A note on some inequalities for the Tutte polynomial of a matroid

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Abstract

We prove that if a matroid M contains two disjoint bases (or, dually, if its ground set is the union of two bases), then $T_M(a,a) \leq \max\{T_M(2a,0), T_M(0,2a)\}$ for $a \geq 2$. This resembles the conjecture that appears in C. Merino and D.J.A. Welsh, Forests, colourings and acyclic orientations of the square lattice, *Annals of Combinatorics* **3** (1999) pp. 417–429: If G is a 2-connected graph with no loops, then $T_G(1,1) \leq \max\{T_G(2,0), T(0,2)\}$. We conjecture that $T_M(1,1) \leq \max\{T_M(2,0), T_M(0,2)\}$ for matroids which contains two disjoint bases or its ground set is the union of two bases. We also prove the latter for some families of graphs and matroids.

1 Introduction

The Tutte polynomial is a two variable polynomial which can be defined for a graph G or, more general, a matroid M. The Tutte polynomial has many interesting combinatorial interpretations when evaluated on different

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points $(x,\ y)$ and along several algebraic curves. For example, for a graph G, the Tutte polynomial along the line $y = 0$ is the chromatic polynomial, after a suitable change of variable and the multiplication by an easy term. Similarly, we can get the flow polynomial of a graph and the all terminal realibility of a network and the partition function of a Q-state Potts model. When considering a GF(q)-representable matroid, the Tutte polynomial give us the weight enumerator of liner codes over GF(q) associated to M.

Thus, the Tutte polynomial has receive a fair amount of attention by trying to get new interpretations or by understanding the structure of the Tutte polynomial. About the latter we can mentioned the recent efforts to finding zeros of the chromatic polynomial. This paper is a small step on trying to understand the maximal and minima of the Tutte polynomial.

2 Preliminaries

For matroid theory we follow Oxley's book [19] and for graph theory we follow Diestel's book [10].

The Tutte polynomial is a matroid invariant over the ring $\mathbb{Z}[x, y]$. Further details of many of the concepts treated here can be found in Welsh [24] and Oxley and Brylawski [7].

Some of the richness of the Tutte polynomial is due to its numerous equivalent definitions. One of the simplest definitions, which is often the easiest way to prove properties of the Tutte polynomial, uses the notion of rank.

If $M = (E, r)$ is a matroid, where r is the rank-function of M, and $A \subseteq E$, we denote $r(E) - r(A)$ by $z(A)$ and $|A| - r(A)$ by $n(A)$ (the function $n(A)$ is called the nullity of A).

Definition 2.1. *The Tutte polynomial of M, $T_M(x, y)$, has the following expansion*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{z(A)}(y-1)^{n(A)} \ . \tag{1}$$

Almost immediately we get that

- $T_G(1, 1)$ equals the number of bases of M and
- $T_G(2, 2)$ equals $2^{|E|}$.

Recall that if $M = (E, r)$ is a matroid, then $M^* = (E, r^*)$ is the dual matroid, where $r^*(A) = |A| - r(E) + r(E \setminus A)$. It is not difficult to prove that $T_M(x, y) = T_{M^*}(y, x)$.

The Tutte polynomial may be also defined by a linear recursion relation given by deleting and contracting elements that are neither loops nor isthmuses.

Definition 2.2. *If M is a matroid, and e is an element that is neither an isthmus nor a loop, then*

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y). \tag{2}$$

If there is no such element e, then $T_M(x, y) = x^i y^j$ where M has i isthmuses and j loops.

The proof that Definition 2.1 and 2.2 are equivalent can found in [7]. We still require another (equivalent) definition of the Tutte polynomial but first we introduce the relevant notions.

Let us take a fixed ordering $\prec$ on the elements of M, say $E = \{e_1, \dots, e_m\}$, where $e_i \prec e_j$ if $i < j$. Given a fixed basis S, an element e is called *internally active* if $e \in S$ and it is the smallest edge in the only cocircuit disjoint from $S \setminus \{e\}$. Dually, an element f is *externally active* if $f \notin S$ and it is the smallest element in the only circuit contained in $S \cup \{f\}$. We define t_{ij} to be the number of bases with i internally activity elements and j externally activity elements. In [22] Tutte defined T_M using these concepts. A proof of the equivalence with Definition 2.1 can be found in [3].

Definition 2.3. *If $M = (E, r)$ is a matroid with a total order on its ground set, then*

$$T_M(x, y) = \sum_{ij} t_{ij} x^i y^j \ . \tag{3}$$

In particular, the terms t_{ij} are independent of the total order used on the ground set.

By an induction argument using equation (2), it can be proved that $t_{10} = t_{01}$ when $E(M) \geq 2$. This is one of a number of identities that hold for the coefficients t_{ij}. For a complete characterization of all the affine linear relations that hold among the coefficients t_{ij} see Theorem 6.2.13 in [7]. From there we extract the relations that we need.

Theorem 2.4. *If a rank-r matroid M with m elements has neither loops nor isthmuses, then*

(a) $t_{ij} = 0$*, whenever* $i > r$ *or* $j > m - r$*;*

(b) $t_{r0} = 1$ *and* $t_{0,m-r} = 1$*;*

(c) $t_{rj} = 0$ *for all* $j > 0$ *and* $t_{i,m-r} = 0$ *for all* $i > 0$*.*

The previous result follows easily from Definition 2.3. In [7] the statement is for simple matroids (geometries) but it is easy to extend it to matroids with parallel elements.

3 Some inequalities for the Tutte polynomial

From the results in the previous section it is easy to prove the following:

Theorem 3.1. *If a matroid M has neither loops nor isthmuses, then*

$$\max\{T_M(4,0), T_M(0,4)\} \geq T_M(2,2).$$

Proof. Let r be the rank and m the number of elements of M.

$$\begin{aligned}\max\{T_M(4,0), T_M(0,4)\} &\geq \max\{4^r, 4^{m-r}\} \\ &= \max\{2^{2r}, 2^{2(m-r)}\} \\ &\geq 2^m = T_M(2,2)\ ,\end{aligned}$$

where the first inequality follows from equation (3) combine with (b) in Theorem 2.4. □

Note that, for a matroid $M = (E, r)$ with dual $M^* = (E, r^*)$, the following inequalities are equivalent for any $A \subseteq E$.

$$\begin{aligned}|A| &\leq |E| - 2(r(E) - r(A)), & (4)\\ |E \setminus A| &\leq 2r^*(E \setminus A) \text{ and} & (5)\\ z(A) + n(A) &\leq m - r. & (6)\end{aligned}$$

We now restrict attention to matroids M in which all subsets A of the ground set E satisfy the (equivalent) inequalities above. By a classical result of J. Edmonds [11], these are the matroids that contain two disjoint bases;

by duality, these are the matroids M whose ground set is the union of two bases of M^*.

As every term $(x-1)^{z(A)}(y-1)^{n(A)}$ in T_M has $x^{z(A)}y^{n(A)}$ as its monomial of maximum degree, the following theorem follows directly from the above set of inequalities.

Theorem 3.2. *If a matroid M contains two disjoint bases, then $t_{ij}=0$, for all i and j such that $i+j>m-r$. Dually, if its ground set is the union of two bases, then $t_{ij}=0$, for all i and j such that $i+j>r$.*

Theorem 3.3. *If a matroid M contains two disjoint bases, then*

$$T_M(0,2a) \geq T_M(a,a)\ , \tag{7}$$

for all $a \geq 2$. Dually, if its ground set is the union of two bases of M^*, then

$$T_M(2a,0) \geq T_M(a,a), \tag{8}$$

for all $a \geq 2$.

Proof. Let us consider just the case when M has two disjoint bases; the other case follows by duality. In this situation $m-r \geq r$. From the proof of Theorem 3.1 and equation (3) we have that $4^{m-r} \geq T_M(2,2) = \sum_{ij} t_{ij}2^{i+j}$. Multiplying this inequality by $(a/2)^{m-r}$ we get

$$(2a)^{m-r} \geq \sum_{ij} t_{ij}\left(\frac{a}{2}\right)^{m-r} 2^{i+j} \geq \sum_{ij} t_{ij}\left(\frac{a}{2}\right)^{i+j} 2^{i+j} = \sum_{ij} t_{ij}a^{i+j}.$$

The second inequality follows from the inequalities $i+j \leq m-r$ for all $t_{ij}>0$. Thus

$$T_M(0,2a) \geq (2a)^{m-r} \geq \sum_{ij} t_{ij}a^{i+j} = T_M(a,a).$$

□

In particular we get the following corolaries.

Corollary 3.4. *For a matroid M, we have that*

$$\max\{T_M(2a,0), T_M(0,2a)\} \geq T_M(a,a),$$

for all $a \geq 2$ whenever M is one of the following:

- *an identically self-dual matroid M,*
- *a paving matroid,*
- *the uniform matroid $U_{r,n}$ for $0 \leq r \leq n$.*
- *a rank-r projective geometry over GF(q) or its dual, for $r \geq 2$.*

Proof. A matroid $M = (E, r)$ is identically self-dual matroid if $M = M^*$, so, B is a basis of M if and only if $E - B$ is a basis of M.

If M is a paving matroid, all the circuits of M have size $r(M)$ or $r(M)+1$. So, any subset of E of size $r(E)$ is either a basis or a circuit. Let us assume that $2r(E) \leq |E|$ and take any basis B of M and any set B' of size $r(E)$ disjoint from B. The set B' is either a basis, or a circuit. Then, for any $e \in B'$, the fundamental circuit in $B \cup e$ is the whole set $B \cup e$. As $B' \setminus e$ is independent of size $r(E) - 1$, there exits an $f \in B$ such that $B' \setminus e \cup f$ is a basis. The set $B \cup e \setminus f$ is also a basis and M contains two disjoint basis. If $2r(E) > |E|$, the complement of a basis B in M has cardinality strictly less than $r(E)$ and cannot contain a circuit, so its an independent set of M. Thus,the ground set of M is the union of two basis.

The matroid $U_{r,n}$ has two disjoint bases if $2r \leq n$ and its ground set is the union of two bases if $n < 2r$.

The matroid PG(r, q) contains W_r as a submatroid for $r \geq 3$. The latter is self-dual and so it contains two disjoint bases. Thus, PG(r, q) contains two disjoint bases. If the matroid is a projective plane of order $m \geq 4$, then it contains $U_{2,4} \oplus_2 U_{2,4}$ as a submatroid. Again, the latter is self-dual and contains two disjoint bases. Thus, projective plane contains two disjoint bases. The only projective plane of order 3 is the Fano matroid and it clearly contains two disjoint bases. □

Corollary 3.5. *For a loopless, bridgeless graph G, we have that*

$$\max\{T_M(2a, 0), T_M(0, 2a)\} \geq T_M(a, a),$$

for all $a \geq 2$ whenever G is one of the following:

- *a 4-edge-connected graph,*
- *a 2-connected chordal graph,*
- *a complete bipartite graph,*

- *a series-parallel graph,*
- *a cubic graph,*
- *a bipartite planar,*
- *a Laman graph,*
- *a triangulation,*
- *the wheel graph W_n, for $n \geq 2$,*
- *the square lattice L_n, for $n \geq 2$,*
- *an n-cycle $n \geq 2$,*
- *a tree with n edges, for $n \geq 1$.*

Proof. By the classical result in [23] every 4-edge-connected graphs has two edge-disjoint spanning trees. It is easy to see that 2-connected chordal and wheel graphs have two edge-disjoint spanning trees. Using the expression for computing the arboricity of a graph given in [18] we get that series-parallel, cubic, bipartite planar, and Laman graphs all have arboricity two, which is equivalent to having two spanning trees that cover all the edges of the graph. Triangulations are geometric duals of cubic planar graphs, so they have two edge-disjoint spanning trees.

It is easy to see that each of $K_{2,m}$ for $m \geq 2$, $K_{3,3}$, the square lattice L_n for $n \geq 2$, the n-cycle for $n \geq 2$, and a tree have two spanning trees which cover all the edges in the graph. For $K_{n,m}$ with n or m at least 4, you can always get two edge-disjoint spanning trees. □

4 A conjecture

After Theorem 3.3, it is natural to ask if inequalities (7) and (8) are valid for $a = 1$. For the case of graphic matroids this is almost the conjecture made in [16].

Conjecture 4.1. *Let G be a 2-connected graph with no loops, then*

$$\max\{T_G(2,0),\ T_G(0,2)\} \geq T_G(1,1)\ . \tag{9}$$

Certainly, equation (9) is not true for graphs in general, by adding loops or bridges to some simple graphs you can construct examples of graph that does not satisfy the conjecture. However, the condition on the connectivity may not be the most natural since if a planar graph G satisfies equation (9), then any geometric dual G^* also satisfies equation (9) but G^* may not be 2-connected, for example if G has parallel edges.

Thus, Theorem 3.3 suggest that a more suitable conjecture for graphs and matroids is the following.

Conjecture 4.2. *If a cosimple matroid M contains two disjoint bases then*

$$T_M(0,2) \geq T_M(1,1) \ . \tag{10}$$

Dually, if the ground set of a simple M is the union of two bases, then

$$T_M(2,0) \geq T_M(1,1) \ . \tag{11}$$

By adding paralel elements to M, we increase the number of its bases but the evaluation $T_M(2,0)$ will remain the same. That is the reason for asking M to be simple if its ground set is the union of two bases. However, this may not be necessary, as in such a matroid all the parallel clases are of size at most two, and there are not many elements in parallel, certainly no more than $|E| - r(E) \leq r(E)$.

We prove Conjecture 4.2 for some classes of matroids and for some of the families of graphs listed in Theorem 3.5.

Note: for a graphic matroid $M(G)$, the evaluation of the Tutte polynomial at $(2,0)$ and $(0,2)$ have the interpretation of being the number of acyclic orientations and the number of totally cyclic orientations of G, respectively. An acyclic orientation of a graph G is an orientation where there are not directed cycles. A totally cyclic orientation is an orientation where every edge is in a directed cyclic. See [7] for a proof of this result. In this situation we use the notation of $\alpha(G)$ for $T_G(2,0)$ and $\alpha^*(G)$ for $T_G(0,2)$. If G is connected, the number of spanning trees of G is the evaluation of the Tutte polynomial at $(1,1)$ and this quantity is denoted by $\tau(G)$.

4.1 Uniform matroids

Using equation (1) the Tutte polynomial of the uniform matroid $U_{r,m}$ is given by the following expression.

$$\begin{aligned} T_{U_{r,m}}(x,y) = (x-1)^r + \binom{m}{1}(x-1)^{r-1} + \ldots + \binom{m}{r-1}(x-1) + \binom{m}{r} \\ + \binom{m}{r+1}(y-1) + \ldots + \binom{m}{m}(y-1)^{m-r}. \end{aligned} \tag{12}$$

Let us suppose that $r \leq m$, in this case the matroid $U_{r,m}$ contains two disjoint bases. The evaluation $T_{U_{r,m}}(1,1)$ corresponds to the number of bases and is $\binom{m}{r}$. The evaluation $T_{U_{r,m}}(2,0)$ is, according to the previous formula,

$$\begin{aligned} T_{U_{r,m}}(2,0) = 1 + \binom{m}{1} + \ldots + \binom{m}{r-1} + \binom{m}{r} \\ - \binom{m}{r+1} + \binom{m}{r+2} \ldots + (-1)^{m-r}. \end{aligned} \tag{13}$$

Then, $T_{U_{r,m}}(2,0) - \binom{m}{r}$ is positive, so $T_{U_{r,m}}$ satisfies equation (11). The case $r > m$ follows by duality and we get the following

Theorem 4.3. *Uniform matroids satisfy Conjecture 4.2.*

4.2 Wheels and whirls

For this subsection we consider wheels, a well-known class of self-dual planar graphs, and whirls, a related class of matroids which are also self-dual. The wheel graph W_n has $n+1$ vertices and $2n$ edges. The vertices $\{1, \cdots, n\}$ form an n-cycle while the vertex 0 is adjacent to every vertex in this cycle. The whirl W^n is the unique relaxation of the matroid $M(W_n)$. See [19] for definition of relaxation.

It is well-known that $\tau(W_n) = L_{2n} - 2$, for $n \geq 1$, where L_k is the kth-Lucas number which is defined recursively by $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 3$. This result was proved by Sedláček [20] and also by Myers [17]. Using the analogy of Binet's Fibonacci formula for Lucas numbers we get

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

The same formula can be obtained directly by using equation (2) for $T(W_n;1,1)$ and then solving the corresponding recurrence relation.

The chromatic polynomial of W_n is known, see [2], and is equal to $\chi(W_n;x) = x(x-2)^n + (-1)^n x(x-2)$. Now, applying the famous result of R. Stanley [21] that relates the number of acyclic orientations and the chromatic polynomial, namely $\alpha(G) = |\chi(G;-1)|$, we get that $\alpha(W_n) = 3^n - 3$. These results together yield the following

Theorem 4.4. *For all $n \geq 2$, $\alpha(W_n) \geq \tau(W_n)$ and W_n satisfies Conjecture 4.2.*

It is not difficult to prove that if M' is a relaxation of the matroid M, then $T_{M'}(x,y) = T_M(x,y) - xy + x + y$. As W^n is a relaxation of $M(W_n)$, we get that

Theorem 4.5. *For all $n \geq 2$, $T_{W^n}(2,0) \geq T_{W^n}(1,1)$ and W^n satisfies Conjecture 4.2.*

4.3 3-regular graphs

As a family of graphs such that their edge-set is the union of two forests but are not self-dual, we consider 3-regular graphs G with girth at least 5. In this case the general bound for the number of acyclic orientations,

$$\alpha(G) \geq (2^{3/8}3^{3/8}4^{1/8})^n ,$$

is given in [13], where n is the number of vertices. And on the other hand the general upper bound for the number of spanning trees in a 3-regular graph G,

$$\tau(G) \leq \frac{2\beta}{3n} e^{\frac{12}{\sqrt{\pi}}\left(\frac{1}{\beta}\right)^{\frac{5}{2}}} \left(\frac{4}{\sqrt{3}}\right)^n ,$$

where $\beta = \lceil \ln(n)/\ln(9/8) \rceil$, is given in [8]. From the formulae we obtain the following:

Theorem 4.6. *If G is a cubic graph of girth at least 5, we have that $\tau(G) < \alpha(G)$ and $M(G)$ and $M^*(G)$ satisfy Conjecture 4.2.*

4.4 Complete graphs and complete bipartite graphs

It is natural to check if Conjecture 4.1 or 4.2 are true for complete graphs and complete bipartite graphs.

It is a clasical result due to Cayley that $\tau(K_n) = n^{n-2}$. The edge-set of K_3 is the union of two spanning trees and $\alpha(K_3) = 6 > 3 = \tau(K_3)$, thus K_3 satisfy Conjecture 4.2. For $n \geq 4$, K_n has two disjoint spanning trees.

We use the following lemma that has an easy proof, see [9].

Lemma 4.7. *If G is a 2-connected graph with a vertex v of degree d, then $(2^d - 2)\alpha^*(G - v) \leq \alpha^*(G)$.*

We will prove that $\alpha^*(K_n) \geq n^{n-2}$, for $n \geq 4$. For $n = 4$ this is true, in fact, $\alpha^*(K_4) = 24 > 16 = \tau(K_4)$. We proceed by induction on n.

$$\begin{aligned}\tau(K_{n+1}) = (n+1)^{n-1} &= \left(\frac{n+1}{n}\right)^n \left(\frac{n}{n+1}\right)^2 (n+1)\tau(K_n) \\ &\leq e(n+1)\tau(K_n) \leq (2^n - 2)\tau(K_n) \\ &\leq (2^n - 2)\alpha^*(K_n).\end{aligned}$$

The last quantity is less than or equal $\alpha^*(K_{n+1})$ by the previous lemma.

Theorem 4.8. *For all $n \geq 3$, $M(K_n)$ and $M^*(K_n)$ satisfy Conjecture 4.2.*

The technique used for complete graphs can be used to prove the Conjecture 4.1 in the case of threshold graphs, a type of chordal graphs, see [9].

Using the matrix-tree theorem it is easy to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$. The Tutte polynomial of $K_{2,m}$ could be easily computed by using the general formula for the Tutte polynomial of the tensor product of graphs G_1 and G_2, where here, G_1 is the graph P_2^m, that is m parallel edges, and G_2 is the path of length two P_3. This particular case of the tensor product is also referred to as the stretching of G_1; see [14]. We get that

$$T(K_{2,m}; x, y) = (x+1)^{m-1} \left(\sum_{i=1}^{m-1} \left(\frac{x+y}{x+1}\right)^i + x^2 \right). \tag{14}$$

Thus, $\tau(K_{2,m}) = 2^{m-1}m \leq 2(3)^m - 2^m = \alpha(K_{2,m})$, for $m \geq 2$. For $K_{3,3}$, we compute its Tutte polynomial and get that $T(K_{3,3}; x, y) = x^5 +$

$4x^4 + 10x^3 + 11x^2 + 5x + 9x^2y + 15xy + 5y + 6xy^2 + 9y^2 + 5y^3 + y^4$, so $\tau(K_{3,3}) = 81 < 230 = \alpha(K_{3,3})$. Note that in these cases, the complete bipartite graphs have two spanning trees which cover all the edges.

For the rest of the cases, complete bipartite graphs have two disjoint spanning trees. For $K_{3,m}$, the number of totally cyclic orientations is at least $6^m - 6(5)^{m-1}$, see [9]. As $6(6^{m-1} - 5^{m-1}) > 6m5^{m-2}$, $\tau(K_{3,m}) \leq \alpha^*(K_{3,m})$ if $(\frac{5}{3})^{m-2} \geq \frac{m}{2}$, but this is true as $m \geq 4$.

For $n \geq 3$ we have the following recursive relations: $\tau(K_{n+1,m}) = m(1 + 1/n)^{m-1}\tau(K_{n,m})$ and $\alpha^*(K_{n+1,m}) \geq (2^m - 2)\alpha^*(K_{n,m})$. This last inequality follows by Lemma 4.7. As $n \geq 3$, $\ln(1 + 1/n) \leq 3/(3n + 1) \leq 3/10$. Thus, $\tau(K_{n+1,m}) \leq \alpha^*(K_{n+1,m})$ follows by induction as $\ln(m) + 3/10(m - 1) \leq \ln(2^m - 2)$ for $m \geq 4$. Putting these together we have the following

Theorem 4.9. *For all $m \geq n \geq 2$, $M(K_{n,m})$ and $M^*(K_{n,m})$ satisfy Conjecture 4.2.*

5 Catalan matroids

A *Dyck path* of length $2n$ is a path in the plane from (0,0) to ($2n$,0), with steps (1,1), called *up-step*, and (1,-1), called *down-step*. It is well-known that the number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Each Dyck path P defines an *up-step set*, consisting of the integers i, $1 \leq i \leq 2n$, for which the i-step of P is an up-step. The collection of up-step sets of all Dyck paths of length $2n$ form the bases of a matroid M_n over $\{1, 2, \ldots, 2n\}$. These matroids are called Catalan matroids and have been study extensively recently, see [5] or [1].

We consider the matroids N_n, $n \geq 2$, obtained form M_n by deleting the elements 1 and $2n$. This corresponds to deleting the loop and isthmus of M_n. From the results in [5] it follows that the matroid N_n is self-dual, but not identically self-dual. The matroid N_n can be constructed from the empty matroid by alternatively adding a isthmus and taking a free extension $n - 1$ times. An expression for the Tutte polynomial of N_n follows from Corollary 5.8 of [5].

$$\sum_{i,j>0} \frac{i+j-2}{n} \binom{2n-i-j+1}{n-i-j+2} x^{i-1} y^{j-1}.$$

After some algebraic manipulations we get a formula for the evaluation at

(2,0) and (0,2).

$$T_{N_n}(2,0) = T_{N_n}(0,2) = \sum_{k=0}^{m} \frac{k}{m} \binom{2m-k-1}{m-k} 2^k,$$

where $m = n-1$. The latter formula equals the central binomial coefficient $\binom{2m}{m}$. Note that N_n has two disjoint bases, namely $\{2, 4, \ldots, 2n-2\}$ and $\{3, 5, \ldots, 2n-1\}$.

Theorem 5.1. *For all $n \geq 2$, N_n satisfies Conjecture 4.2.*

The reason to bring up Catalan matroids is because the construction using free extensions. If an rank-r M contains two disjoint bases, so does its free extension $M+e$. The Tutte polynomial of $M+e$ can be expressed in term of the Tutte polynomial of M by

$$T_{M+e}(x,y) = \frac{x}{x-1} T_M(x,y) + \left(y - \frac{x}{x-1}\right) T_M(1,y).$$

And then $T_{M+e}(0,2) = 2T_M(1,2)$, that is twice the number of spanning sets of M. But the number of bases of $M+e$ is just the number of independent sets of size r and $r-1$. And the latter is much smaller than the former.

Theorem 5.2. *If a matroid M contains two disjoint bases, then its free extension $M+e$ satisfies Conjecture 4.2. Dually, if its ground set is the union of two bases, then the free coextension satisfies Conjecture 4.2.*

6 Conclusion

We proved Conjecture 4.2 for some families of graphs and matroids. There are some more families; for example Marc Noy (private communication) proved that $\tau(G) \leq \alpha(G)$ when G is a maximal outerplanar graph by using equation (2).

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References

[1] F. Ardila, The Catalan matroid. *Journal of Combinatorial Theory, Series A*, **104** (2003), pp. 49-62.

[2] N. Biggs, Algebraic Graph Theory. Cambridge University Press, Cambridge, 1996.

[3] A. Björner, Homology and shellability of matroids and geometric lattices. In: White, N. (ed) Matroid Applications, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

[4] J. Bonin and A. de Mier, Tutte polynomials of generalized parallel connections. *Adv. in Appl. Math.*, **32** (2004), pp. 31–43.

[5] J. Bonin, A. de Mier and M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials. *Journal of Combinatorial Theory, Series A*, **104** (2003), pp. 63–94.

[6] T. Brylawski, A decomposition for combinatorial geometries. *Trans. Amer. Math. Soc.*, **171** (1972), pp. 235–282.

[7] T. Brylawski and J. Oxley, The Tutte polynomial and its applications. In N. White, editor, *Matroid Applications*, Encyclopedia of Mathematics and its Applications, pp. 123–225. Cambridge University Press, Cambridge, 1992.

[8] F. Chung and S. T. Yau, Coverings, heat kernels and spanning trees. *Elect. J. Combinatorics*, **6**(1) (1999).

[9] R. Conde and C. Merino, Cyclic and totally cyclic orientations vs. spanning trees in threshold graphs. Preprint.

[10] R. Diestel, Graph Theory, Graduate Texts in Mathematics. Springer, New York, 2000.

[11] J. Edmonds, Lehman's Switching Game and a Theorem of Tutte and Nash-Williams. *J. Res. Natl. Bur. Stand.*, **69B** (1965), pp. 73–77.

[12] R. Grone and R. Merris, A bound for the complexity of a simple graph. *Discrete Math.*, **69** (1988), pp. 97–99.

[13] N. E. Kahale and L. J. Schulman, Bounds on the chromatic polynomial and on the number of acyclic orientations of a graph. *Combinatorica*, **16** (1996), pp. 383–397.

[14] F. Jaeger, D. L. Vertigan and D. J. A. Welsh, On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Camb. Phil. Soc.*, **108** (1990), pp. 35–53.

[15] R. Merris, Degree maximal graphs are Laplacian integral. *Linear Algebra Appl.*, **199** (1994), pp. 381–389.

[16] C. Merino and D.J.A. Welsh, Forests, colourings and acyclic orientations of the square lattice. *Annals of Combinatorics*, **3** (1999), pp. 417–429.

[17] B.R. Myers, Number of spanning trees in a wheel. *IEEE Trans. Circuit Theory*, **CT-18** (1971), pp. 387–391.

[18] C. St. J. A. Nash-Williams, Decomposition of finite graphs into forests. *J. London Math. Soc.*, **39** (1964), 12.

[19] J. G. Oxley, Matroid Theory. Oxford University Press, New York, 1992.

[20] J. Sedláček, Lucas numbers in graph theory (Czech. English summary). Mathematics (Geometry and Graph Theory) (Czech) pp. 111–115, Univ. Karlova, Prague, 1970.

[21] R.P. Stanley, Acyclic orientations of graphs. *Disc. Math.*, **5** (1973), pp. 172–178.

[22] W. T. Tutte, A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, **6**, (1954), pp. 80–91.

[23] W. T. Tutte, On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.*, **36** (1961), pp. 221–230.

[24] D. J. A. Welsh, Complexity: Knots, Colourings and Counting. Cambridge University Press, Cambridge, 1993.