DIAGRAMMATIC FORMULATION OF MULTI-BRAIDED QUANTUM GROUPS

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Abstract. A diagrammatic formulation of a generalized theory of Hopf algebras is outlined. The formalism covers various examples of quantum spaces equipped with a group-like structure, that are not includable in a braided category: Instead of one braid morphism, an infinite family of mutually related braid operators emerges, naturally expressing twisting properties of the product and the coproduct.

1. Introduction

In this article we are going to sketch basic ideas of a new diagrammatic formalism for generalized quantum groups.

Our main motivation comes from non-commutative differential geometry [2, 3] which provides a variety of interesting 'quantum spaces' possessing a natural group-like structure, which are however not includable in the framework of standard braided Hopf algebras [12].

A couple of words about non-commutative geometry and its philosophy. Non-commutative geometry deals with 'quantum spaces'. It is assumed that these spaces are represented by appropriate (non-commutative in general) complex *-algebras (interpreted as 'functions' over the associated spaces). When algebras are commutative, we are back in classical geometry. One of the most important features that distinguishes quantum spaces from classical spaces is that quantum spaces can not be viewed as *collections of points* equipped with an appropriate additional geometrical structure. In most cases 'true quantum spaces' have *no points at all*.

However, quantum spaces coming from standard quantum groups are always *inherently inhomogeneous* as there is always one classical point–corresponding to the 'neutral element' represented by the counit map (indeed, taking the classical Gelfand-Neimark theory into account, it is natural to assume that 'points' of a quantum space correspond to characters of the associated *-algebra, and in the standard theory the counit is necessarily a character).

This fact can be understood as a consequence of a too strong set of axioms imposed within the standard formulation. In particular, any 'group-like' quantum object based on a simple algebra is not includable within the standard approach. Such things are clearly not acceptable from both conceptual and aestetic point of views—quantum analogs of groups should retain a similar kind of internal homogeneity and isotropy, as their classical counterparts.

The mentioned inhomogeneity of standard quantum groups becomes transparent if we consider the theory of locally trivial quantum principal bundles over classical smooth manifolds [6], possessing compact quantum structure groups [17]. Surprisingly at a first sight, the classification problem of such structures reduces to the classification of classical principal bundles, over the same base space, the structure group of which is given by *the set of points* of the initial quantum group. This 'classical part' of a compact quantum group is a standard compact group, in a natural manner.

A geometrical explanation of the bundle reduction phenomenon is simple: Quantum groups are always interpretable as spaces consisting of two parts—classical and quantum (completely 'pointless' and described by the ideal over all commutators). Now, if we try to constuct a locally trivial quantum principal bundle over a classical space, we have to introduce a system of 'transition funcions' that would glue the fibers over the intersections of locally-trivializing regions. These transition functions preserve the structure of the fiber, and as a consequence they map the classical part into the classical part and the quantum part into the quantum part. In addition, the transition functions are covariant, with respect to the quantum group structure. It follows therefore that they are completely determined by their reductions on the corresponding classical part of the quantum structure group. The reduced transition functions give us the corresponding classical cocycle.

Another geometrical indication that standard Hopf/braided-Hopf algebras are insufficient to address the diversity of group-like quantum spaces comes from K-theory and cyclic cohomology [2, 3]. If two C*-algebras are strongly Morita equivalent, they have the same K-groups and cyclic cohomology groups. At the geometrical level, it means that the underlying quantum spaces are *very* similar (in fact, the formulation of [2, 3] assumes that strongly Morita-equivalent C*-algebras define exactly the same quantum space). From this point of view, it would be natural to expect that a proper geometrical formulation of quantum groups is closed under *stabilizations*. In particular if $\mathcal{A}$ is a 'good' Hopf algebra, then the matrix algebra $\mathrm{M}_n(\mathcal{A})$ should be acceptable as well. However, the standard theory is not closed under stabilizations.

Let us list some interesting examples of quantum objects that are naturally includable in the present formalism.

- Quantum tori—equivalently, rotation algebras [16] (both irrational and rational);
- General braided Clifford algebras (including standard Clifford and Weyl algebras, and bundles) [10][11][15].
- Matrix algebras over arbitrary quantum groups (in other words, the theory is closed under tensoring with matrix algebras).
- Various algebras emerging from quantum and classical combinatorics (as Hecke algebras and their approriate extensions, for example).
- Gauge bundles associated to general quantum principal bundles [7].
- Structures appearing when deforming the co-product/product or both, of a given quantum group (in other words, the theory is closed under the appropriate deformations).
- Cuntz algebras of partial isometries [4];
- Subquantum extensions of quantum observable algebras [5].

All these structures can not be viewed as standard Hopf algebras (including Hopf algebras in braided categories).

The paper is organized in the following way:

Our exposition begins by introducing the main diagrammatic category which can be viewed as a *blueprint* for all quantum groups. This is a monoidal category [1] generated by a single object (so that all objects are labeled by natural numbers) and two morphisms of type $1|2$ and $2|1$, that play the role of abstract coproduct and product. Because of its universal nature, calculations performed in $\mathcal{M}$ are automatically valid in any concrete realization.

We shall then consider some elementary consequences of the introduced axioms. All considerations will be performed in terms of *diagrams-pictures* representing morphisms in our category, and their algebraic interrelations. In particular, we shall introduce left and right 'transfer operators' and prove the existence of the co-unit and the antipode. It is important to stress that in our formulation we do not require the existence of any 'base field' object (no 0-object in our category). In terms of the standard language of Hopf algebras, this means that we would rather consider the composition of the counit and the unit, instead of dealing with them separately. This unified morphism will be called *the co-unit* .

After presenting the basic diagrammatic setup, we shall discuss a very interesting problematics of a possibility to play a *flipping game*: to introduce certain morphisms that allow us to transpose objects and morphisms, moving them from one position in a diagram to another.

In particular, as we shall see, there is always a canonical flip-over morphism (acting in the object 2, which expresses the 'multiplicativity property' of the coproduct moprhism.

If this flip-over morphism satisfies some additional conditions (invertibility and two octagonal diagrams) then it is possible to introduce an infinite family of twist morphisms (indexed by pairs of independent integers) that allow us to express twisting properties of the product and the coproduct (and hence, of all other morphisms) in a concise and elegant way. The initial flip-over morphism corresponds to the index (1,1). Furthermore, all these twist morphisms are mutually maximally compatible, in a braided sense. In particular, they all satisfy the braid equation.

An important feature of our theory is that, in contrary to braided categories formulation [12], we do not postulate any braid equation, nor any a priori given type of a twisting property. All such properties come out as theorems in our approach. The standard formulation of braided Hopf algebras can be viewed as a very special case of the formulation we present here, when all twist operators collapse to a single morphism (and when the co-unit is recognized as a composite object).

Finally, in the last section an alternative diagrammatic axiomatics is presented, and some concluding remarks are made.

This article follows the formalism developed in [9], substantially generalizing it further to include various additional structures. Technical proofs, as well as discussions of examples, will be presented in a separate publication.

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2. Blueprint Category

We are going to introduce our main structure–a monoidal category $\mathcal{M}$ of appropriate diagrams. Objects of $\mathcal{M}$ are labeled by natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5 \dots\}$ so that the monoidal product is simply the addition of natural numbers.

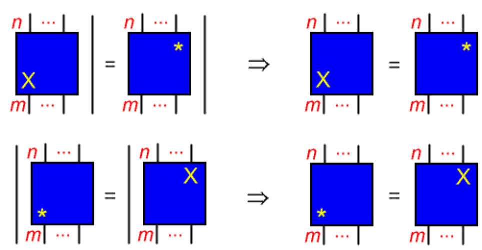
As usual in diagrammatic considerations, the arrows in our category will be represented by certain *pictures* possessing the appropriate number of entry lines (for the domain object) and exit lines (for the codomain object).

The set of morphisms between objects m and n is denoted by $\mathrm{Mor}[m,n]$. If $f \in \mathrm{Mor}[m,n]$, then we shall say that a morphism f is of type $m|n$. In the following picture, a morphism of type $n|m$ is shown

together with the identity morphism acting on the object k, represented simply by k parallel vertical lines.

The composition of morphisms is represented by connecting the final edges of the first diagram with the starting edges of the second. Our diagrams have vertical orientation, and 'time' goes from up to the bottom. The tensor product of morphisms is represented simply by putting diagrams next to each other (juxtaposition).

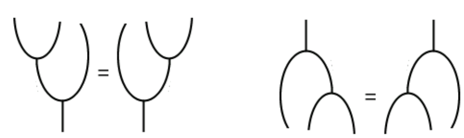
Let us assume that $\mathcal{M}$ is such that the following *cancellation* properties hold:



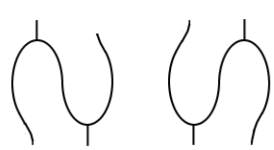
Furthermore, we shall assume that $\mathcal{M}$ possesses two distinguished morphisms:

These morphisms will be called coproduct and product respectively. Our further axioms will be

(i) **Associativity** & **Coassociativity**:



(ii) **Regularity**: Consider moprphisms



of type $2|2$. They are isomorphisms in $\mathcal{M}$.

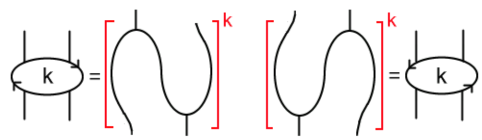
Our motivation for the regularity property comes from the C*-algebraic formulation of compact quantum groups [17]. In what follows, all results will be valid in any monoidal category satisfying the above mentioned simple properties. The adjective 'blueprint' is justified by fixing $\mathcal{M}$ to be *the minimal* monoidal category within the class. So it makes sense to formulate

Definition 1. *Let $\mathcal{C}$ be a monoidal category. A quantum group in $\mathcal{C}$ is a covariant functor from $\mathcal{M}$ with values in $\mathcal{C}$.*

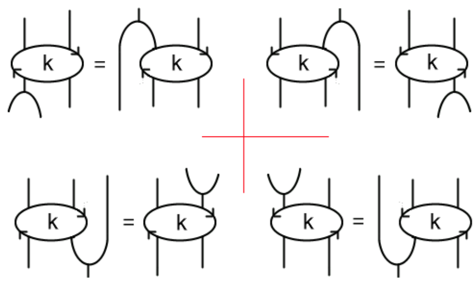
By construction $\mathcal{M}$ is a self-dual category.

3. Elementary Consequences

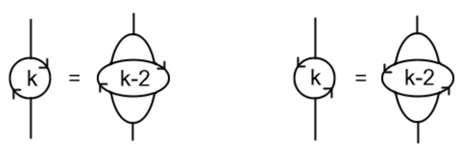
We are going to introduce a couple of auxiliary morphisms, that will enable us to construct the counterparts of the antipode and the co-unit. At first, for every integer $k \in \mathbb{Z}$ let us define the following 'elliptic' $2|2$-type morphisms



Using the coassociativity and associativity property, it is easy to verify that the following commutation identities hold:

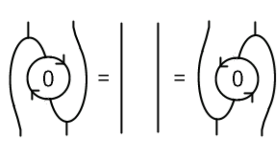


Secondly, let us consider the following 'circular' $1|1$-type morphisms:

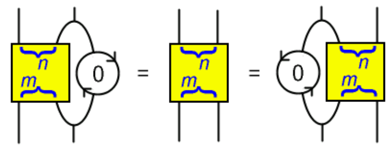


Applying the above commutation identities, it is easy to prove that

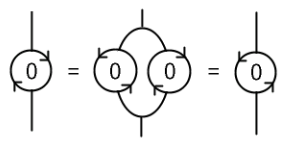
for each $k \in \mathbb{Z}$. In particular, a nice 'jumping property' holds:



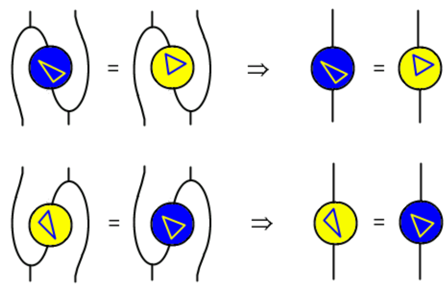
Now, consider an arbitrary $n|m$-type morphism in $\mathcal{M}$. Using the above jumping property, the coassociativity and associativity properties, the cancellation law, and performing some elementary transformations we get



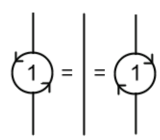
which implies



It follows immediately that the following *diagonal cancellation* properties hold



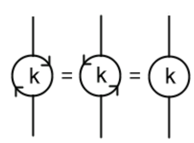
for 1|1-type morphisms in $\mathcal{M}$. In particular, we conclude that



which can be viewed as a kind of a normalization property.

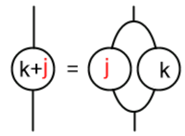
Now, it is easy to verify that

(i) All left-oriented circle-morphisms coincide with the correspoining right-oriented 1|1-type morphisms



and consequently we can *drop the orientation* from our notation.

(ii) The following convolution identity holds:



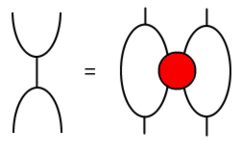
Note that elliptic 2|2 morphisms are inherently oriented (except the 0-case, when they reduce to the identity morphism).

Definition 2. *The circular morphism corresponding to $k = 0$ is called the co-unit. The one that is labeled by $k = -1$ is called the antipode.*

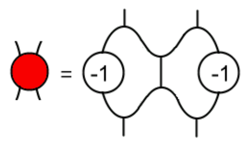
It is easy to see that the entire system of circular morphisms is uniquely fixed by the convolution property and the normalization condition (for $k = 1$ we get $\mathrm{id}\colon 1 \to 1$).

4. Canonical Twist and Beyond

We would like to introduce a new morphism in the game, playing the role of an elementary twist, expressing a 'multiplicativity property' of the coproduct map:



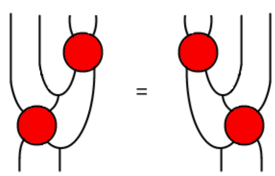
However it is straightforward to verify that the above property is equivalent to



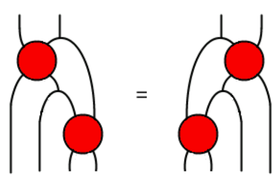
which is nothing but *the definition* of the twist operator.

In other words, it is always possible to introduce a flip-over morphism, so that the coproduct and product are co/multiplicative one to each other.

It is important to observe that such a flip morphism will not in general obey the braid equation. The most interesting theory appears if we assume that the flip-over operator is *invertible* and postulate the following *additional octagonal identity*



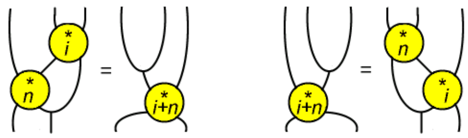
together with its *dual counterpart*



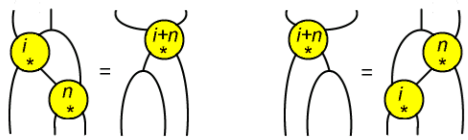
We shall denote by $\mathcal{M}_2$ the category obtained from $\mathcal{M}$ by adding these conditions. If the above mentioned octagonal/invertibility properties hold, then it is possible to play a very interesting twisting game. All morphisms become 'twistable'!

However, in order to properly express the twisting properties of the product and coproduct maps, it is not enough to use this single flip-over operator only. A more detailed analysis shows that the flip-over morphism is naturally included in an infinite family of 'twist-type' isomorphisms

the elements of which are indexed by a 2-dimensional integer lattice, so that we have the following additive twisting rule for the product

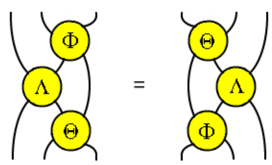


as well as the corresponding dual twisting properties



for the coproduct morphism. Here, the symbol * represents an arbitrary, but fixed index. The above twisting properties, together with the normalization condition, uniquely fix the entire family of extra flip-over isomorphisms.

Moreover, it turns out that all these isomorphisms are mutually maximally compatible, in a braided sense. This means that

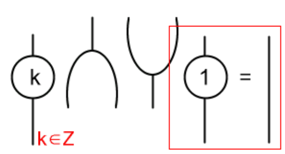


with $\Theta, \Lambda, \Phi \in \mathbb{Z} \times \mathbb{Z}$. In particular, all these twists satisfy the braid equation.

We see that our structure is not includable into the conceptual framework of braided categories, which emerges as a very special case of the formalism, when *all twist operators coincide.* In general, we need an infinity of braidings in order to express all twisting properties in the approriate way.

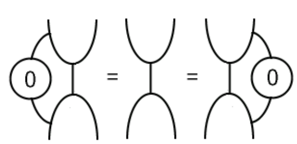
5. Variations & Concluding Remarks

We can obtain the same theory by using a slightly different set of axioms, going somewhat backwards. As in our initial formulation, we can start from a monoidal category $\mathcal{M}$ over natural numbers, with generating morphisms



and assume:

- **Standard Jumping Property of Co-unit**
- **Non-standard Jumping Property**

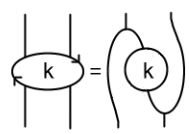


- **Convolution Property**
- **Associativity & Coassociativity**

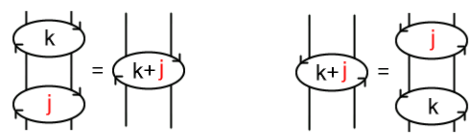
In this formulation we do not have to postulate the cancellation property. It follows as a simple theorem, from the above non-standard jumping property of the co-unit, together with the existence of the twist.

Indeed, the jumping property allows us to introduce the twist operator the same way as in the previous section. Using the twisting operator, and performing some elementary transformations, it is easy to see that the co-unit acts trivially on the left and on the right, on any morphism in our category. This implies the cancellation law.

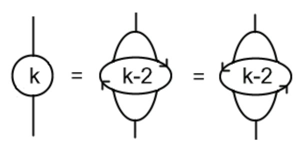
Furthermore, is easy to see that the circlular $1|1$ morphisms naturally give rise to our previously introduced elliptic $2|2$ morphisms



so that we have



and in particular



which closes a circle, establishing the equivalence with the first formulation, presented in Section 2.

In conclusion, we can define multi-braided quantum groups as certain 'images' of this game:

Definition 3. *Let $\mathcal{C}$ be an arbitrary monoidal category. A multi-braided quantum group in $\mathcal{C}$ is any covariant functor from $\mathcal{M}_2$ to $\mathcal{C}$.*

It is worth mentioning that octagonal conditions we introduced in the previous section are equivalent to the co/associativity of the canonical co/product maps, that we can naturally define on the composite object $2 = 1 \otimes 1$.

The formulation we presented essentially relies on the fact that both product and coproduct are co/associative morphisms. This assumption is especially important in considerations involving twist operators, and various jumping properties.

An interesting possibility is to completely drop co/associativity property, replacing it by a single weaker self-dual axiom, expressing the equality between two $3|3$-type morphisms [13]. In such a framework generalized Hopf algebras can be defined by requiring the existence of an antipodal morphism (see also [14]). From the point of view of our initial non-commutative geometric motivations, such objects look probably too general to be interpretable as quantum groups.

Geometrical features of standard quantum groups that are mentioned in the introduction show up in certain special situations only. One can build a general theory of bundles and classifying spaces for quantum groups, and these objects can be very far away from any classical structure [6, 8].

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