

$$\Rightarrow \quad Gal(L\backslash F)/\ker\Phi \cong Im(\Phi) \leq Gal(K\backslash F)$$

$$\ker \Phi = \{ \sigma \in Gal(L \backslash F) \mid \sigma|_K = Id \}$$

$$= Gal(L \backslash K).$$

Por otro lado

$$|Im(\Phi)| = \frac{|Gal(E \backslash F)|}{|Gal(L \backslash K)|}$$

$L \subseteq F$
$K \subseteq L$
son Galois

$$= \frac{[L : F]}{[L : K]} = [K : F]$$

$$= |Gal(K \backslash F)|$$

$F \subseteq K$
es Galois

$$\Rightarrow Im(\Phi) = Gal(K \backslash F)$$

V

Eg.: $D_4 \cong \langle \sigma, \tau \rangle$ $\qquad L = \mathbb{Q}(i, \sqrt[4]{2})$

$\sigma(i) = i \qquad \tau(i) = -i$

$\sigma(\sqrt[4]{2}) = i\sqrt[4]{2} \qquad \tau(\sqrt[4]{2}) = \sqrt[4]{2}$

$|D_4| = 8$ actua en el cuadrado (4-ágono regular)

$$
\begin{tikzcd}
 & & \{e\} \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\langle \tau \rangle \arrow[dr, dash] & \langle \sigma^2\tau \rangle \arrow[d, dash] & \langle \sigma^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle \sigma\tau \rangle \arrow[d, dash] & \langle \sigma^3\tau \rangle \arrow[dl, dash] \\
 & \langle \sigma^2, \tau \rangle \arrow[dr, dash] & \langle \sigma \rangle \arrow[d, dash] & \langle \sigma^2, \sigma\tau \rangle \arrow[dl, dash] & \\
 & & \mathrm{Gal}(L \backslash \mathbb{Q}) & &
\end{tikzcd}
$$

Hemos visto hasta este momento:

$$
\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[d, dash] & \mathbb{Q}(i\sqrt[4]{2}) \arrow[dl, dash] & \mathbb{Q}(i, \sqrt{2}) \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] & \mathbb{Q}(i) \arrow[d, dash] & \mathbb{Q}(i\sqrt{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
$$

$L_{\langle \sigma\tau \rangle} = ?$ $\qquad L_{\langle \sigma^3\tau \rangle} = ?$

TAREA