INTRODUCTION TO QUANTUM PRINCIPAL BUNDLES

MICHO ĐURĐEVICH

1. Introduction

In diversity of mathematical concepts and theories a fundamental role is played by those giving a unified treatment of different and at a first sight mutually independent circles of problems.

As far as classical differential geometry is concerned, such a fundamental role is given to the theory of principal bundles. Various basic concepts of theoretical physics are also naturally expressible in the language of principal bundles. Classical gauge theory and general relativity theory are paradigmic examples.

But classical geometry is just a very special case of a much deeper *quantum geometry.*

So it is natural to ask what would be the analogs of principal bundles in quantum geometry. And it is reasonable to expect that such *quantum principal bundles* would play a similar fundamental role in quantum geometry, as is the role of classical principal bundles in classical geometry.

A general theory of quantum principal bundles, where quantum groups play the role of structure groups and general quantum spaces play the role of base manifolds, has been developed in [3, 4]. A very brief exposition (without proofs) can be found in [2].

Here we shall discuss basic ideas of the theory of quantum principal bundles, trying to speak informally and paying a special attention to interesting purely quantum phenomenas appearing in the game.

It is not difficult to incorporate the basic geometrical idea of a principal bundle, into the noncommutative context. Let G be a compact matrix quantum group [20], represented by a Hopf *-algebra $\mathcal{A}$ (here, $\mathcal{A}$ plays the role of polynomial functions over G). Let us consider a quantum space P, represented by a *-algebra $\mathcal{B}$. Let us assume that G acts on P on the right. This means that the appropriate *-homomorphism $F:\mathcal{B}\rightarrow\mathcal{B}\otimes\mathcal{A}$ is given (giving a coaction of $\mathcal{A}$ on $\mathcal{B}$). The main requirement is that the diagram

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{F} & \mathcal{B}\otimes\mathcal{A} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\
\mathcal{B}\otimes\mathcal{A} & \xrightarrow[F\otimes\mathrm{id}]{} & \mathcal{B}\otimes\mathcal{A}\otimes\mathcal{A}
\end{array}$$

is commutative. Here $\phi:\mathcal{A}\rightarrow\mathcal{A}\otimes\mathcal{A}$ is the coproduct map in $\mathcal{A}$. Following classical theory, we would like to express the idea that the action of G is 'free'. It turns out that the freeness condition is expressible as the property that for every $a \in \mathcal{A}$ there exist elements $q_k, b_k \in \mathcal{B}$ such that

$$\sum_k q_k F(b_k) = 1 \otimes a.$$

Now having the action of G on P, we can define the base manifold M as the corresponding 'orbit space'. At the dual level, this means that the *-algebra $\mathcal{V}$ representing M is just the fixed point algebra for the action F. More precisely,

$$\mathcal{V} = \{b \in \mathcal{B} | F(b) = b \otimes 1\}$$

Geometrically, the idea is that smooth functions on M are just smooth functions on P, constant along the action orbits.

In such a way, we arrive to *quantum principal bundles.*

2. Differential Calculus

A careful analysis shows that the most appropriate approach to the foundations of differential calculus on quantum principal bundles is the axiomatic one. Actually, this becomes transparent already in the foundations of differential calculus on quantum groups [21]. What we have to do is to introduce axiomatically a class of graded-differential *-algebras, representing differential forms over a given quantum principal bundle P.

But before introducing differential calculus on P, it is necessary to introduce differential calculus on the structure group G. Following [21] let us assume that Γ is a bicovariant *-calculus over G, and let $\Gamma^\wedge$ be an appropriate graded-differential *-algebra built over Γ (for example we can assume that $\Gamma^\wedge$ is the braided-exterior algebra of [21] or the universal envelope described in [2]-Appendix B).

The elements of Γ play the role of first-order differential forms over G and the elements of $\Gamma^\wedge$ are interpreted as all possible differential forms.

We shall assume that the calculus on P is based on an arbitrary graded differential *-algebra $\Omega(P)$ satisfying the following three conditions:

(i) We have $\Omega(P)^0 = \mathcal{B}$. This means that 0th order forms are just functions on the bundle. Furthermore, the algebra $\mathcal{B}$ generates the differential algebra $\Omega(P)$. In other words, the spaces $\Omega(P)^n$ are linearly spanned by the elements of the form $w = b_0 d(b_1) \dots d(b_n)$ where $b_i \in \mathcal{B}$ and $d : \Omega(P) \to \Omega(P)$ is the differential. This is a kind of a minimality condition of $\Omega(P)$ and it ensures uniqueness of many entities associated to the calculus.

(ii) The action map F admits (necessarily unique, grade-preserving and hermitian) extension $\widehat{F} : \Omega(P) \to \Omega(P) \mathbin{\widehat{\otimes}} \Gamma^\wedge$, which is a differential algebra homomorphism.

All the above conditions are satisfied in the classical theory, where all the algebras are commutative and we play with the standard differential forms. In classical geometry differential calculus is considered as something intrinsically associated to the space—and consequently only the classical calculus is considered. The situation is very different in non-commutative geometry. For a given quantum principal bundle P we will have a variety of different calculi $\Omega(P)$. In particular it is possible to construct a *non-classical* differential calculus over a *classical* principal bundle P.

Such quantum-type differential structures play a very interesting role in the study of characteristic classes and topological properties of classical spaces (see below). A general phenomena is that in quantum geometry there exist no a unique and distinguished way of constructing differential calculus on quantum spaces. Differential calculus is *context-dependent*.

It is always possible to construct a differential algebra satisfying the above listed properties—for example as a trivial solution we can define $\Omega(P)$ as the universal differential envelope of $\mathcal{B}$.

Let us assume that a differential calculus $\Omega(P)$ is fixed. Then we can naturally associate to it two important algebras: The first one is the graded *-algebra of *quantum horizontal forms* defined by

$$\mathfrak{hor}(P)=\widehat{F}^{-1}[\Omega(P)\otimes\mathcal{A}].$$

The second algebra is a graded-differential *-algebra $\Omega(M)$ representing the calculus on the base M. It is defined as the $\widehat{F}$-fixed point subalgebra of $\Omega(P)$. It is a subalgebra of $\mathfrak{hor}(P)$. It is easy to see [4] that the map $\widehat{F}$ restricted to horizontal forms gives a map $F^\wedge:\mathfrak{hor}(P)\rightarrow\mathfrak{hor}(P)\otimes\mathcal{A}$, the right action of the structure group G on horizontal forms.

3. The Formalism of Connections

The fundamental concept of connections on a quantum principal bundle is defined as follows. Let Γ_{inv} be the space of left-invariant elements [21] of the first-order calculus Γ. This space is the analog of the dual space of the Lie algebra of the structure group G. A connection on P is every first-order hermitian linear map $\omega:\Gamma_{inv}\rightarrow\Omega(P)$ satisfying

$$\widehat{F}[\omega(\theta)]=\sum_k\omega(\vartheta_k)\otimes c_k+1\otimes\vartheta,$$

where $\sum_k\vartheta_k\otimes c_k=\mathrm{ad}(\vartheta)$ and $\mathrm{ad}:\Gamma_{inv}\rightarrow\Gamma_{inv}\otimes\mathcal{A}$ is the corresponding *quantum adjoint action* of G.

The first summand in the above formula corresponds to the pseudotensoriality property of connections. The second summand plays the role of the classical requirement that connections map fundamental vector fields into their generators [17]. It can be shown [4] that every quantum principal bundle admits a connection.

In classical geometry, fixing a connection means that we can speak about vertical differential forms (relative to a given connection)—horizontality is an intrinsic property but verticality is connection-dependent. Furthermore, the whole algebra of differential forms splits into the tensor product of horizontal and vertical forms.

A similar situation holds in quantum theory. As explained in [4], every connection ω on a quantum principal bundle P naturally induces a horizontal-vertical splitting of the form

$$m_\omega:\Omega(P)\leftrightarrow\mathfrak{hor}(P)\otimes\Gamma_{inv}^\wedge$$

where $\Gamma_{inv}^\wedge$ is the algebra of left-invariant forms on the group (the left-invariant part of $\Gamma^\wedge$, equivalently the subalgebra of $\Gamma^\wedge$ generated by Γ_{inv}). In contrast to the classical case however, the map m_ω is not multiplicative, it is only left $\mathfrak{hor}(P)$-linear.

With the help of the decomposition m_ω, we can now define the *horizontal projection operator* $h_\omega : \Omega(P) \to \mathfrak{hor}(P)$ simply by annihilating the vertical components of differential forms. And having the operator of horizontal projection, we can define the covariant derivative

$$D_\omega = h_\omega d : \Omega(P) \to \mathfrak{hor}(P)$$

and the curvature tensor

$$R_\omega = D_\omega \omega$$

the same way as in the classical geometry. It can be shown [4] that all basic classical identities with the curvature and covariant derivative have quantum counterparts. In particular, we have a quantum Bianchi identity

$$D_\omega R_\omega = \{\text{purely q-terms1}\}$$

and a generalized Leibniz rule for the covariant derivative

$$D_\omega(\varphi\psi) = D_\omega(\varphi)\psi + (-1)^{\partial\varphi}\varphi D_\omega(\psi) + \{\text{purely q-terms2}\}$$

where $\varphi, \psi \in \mathfrak{hor}(P)$. The above purely quantum terms vanish, if the connection ω is sufficiently compatible with the geometrical structure of the bundle, in the appropriate sense. Such connections are called *regular connections.* By definition, in the classical theory all connections are regular.

It is also possible to prove the first structure equation

$$R_\omega = d\omega - \langle\omega, \omega\rangle,$$

where the brackets $\langle\rangle$ are induced from the appropriate [21, 4] analog of the Lie commutator for G.

4. Quantum Characteristic Classes

We shall now outline the construction of characteristic classes and the Weil homomorphism for quantum principal bundles. A detailed theory is presented in [9, 10]. It turns out that there exist two very different natural ways of incorporating classical Weil theory into the quantum context.

4.1. **Regular Connections**

The first method works for quantum bundles admitting *regular connections* . As we mentioned in the previous section, for such connections the covariant derivative $D_\omega : \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ satisfies the graded Leibniz rule, and the standard form of the Bianchi identity holds.

As explained in [21], there exists a natural *braid operator* $\sigma : \Gamma_{inv}^{\otimes 2} \to \Gamma_{inv}^{\otimes 2}$. This map is a proper replacement for the classical transposition. It intertwines the adjoint action of G on $\Gamma_{inv}^{\otimes 2}$.

Let Σ be the braided-symmetric algebra built over Γ_{inv}. In other words, it is a *-algebra generated by the vector space Γ_{inv} and the braided-symmetricity quadratic relations

$$\mathrm{im}(I - \sigma) \subset \Gamma_{inv} \otimes \Gamma_{inv}.$$

It turns out [4] that the curvature map R_ω admits a unique extension to a *-homomorphism $W_\omega : \Sigma \to \mathfrak{hor}(P)$. This map intertwines the natural adjoint action of G on Σ and the right action $F^\wedge$. In particular, this means that $W_\omega(\Upsilon) \subseteq \Omega(M)$, where $\Upsilon \subseteq \Sigma$ is the subalgebra of adjointly-invariant elements.

It can be shown that the image of W_ω is contained in *closed* elements of $\Omega(M)$. Moreover, if we pass to cohomology classes of $\Omega(M)$, it turns out that such a factorized map W_ω does not depend of the choice of a regular connection ω. In such a way we obtain the *quantum Weil homomorphism* as an intrinsic map

$$W : \Upsilon \to H[\Omega(M)].$$

As we already mentioned, the space Γ_{inv} plays the role of the dual space of the Lie algebra of G. Accordingly, Σ plays the role of the polynomial functions over the Lie algebra of G, and Υ plays the role of the invariant polynomials for G–this is the quantum analog of *universal chracteristic classes.*

4.2. **General Bundles**

In the case of general quantum principal bundles (where we are not interested whether or not the bundle admits regular connections) the regular-case construction does not work and another approach is necessary. The main idea is to define quantum characteristic classes as generated by those *generic algebraic expressions* built from an arbitrary connection ω and its differential $d\omega$, that are closed elements of $\Omega(M)$.

More preciesly, we start from the free differential algebra Ω built over Γ_{inv}. It is possible to introduce a differential version of the adjoint action, as a graded-differential *-homomorphism $\widehat{\mathrm{ad}} : \Omega \to \Omega \mathbin{\widehat{\otimes}} \Gamma^\wedge$. By construction, every connection ω extends uniquely to a graded-differential *-homomorphism $\widehat{\omega} : \Omega \to \Omega(P)$. This homomorphism intertwines the maps $\widehat{\mathrm{ad}}$ and $\widehat{F}$, and hence it maps the $\widehat{\mathrm{ad}}$-invariants $I(\Omega)$ into $\Omega(M)$. The algebra $I(\Omega)$ is a graded-differential *-subalgebra of Ω and the restricted map $\widehat{\omega} : I(\Omega) \to \Omega(M)$ is a graded-differential *-homomorphim–so we can pass to cohomology classes. It turns out that the cohomology map does not depend of the choice of a connection ω, and we arrive to an intrinsic *-homomorphism $W : H[I(\Omega)] \to H[\Omega(M)]$.

In this context, the algebra $H[I(\Omega)]$ plays the role of universal characteristic classes for G. Various very interesting purely quantum phenomenas appear. At first, there exist a diversity of quantum principal bundles with a very nontrivial topological structure, and in many cases without any 'classical limit'. In particular, there exist examples of quantum principal bundles with non-trivial *odd-dimensional* characteristic classes. This is impossible in classical geometry, where all characteristic classes are expressed via the curvature tensor (it is also impossible in the quantum-regular case).

Secondly, it is very interesting to consider quantum bundles with classical structure groups over classical manifolds, and to analyze their characteristic classes. Since we have an additional freedom in constructing a differential calculus over the bundle and the group, new cohomology classes (that are not interpretable as characteristic classes in the classical sense) are now included in the framework of quantum characteristic classes.

It is also very interesting to analyze relations between regular and general universal characteristic classes. In general, two algebras of universal characteristic classes will be different. The difference between them is encoded in the algebraic properties of the braid operator $\sigma : \Gamma_{inv}^{\otimes 2} \to \Gamma_{inv}^{\otimes 2}$.

5. Quantum Classifying Spaces

Let us assume that G is a classical Lie group. The classification problem of classical principal G-bundles is solved by constructing the classifying space B_G, together with a principal G-bundle E_G over B_G. The pair (B_G, E_G) has the following universal property: Every classical principal G-bundle P over a given classical space M can be obtained as a pull back of the bundle E_G, via an appropriately chosen map $f : M \to B_G$. Characteristic classes of P can be obtained as pull-backs of the cohomology classes of B_G, via the classification maps.

Moreover, *isomorphism classes* of principal G-bundles over M are in a natural one-to-one correspondence with homotopy classes of continuous maps from M to B_G.

The construction of classifying spaces can be incorporated in the quantum context [8]. Starting from an arbitrary compact quantum group G, we can construct a quantum space $\widetilde{B}_G$ and a quantum principal G-bundle $\widetilde{E}_G$ over $\widetilde{B}_G$, such that every quantum principal bundle P over a quantum space M can be obtained as a pull back of the bundle $\widetilde{E}_G$, via a classifying map from M to $\widetilde{B}_G$.

A quantum version of the classification theorem is [8] that there exists a natural bijection between *homotopic classes* of quantum principal G-bundles P over M and homotopic classes of classifying maps from M to $\widetilde{B}_G$. Furthermore, all quantum characteristic classes of P can be obtained from the cohomology classes of $\widetilde{B}_G$ via the corresponding pull-backs, as in the classical case (however it is necessary to take care of differential calculi over various bundles figuring in the game).

Let us observe the difference in the formulations of classical and quantum classification theorems: In the quantum case we have homotopy classes of quantum bundles P, and in the classical case we have isomorphism classes of classical bundles P. The explanation is that in classical geometry homotopic bundles over the same space (and with the same structure group) are always isomorphic. This is not true in the quantum case. For instance, deformation quantization theory gives us examples of homotopic but not isomorphic quantum spaces and bundles.

In particular, a given classical bundle could be homotopically equivalent to a truly quantum bundle (the base and the structure group remain the same–classical). Furthermore, there exist quantum bundles over a classical manifold M and with the classical structure group G that are not homotopically equivalent to any classical bundle. This means that the classification problem, for classical Lie group G and a classical base manifold M, is essentially different depending on the context—whether we are considering the classification within the framework of classical or quantum geometry. Indeed, the quantum classifying space $\widetilde{B}_G$ associated to a classical Lie group G is still an intrinsically quantum object–described by a highly non-commutative algebra. The classical classifying space B_G can be viewed as *the classical part* of $\widetilde{B}_G$, consisting of all the points of $\widetilde{B}_G$.

6. Quantum Frame Bundles

A very interesting class of quantum principal bundles is given by *frame bundles.* They provide a nice working framework to incorporate various geometrical structures (including Riemannian, symplectic, complex and spin structures) into the quantum context. The main idea is to define quantum frame bundles axiomatically, as in the classical geometry [17], starting from the idea of cannonical *coordinate first-order horizontal forms.* In classical geometry, the algebra of horizontal forms is realizable as the tensor product of the functions on the bundle with the exterior algebra associated to the space of coordinate first-order forms. This property can be used [11] as a starting point in defining quantum frame bundles. In quantum case the space of horizontal forms is defined as

$$\mathfrak{hor}(P) = B \otimes \mathbb{V}^{\wedge},$$

where $\mathbb{V}$ is a vector space defining quantum first-order horizontal forms, and $\mathbb{V}^{\wedge}$ is a braided exterior algebra built over $\mathbb{V}$, relative to the appropriate braid operator $\tau : \mathbb{V}^{\otimes 2} \to \mathbb{V}^{\otimes 2}$.

This braid operator comes from the theory of bicovariant bimodules [21]–because it is assumed that $\mathbb{V}$ is a left-invariant part of a bicovariant *-bimodule Ψ over G. This also implies that G is naturally acting by a *-homomorphism $\chi : \mathbb{V}^{\wedge} \to \mathbb{V}^{\wedge} \otimes \mathcal{A}$. Taking a product of the actions F and χ we can define the action

$$F^{\wedge} : \mathfrak{hor}(P) \to \mathfrak{hor}(P) \otimes \mathcal{A}.$$

It is important to mention that the algebra structure on $\mathfrak{hor}(P)$ is not a simple tensor product, but the appropriate *cross-product* of $\mathcal{B}$ and $\mathbb{V}^{\wedge}$, so that $F^{\wedge}$ is a *-homomorphism. To complete the geometrical picture of a quantum frame bundle, it is necessary to postulate the existence of the appropriate differential calculus on the *-algebra $\Omega(M)$ of $F^{\wedge}$-invariant elements of $\mathfrak{hor}(P)$.

Quantum frame bundles also provide a class of examples of principal bundles where it is possible to built an intrinsic differential caclulus $\Omega(P)$ on the bundle, applying a general constructive approach to differential calculi, developed in [7]. In the framework of such a calculus, it is possible to incorporate into the quantum context the entire formalism of torsion operators, and to generalize various important constructions of clasical Riemanian/spin geometry–including the study of Levi-Civita connections and constructions of a quantum Dirac operator [11, 12].

7. Associated Vector Bundles

Starting from a quantum principal bundle P it is possible to define the concept of the associated vector bundle, essentially the same way as in the classical theory. In the framework of the formalism, the associated bundles appear as certain $\mathcal{V}$-bimodules. More precisely, to every representation u of G in a finite-dimensional vector space V, we can associate a $\mathcal{V}$-bimodule $\mathcal{E}_u$ consisiting of all intertwiners $f : V \to \mathcal{B}$ between the representation $u : V \to V \otimes \mathcal{A}$ and the action F. In other words, the maps f are such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ u\ } & V\otimes\mathcal{A} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\otimes\mathrm{id}} \\ \mathcal{B} & \xrightarrow[F]{} & \mathcal{B}\otimes\mathcal{A} \end{array}$$

is commutative.

From the point of view of the classical analogy, the elements of $\mathcal{E}_u$ play the role of smooth sections of the actual associated vector bundle.

It turns out that the bimodules $\mathcal{E}_u$ are finite and projective on both sides. The association $u \to \mathcal{E}_u$ preserves the direct sums, products and conjugation operation in the category of representations of G. In other words, we have

$$\mathcal{E}_{u\oplus v} = \mathcal{E}_u \oplus \mathcal{E}_v \qquad \mathcal{E}_{u\times v} = \mathcal{E}_u \otimes_{\mathcal{V}} \mathcal{E}_v \qquad \mathcal{E}_{\bar{u}} = \bar{\mathcal{E}}_u.$$

It is important to point out a difference between this definition of vector bundles, and the definition of vector bundles as finite projective one-sided modules [1] over the base space algebra. Our definition is intrinsically connected with quantum principal bundles. Indeed, it can be shown that the system of all bimodules $\mathcal{E}_u$ contains the *complete information* about the initial quantum principal bundle P, and accordingly it is possible to reconstruct the bundle starting from the system of associated bundles [6]. This construction is based on the quantum version [22] of classical Tannaka-Krein duality theory.

Cohomological invariants of the bundle are naturally expressible in terms of the system of associated bundles. Accordingly, it is possible to introduce the appropriate K-theory and the Chern character. These entities differ from the standard K-theory and the Chern character, defined in the framework of cyclic cohomology theory [1, 16]. However both constructions can be viewed as special cases of KK-theory [15].

8. Gauge Transformations

At the classical level gauge transformations can be defined as *vertical automorphisms* of a given principal bundle P. They can also be defined as the smooth sections of the adjoint[=gauge] bundle $\mathrm{Ad}(P)$, associated to P. The fibers of this gauge bundle are Lie groups isomorphic to the structure group G, and there exists a natural *fiberwise action* $\gamma : \mathrm{Ad}(P) \times_M P \to P$. of $\mathrm{Ad}(P)$ on P. Actually, *the whole information* about gauge transormation is incorporated in the gauge bundle (and its action on P).

Interestingly, the above two approaches to gauge transformations are essentially different at the quantum level. Let us briefly discuss both of them.

8.1. **Gauge transformations as vertical automorphisms**

This is a straightforward generalization of the classical construction. Starting from a given quantum principal bundle P, we define the group of gauge transformations as consisting of the $\mathcal{V}$-linear *-automorphisms $\psi : \mathcal{B} \to \mathcal{B}$ intertwining the action $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$. At the geometrical level, this corresponds to vertical automorphisms of P.

Such a definition of gauge transformations gives surprising results. This is because quantum bundles possess much more 'aprioricallyy given' geometrical structure than the classical ones, and by definition gauge transformations have to preserve all this structure. This gives additional restrictions to possible candidates for gauge transformations. To illustrate this phenomena, let us consider locally-trivial quantum bundles (with a quantum structure group G) over a classical smooth manifold M. The classification of such bundles is reduced [3] to the classification of the classical G_{cl}-bundles over M, where G_{cl} is the *classical part* of G—consisting of the points of G. In fact, the classical bundle that corresponds to a quantum bundle P is simply the classical part P_{cl} of P.

The correspondence $P \leftrightarrow P_{cl}$ has a simple geometrical explanation. Each truly quantum group G is inherently inhomogeneous, because it always possesses a nontrivial classical part G_{cl} consisting of points of G and a nontrivial quantum part, imaginable as the 'complement' to G_{cl} in G. It is clear that 'quantum transition functions' for P, being diffeomorphisms at the level of spaces, preserve this intrinsic decomposition. As a result, because of the right covariance, transition functions are completely determined by their 'restrictions' on G_{cl}—and these restrictions give us a G_{cl}-cocycle determining the classical bundle P_{cl}.

A similar argumentation leads to the conclusion that gauge transformations of P *coincide* with gauge transformations of its classical part P_{cl}.

Another surprising phenomena appear at the level of differential calculus. It is natural to consider differential calculi over P that are invariant under all gauge transformations. We can also consider the calculi that are locally trivial (for a classical base M). Both assumptions are automatically fulfilled in classical geometry. In the quantum case, they lead to the class of calculi that are locally trivialized *whenever* the bundle is locally trivialized. It turns out [3] that this property gives very strong restrictions on the possible calculus on the quantum structure group G. Moreover, there always exists the *minimal admissible* calculus on G and P.

In classical geometry, this minimal admissible calculus is precisely the classical calculus on the structure group and the bundle. In the quantum case, however, some very surprising things appear. For example, if we assume G is the quantum $SU(2)$ group [19] and if the deformation parameter $\mu \in (-1, 1)$ then the minimal admissible calculus over this group is infinite-dimensional with the space Γ_{inv} naturally identified with the algebra of functions over a quantum sphere [18]. The classical part of the quantum $SU(2)$ group is $U(1)$. In the classical case ($\mu = 1$) the minimal admissible calculus is just the classical 3-dimensional one, and the classical part of the group is the whole group itself.

8.2. **The quantum adjoint bundle**

The second natural approach in developing quantum gauge transformations consists in constructing *quantum gauge bundles.* As already mentioned, in classical geometry the whole information about gauge transformations is contained in the adjoint gauge bundle $Ad(P)$ and its action on P. This action is a smooth bundle map $\gamma : Ad(P) \times_M P \to P$. In quantum case, it is possible to construct a quantum analog of the gauge bundle [5, 13], represented by a *-algebra $\mathcal{C}$ equipped with an inclusion of $\mathcal{V}$ into $\mathcal{C}$, representing the fibering over M. All this goes together with a *-homomorphism $\Delta : \mathcal{B} \to \mathcal{C} \otimes_{\mathcal{V}} \mathcal{B}$, corresponding to the bundle action γ.

The quantum gauge bundle is a very interesting geometrical object, in particular because its properties are connected with an intrinsic *braid structure* that exists on every quantum principal bundle [13]. Moreover, the algebra $\mathcal{C}$ possesses a canonical *braided quantum group* [14] structure over $\mathcal{V}$, represented by a natural coproduct map

$$\phi_M : \mathcal{C} \to \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}.$$

References

[1] Connes A: *Noncommutative Geometry,* Academic Press (1994)
[2] Đurđevich M: *Quantum Principal Bundles,* Proceedings of XXII*th* International Conference on Differential-Geometric Methods in Theoretical Physics, Ixtapa-Zijuatanejo, Mexico (1993)
[3] Đurđevich M: *Geometry of Quantum Principal Bundles I*, Commun Math Phys 175 (3) 457–521 (1996)
[4] Đurđevich M: *Geometry of Quantum Principal Bundles II,* Rev Math Phys 9 (5) 531–607 (1997)
[5] Đurđevich M: *Quantum Principal Bundles and Corresponding Gauge Theories,* J Phys A Math Gen 30 2027–2054 (1997)
[6] Đurđevich M: *Quantum Principal Bundles and Tannaka-Krein Duality Theory,* Rep Math Phys 38 (3) 313–324 (1996)
[7] Đurđevich M: *Differential Structures on Quantum Principal Bundles,* Rep Math Phys 41 (1) 91–115 (1998)
[8] Đurđevich M: *Quantum Classifying Spaces and Universal Quantum Characteristic Classes,* Banach Center Publications, 40 315–327 (1997)
[9] Đurđevich M: *Quantum Principal Bundles and Their Characteristic Classes,* Banach Center Publications, 40 303–313 (1997)
[10] Đurđevich M: *Characteristic Classes of Quantum Principal Bundles,* Preprint, Institute of Mathematics, UNAM
[11] Đurđevich M: *General Frame Structures on Quantum Principal Bundles,* Preprint, Institute of Mathematics, UNAM
[12] Đurđevich M: *Classical Spinor Structures on Quantum Spaces,* in: Clifford Algebras and Spinor Structures, Kluwer, 365–377 (1995)
[13] Đurđevich M: *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles,* Preprint, Institute of Mathematics, UNAM
[14] Đurđevich M: *Generalized Braided Quantum Groups,* Isr Jour Math 98 329–348 (1997)
[15] Jensen K K, Thomsen K: *Elements of KK-Theory,* Birkhäuser, Boston (1991)
[16] Kastler D: *Cyclic Cohomology Within Differential Envelope,* Hermann, Paris (1988)
[17] Kobayashi S, Nomizu K: *Foundations of Differential Geometry,* Interscience Publ, New York, London (1963)
[18] Podles P: *Quantum Spheres,* Lett Math Phys 14 193–202 (1987)
[19] Woronowicz S L: *Twisted SU(2) group; An example of a noncommutative differential calculus,* RIMS, Kyoto University 23, 117–181 (1987)
[20] Woronowicz S L: *Compact Matrix Pseudogroups,* Commun Math Phys 111, 613–665 (1987)
[21] Woronowicz S L: *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups),* Commun Math Phys 122, 125–170 (1989)
[22] Woronowicz S L: *Tannaka-Krein Duality for Compact Matrix Pseudogroups; Twisted SU(n) groups,* Invent Math 93, 35–76 (1988)

Instituto de Matematicas, UNAM, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, México DF, CP 04510, MEXICO

E-mail address: micho@matem.unam.mx
http://www.matem.unam.mx/~micho