On spanning trees and cycles of multicolored point sets with few intersections: Appendix

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A Appendix

On the section Minimum weight spanning trees of the paper “On spanning trees and cycles of multicolored point sets with few intersections” the following claim was made.

Lemma A.1. Let $P_i$ and $P_j$ be two sets of points and $T_i$ and $T_j$ the corresponding minimum spanning trees. Then for an edge $e \in T_i$ of length $r$, there are at most $c$ edges of $T_j$ with length at least $r$ which cross $e$. In fact $c \leq 8$.

In this appendix we give an outline proof of Lemma A.1. Let’s recall some basic properties of minimum weight spanning trees.

For an edge $f = (v, w)$, we denote by $\text{Cir}(f, v)$ the circle of radius $|f|$ and center at the point $v$.

Lemma A.2. Let $f = (v, w)$ be an edge of $T_j$. Then, there are no points of $P_j$ in the interior of the region $\text{Cir}(f, v) \cap \text{Cir}(f, w)$.

Corollary A.3. If two edges $(v, w)$ and $(v, u)$ of $T_i$ have an endpoint $v$ in common, then the smaller angle between the segments $vw$ and $vu$ is at least $\frac{\pi}{3}$. In the case that the angle is $\frac{\pi}{3}$, then the segments are of equal length.

One consequence is that the maximum degree in the m.w.s tree is 5 (remember that the points are in general position, thus, if a vertex $v$ has 6 adjacent points, these will define a regular hexagon with center at $v$).
The proof of Lemma A.1 starts as follows. Take an edge $e$ in $T_i$ and edges $f_1, \ldots, f_k$ in $T_j$ that intersect $e$ and have length at least $|e|$. We assume that $e$ is horizontal, then we can partition the edges $f_1, \ldots, f_k$ into the sets $U$ of the edges that have their midpoint above $e$ and $L$ the rest of the edges. We show that $U$ has at most 5 elements.

**Lemma A.4.** Let $f = (v, w)$ and $g = (x, y)$ be two edges in $T_j$, then for $h = (v, x)$ and $i = (w, y)$ we have that

$$\max\{|f|, |g|\} \leq \max\{|h|, |i|\}$$

This lemma says that for any pair of edges $f$ and $g$ there are endpoints $v \in f$ and $w \in g$ that are at distance greater or equal than $|f|$ and $|g|$, we called the segment defined by $v$ and $w$ the *opening* between $f$ and $g$. So, for any pair of consecutive edges in $U$ we have that their opening is either above or below $e$.

Clearly, the worst case is when all the openings of consecutive pairs in $U$ are above $e$.

We have the following technical lemma.

**Lemma A.5.** Let $f = (v, w)$ be an edge of $T_j$ and let $p$ be a point outside $\text{Cir}(f, w)$ but such that the angle $\frac{\pi}{6} \leq \angle pvw \leq \frac{\pi}{3}$. If $g = (p, q)$ is an edge of $T_j$, for some point $q$, then the distance from $g$ to $f$ is at least $|f| \sin(\frac{2\pi}{3} - 2\alpha)$.

The use of the lemma in our proof is better understood by seen figure 1. In the figure the angle $\alpha = \angle pvw$.

The lemma says not just that the distance $d$ is in inverse proportion to the angle $\alpha$, but that the worst case for edges in $U$ is when they all are of roughly the same length.

**Lemma A.6.** Let $e = (v, w)$ be an edge of $T_i$ and let $f = (p, q)$ and $g = (r, s)$ be edges of $T_j$ with their midpoints on the same side of $e$, as in Figure 2. If we have that $\alpha, \beta > \frac{\pi}{3}$ then at most one more edge of $T_j$ could cross $e$ and have its midpoint on the same side of $e$ as $f$ and $g$.

With these restrictions the worst case for a edges in $U$ is given by Figure 3, so at most 5 elements are in $U$. 2
Figure 1: Lemma A.5 gives a relation between the angle $\alpha$ and the distance $d$ of $q$ to the edge $f$. In the draw the segment from $q$ to $f$ is perpendicular to $f$.

Figure 2: From Lemma A.6 we know that there is at most one more edge crossing $e$ between $f$ and $g$ when $\alpha, \beta > \frac{\pi}{3}$.
This immediately gives a bound of 10 for the number of edges of $T_j$ that intersect $e$ and are of length greater or equal than $|e|$.

A more careful analysis shows that in the situation of Figure 3, the set $\mathcal{L}$ has at most two elements. We get a total of 7 crossings. Thus the maximum number of crossings is at most 8, that is when $|\mathcal{U}| = |\mathcal{L}| = 4$.

![Figure 3: In the figure $f_1$ and $f_2$ have a common vertex, they have the same length, the angle between them is $\frac{\pi}{3}$ and the angle between $f_1$ and the horizon is $\epsilon$. Similarly for $f_4$ and $f_5$.](image)

\[\text{Figure 3: In the figure } f_1 \text{ and } f_2 \text{ have a common vertex, they have the same length, the angle between them is } \frac{\pi}{3} \text{ and the angle between } f_1 \text{ and the horizon is } \epsilon. \text{ Similarly for } f_4 \text{ and } f_5.\]