Almost isomorphisms of Markov shifts

(joint work with Mike Boyle and Jérôme Buzzi)

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Universidad Nacional Autónoma de México

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Contents

Countable state Markov shifts

A Markov shift is determined by a countable directed graph

$$G = (\mathcal{V}, \mathcal{E})$$

$\mathcal{V}$ set of vertices

$\mathcal{E}$ set of edges

$$\Sigma = \{x = (x_n) \in \mathcal{E}^{\mathbb{Z}} \mid x_{n+1} \text{ follows } x_n\}$$

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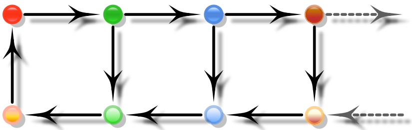

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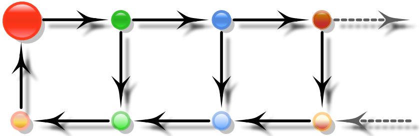

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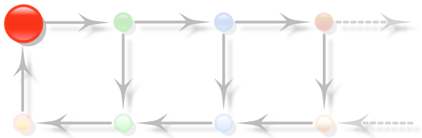

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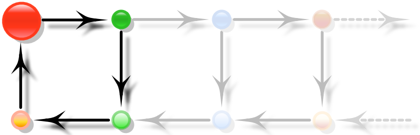

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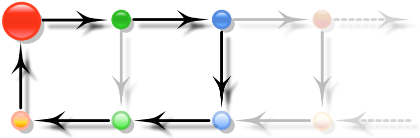

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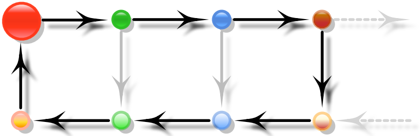

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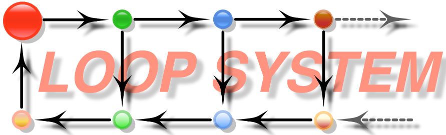

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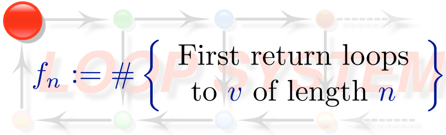


$$f_n := \#\left\{\begin{array}{c}\text{First return loops}\\ \text{to } v \text{ of length } n\end{array}\right\} \Longrightarrow f(z) = \sum_{n=1}^{\infty} f_n z^n$$

Loop shifts

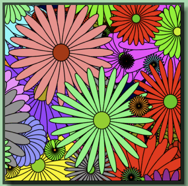
Loop graph

$$f \in \mathbb{Z}_+[[z]], \text{ say } f(z) = \sum_{n=1}^{\infty} f_n z^n$$

- Distinguised vertex v
- f_n first return loops to v
- Every vertex but v lies on a unique loop

Loop shifts

Petal graph



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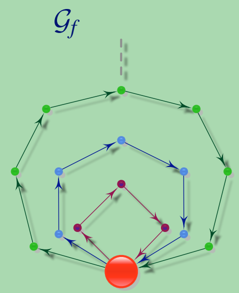

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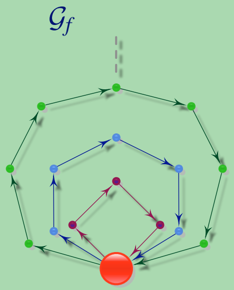

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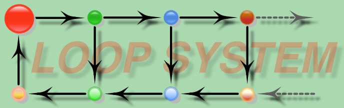

$\Rightarrow f(z) = \sum_{n=1}^{\infty} z^{2n+2}$

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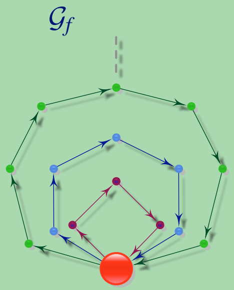

Let the *loop shift* σ_f be $\Sigma(\mathcal{G}_f)$

Recall entropy

Entropy

$$h(\Sigma) = \log(\lambda) = \limsup |t_n|^{1/n}$$

t_n equals the number of loops at an arbitrary vertex

Not necessarily *first return* loops

It is the supremum of the *measure theoretic* entropies over all invariant Borel probabilities (*Gurevich entropy*)

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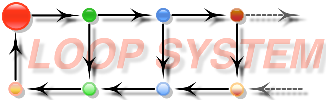

$$\Longrightarrow f(z) = \sum_{n=1}^{\infty} z^{2n+2} = \frac{z^4}{1-z^2} \quad \Rightarrow \quad h = \frac{1}{2}\log\frac{2}{\sqrt{5}-1}$$

Contents

Classification in terms of entropy and loop systems

Σ	Markov shift
$\log(\lambda)$	Entropy
$f(z) = \sum_{n=1}^{\infty} f_n z^n$	Loop system at fixed vertex v

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2. Σ is *recurrent* if $f(1/\lambda) = 1$
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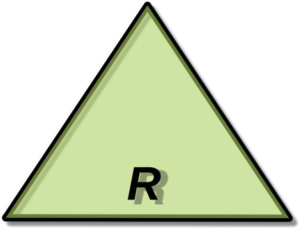

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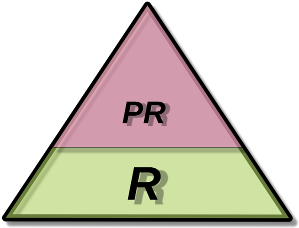

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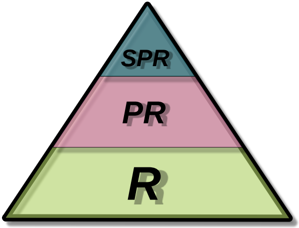

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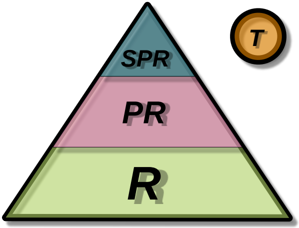

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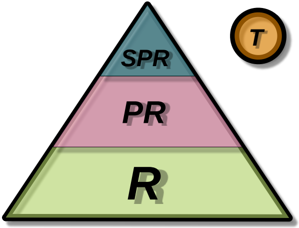

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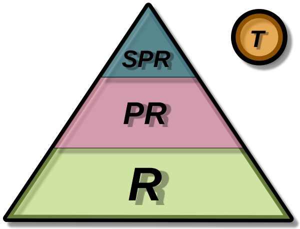

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SPR characterizations

Theorem

TFAE conditions on an irreducible Markov shift Σ

1. Σ is *SPR*
2. Removing an edge strictly lowers entropy [UF-96, S-92, GS-98, R-03]
3. Some (every) *local zeta function* has non-trivial meromorphic extension
4. Σ has a (unique) measure of maximal entropy μ and (Σ, μ) is *exponentially filling*

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▸ EXPONENTIALLY FILLING

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Good finitary isomorphisms

Magic word

$$\varphi \colon (\Sigma_1, \mu) \to (\Sigma_2, \nu)$$

A Σ_2-block W is a *magic word* for φ if

1. **Existence.**
   If $z \in \Sigma_2$ sees W infinitely many times in past and future, then z has a preimage under φ
2. **Uniqueness.**
   If WUW is a Σ_2-block, then for μ-a.e. $x, y \in \Sigma_1$ such that

   $$(\varphi x)[0, |WUW| - 1] = WUW = (\varphi y)[0, |WUW| - 1]$$

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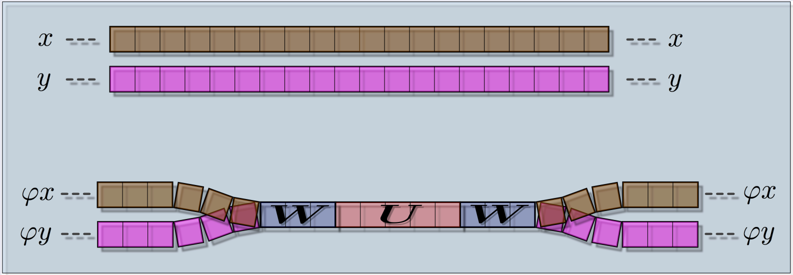

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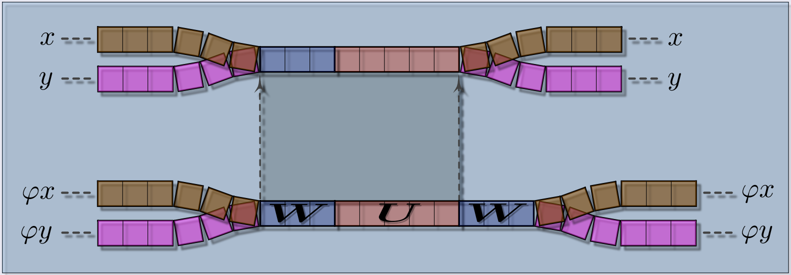

Almost isomorphism

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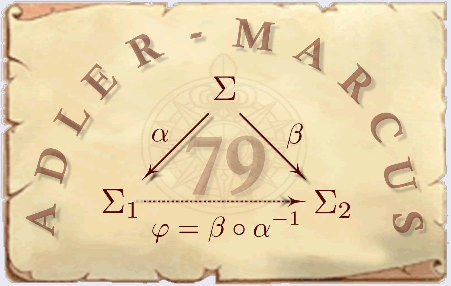

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Almost isomorphism and entropy-conjugacy for *SPR*

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Two SPR irreducible Markov shifts Σ_1 and Σ_2 are almost isomorphic if and only if they are entropy conjugate.

Theorem (MAIN: [BBG-06])

Two SPR irreducible Markov shifts Σ_1 and Σ_2 are almost isomorphic if and only if they have the same entropy and period.

Remark

If Σ_1 is SPR and is almost isomorphic to Σ_2, then Σ_2 is SPR.

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Two SPR irreducible Markov shifts Σ_1 and Σ_2 are almost isomorphic if and only if they have the same entropy and period.

Remark

If Σ_1 is SPR and is almost isomorphic to Σ_2, then Σ_2 is SPR.

Almost isomorphism and entropy-conjugacy for *SPR*

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Given *SPR* irreducible and aperiodic Markov shifts Σ_1 and Σ_2 of equal entropy $\log \lambda$, find an *AI*

- Reduce to loop shifts
- Find a loop shift defined by

$$c \in \mathbb{Z}_+[[z]]$$

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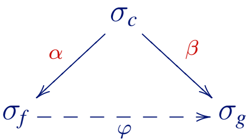

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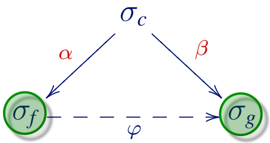

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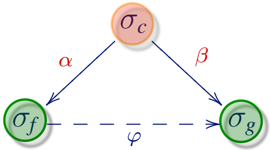

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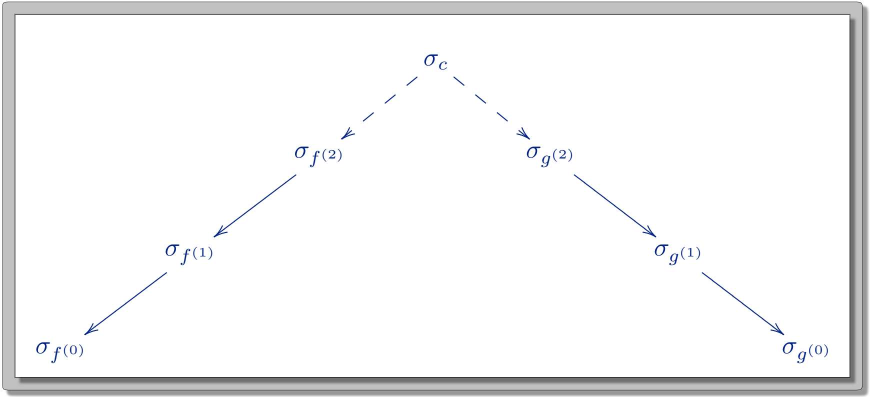

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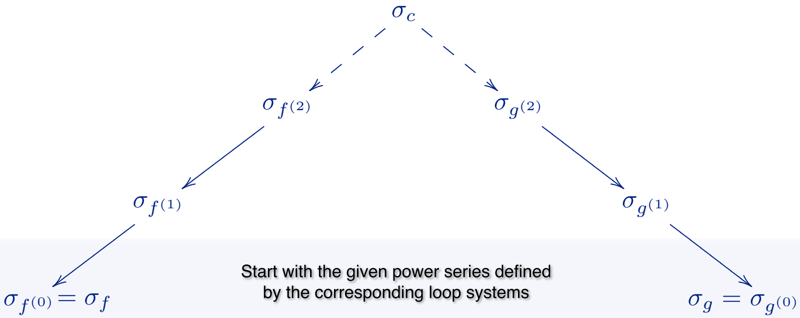
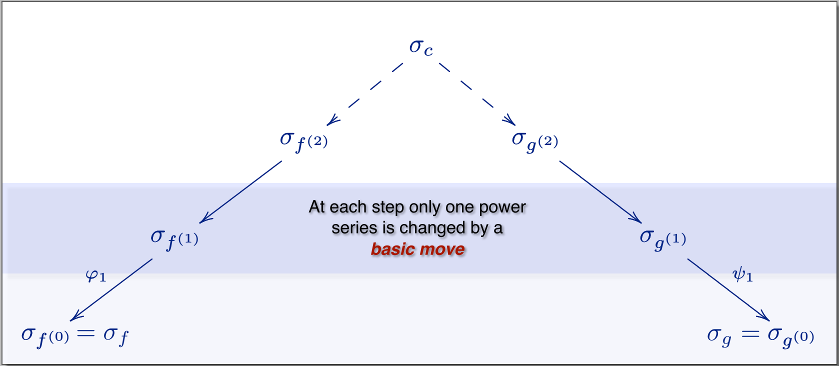


Start with the given power series defined by the corresponding loop systems

An inductive construction



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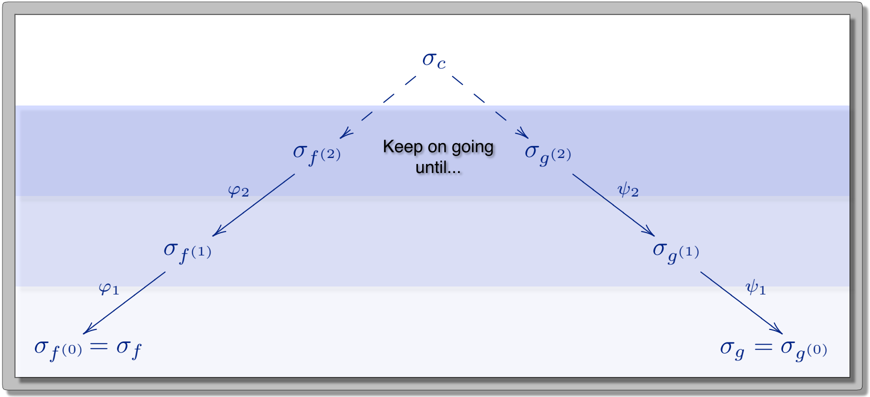

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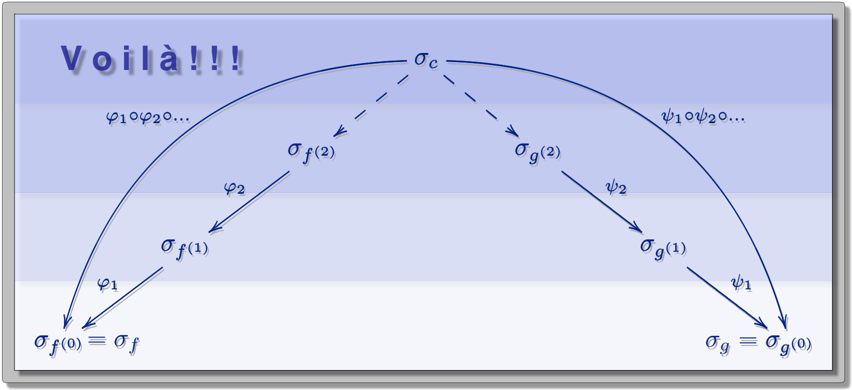

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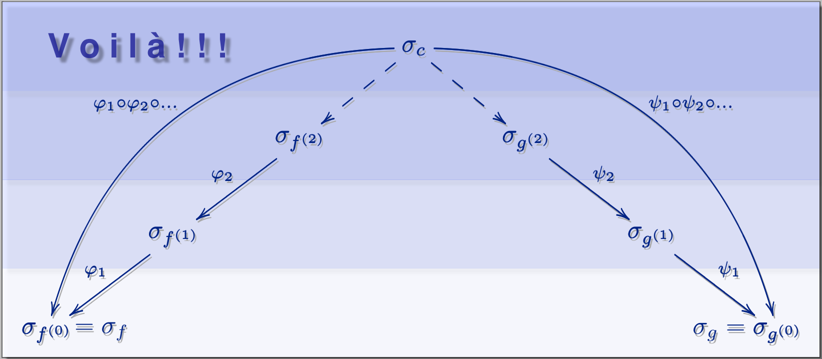

BASIC MOVE

The inductive construction in loops lemma

- Start with $f^{(0)} = f$ and $g^{(0)} = g$
- Suppose that we are given $f^{(k)}$ and $g^{(k)}$
- If $f^{(k)} = g^{(k)}$, let $c = f^{(k)}$ and stop
- Otherwise choose N minimal such that $f_N^{(k)} \neq g_N^{(k)}$
- Suppose $f_N^{(k)} > g_N^{(k)}$, let $g^{(k+1)} = g^{(k)}$ and perform a *basic move* on $f^{(k)}$

$$1 - f^{(k+1)}(z) = \frac{1 - f^{(k)}(z)}{1 - z^N}$$

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REMEMBER

The inductive construction in loops lemma

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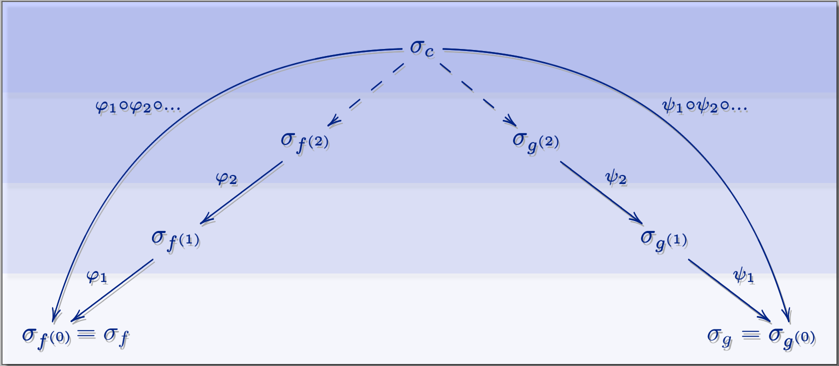
$$1 - f^{(k+1)}(z) = \frac{1 - f^{(k)}(z)}{1 - z^N}$$

$$\lim_{k\to\infty} f^{(k)} = c = \lim_{k\to\infty} g^{(k)}$$

Are the resulting maps good?



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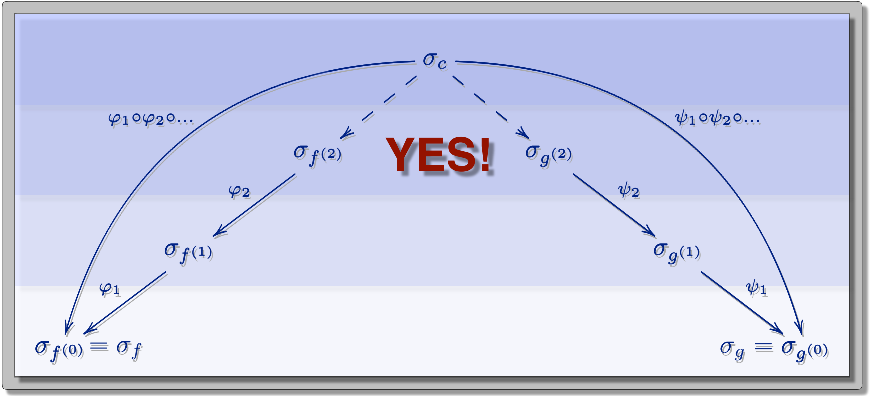

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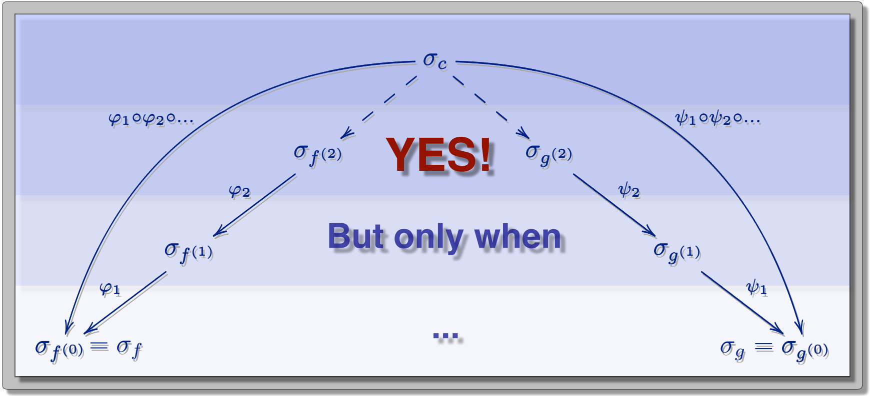

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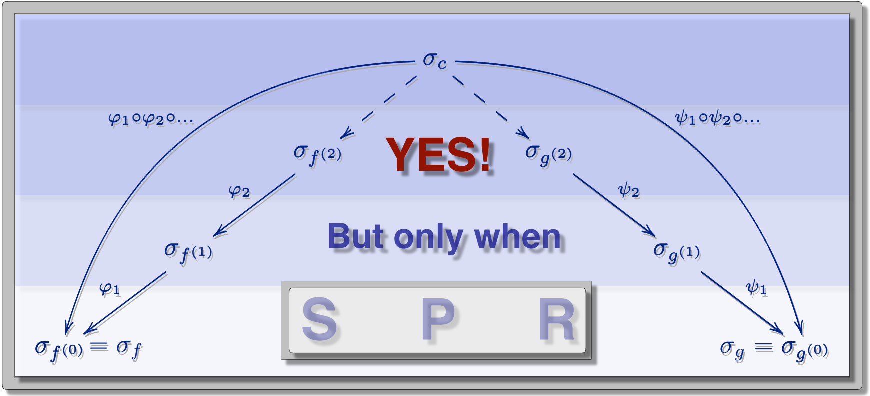

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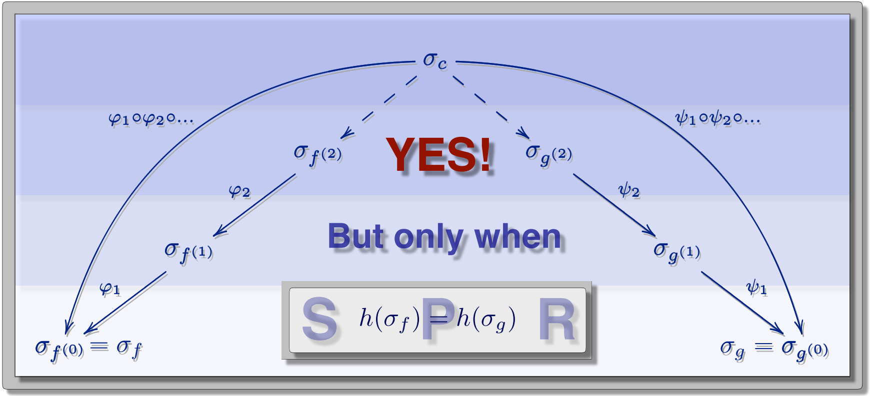

A closer look at the sequence of basic moves

- There is a sequence of nondecreasing integer numbers

$$(r_1, r_2, \dots)$$

with r_n being the length of the loop in turn to be “removed”

- It looks like

$$(\underbrace{1, 1, \dots, 1}_{R_1}, \underbrace{2, 2, \dots, 2}_{R_2}, \dots)$$

- The resulting map possesses a **magic word** if

$$R_n \leq f_n \text{ for } n \geq 1 \quad \text{and} \quad R_n < f_n \text{ for } n = r_1$$

- ...and it is an **entropy conjugacy** if in addition

$$\limsup R_n^{1/n} < \lambda$$

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2. $$\limsup R_n^{1/n} < \lambda$$

Arrange things properly for **magic word**

1

Lemma

We can suppose that σ_f *and* σ_g *are irreducible and aperiodic loop shifts of equal entropy* $\log \lambda > 0$ *and there is* $1 \leq \beta < \lambda$ *such that for sufficiently large* N

1. $|\mathcal{O}_n(\sigma_f)| = |\mathcal{O}_n(\sigma_g)| = 0$ *for all* $n < N$
2. $\min\{f_n, g_n\} \geq \beta^n$ *for all* $n \geq N$

Arrange things properly for entropy conjugacy

2

Lemma

If σ_f *and* σ_g *are strong positive recurrent loop systems of equal entropy* $\log \lambda > 0$*, then for some* $\kappa < \lambda$

$$|\mathcal{O}_n(\sigma_f)| = |\mathcal{O}_n(\sigma_g)| + o(\kappa^n)$$

SPR-ZETA

Contents

Applications to other dynamical systems

Theorem ([BBG-06])

The following measurable dynamical systems have natural extensions which are entropy-conjugate to the disjoint union of finitely many SPR Markov shifts of equal entropy

1. *Subshifts of quasifinite type [Bu-05]*
2. *Piecewise monotonic interval maps with non-zero topological entropy*
3. *The multi-dimensional β-transformations [Bu-97]*
4. *C^∞ smooth entropy-expanding maps*

Beyond...

I. Beyond strong positive recurrence

Main question

Are equal entropy irreducible PR Markov shifts entropy-conjugate?

II. Weights

1. Thermodynamic formalism
2. Markov chains

▸ CHAINS

This is the end...

M E R C I !

Local zeta function

Definition

Let Σ be a Markov shift defined by a graph $G = (\mathcal{V}, \mathcal{E})$.

The local zeta function at $v \in \mathcal{V}$ is

$$\zeta_v(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \#\{\text{Loops through } v \text{ of length } n\}\right)$$

Exponentially recurrent

Definition

(Σ, μ) is *exponentially filling* if for every open set $X \subset \Sigma$ with

$$\mu(X) > 0$$

we have

$$\limsup_{n\to\infty}(\mu\left\{x \in \Sigma \mid x \notin \cup_{k=1}^{n} \sigma^{-k}(X)\right\})^{1/n} < 1$$

Basic move

Lemma

Let σ_f *be a loop shift. Write*

$$f = h + k$$

with $h, k \in \mathbb{Z}_+[[z]]$. *Then*

$$\frac{1-f}{1-k} = 1 - hk^* = 1 - h - hk - hk^2 - hk^3 - \dots$$

and there is an injective one-block code with a magic word

$$\varphi\colon \sigma_{hk^*} \longrightarrow \sigma_f$$

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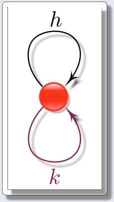

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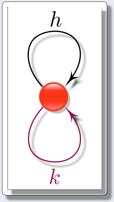

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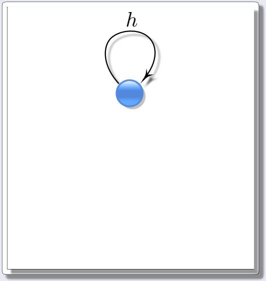

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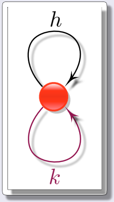

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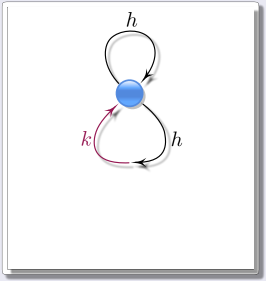

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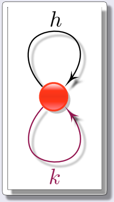

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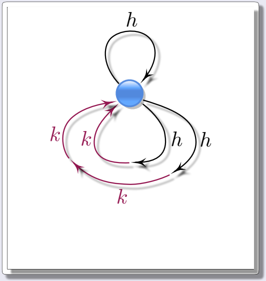

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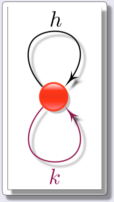

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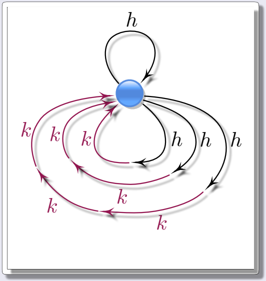

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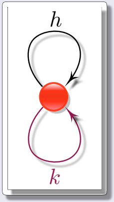

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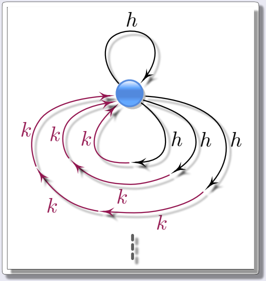
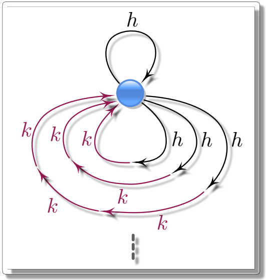


Coding for basic move

$$k^* h = h + kh + k^2 h + k^3 h \dots$$



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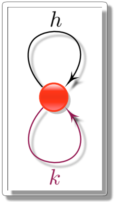

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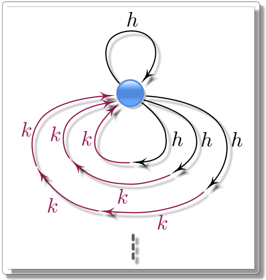

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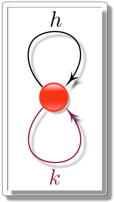

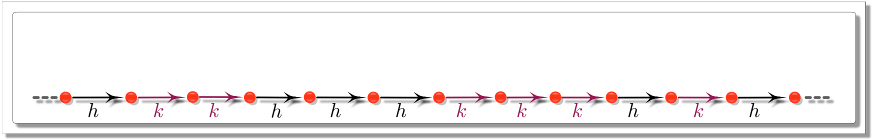

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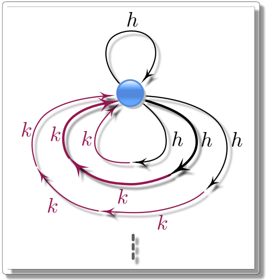

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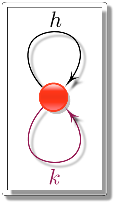

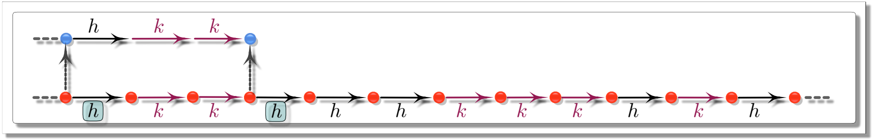

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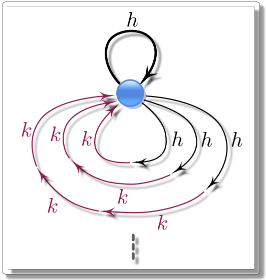

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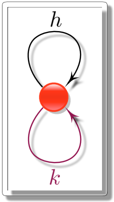

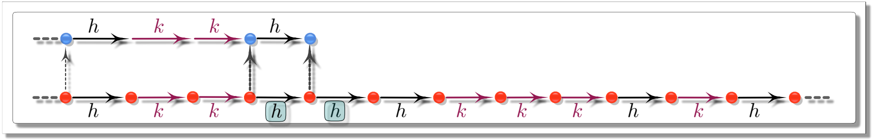

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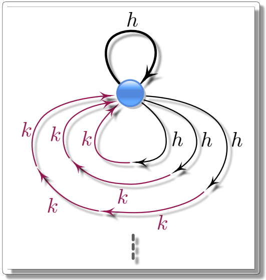
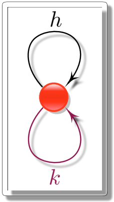
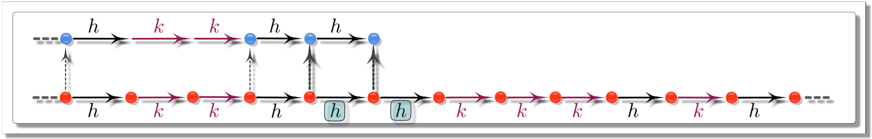


$$f = h + k$$





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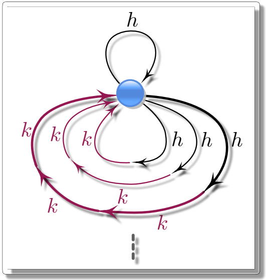

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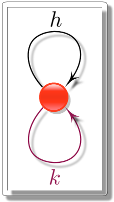

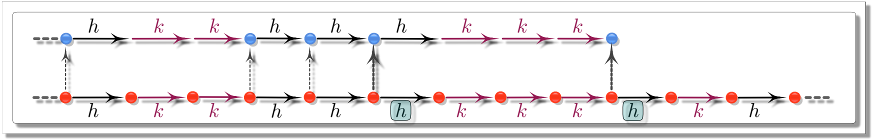

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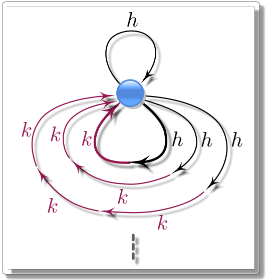

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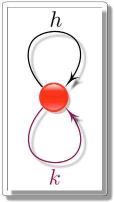

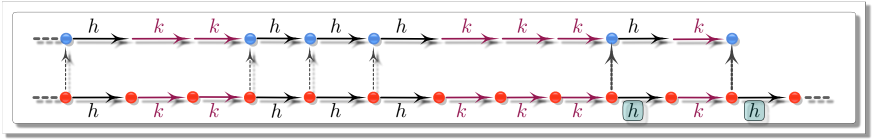

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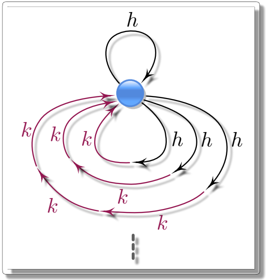

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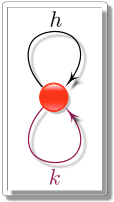

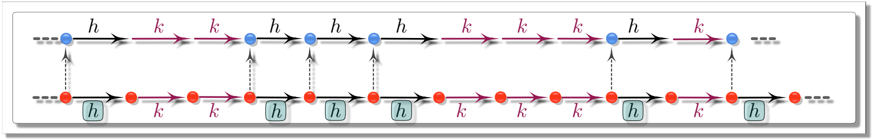

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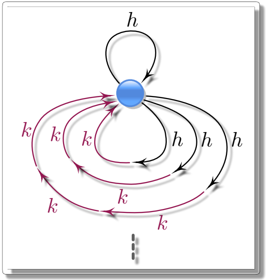

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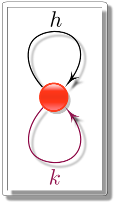
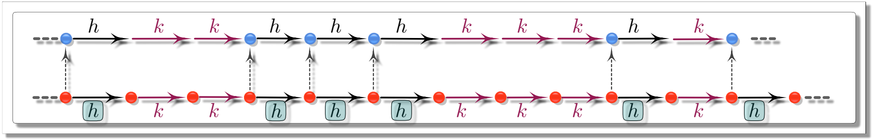


BACK TO INDUCTION



Magic word isomorphisms

Magic word isomorphisms

$$\varphi\colon (\Sigma_1, \mu) \to (\Sigma_2, \nu)$$

φ is a *magic word isomorphism* if both φ and φ^{-1} have magic words

- Magic word isomorphisms of countable state Markov *chains* are classified in [G-03] as compositions of *elementary isomorphisms*
- Structure similar to "positive" algebraic K-theory
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TAMALE

Isomorphisms

Isomorphism

$$\varphi\colon (\Sigma_1, \mu) \to (\Sigma_2, \nu)$$

Full measure subsets

$A \subset \Sigma_1$ and $B \subset \Sigma_2$

Restriction $\varphi|_A : A \to B$ is

- Bijective
- Bimesurable
- Shift-commuting
- Measure preserving

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Finitary

- Homeomorphism

In this case, for μ-a.e. $x \in \Sigma_1$, there exists a minimal $n = n(x)$ such that for μ-a.e. $y \in \Sigma_1$ with

$$x[-n, n] = y[-n, n]$$

we have

$$(\varphi x)_0 = (\varphi y)_0$$

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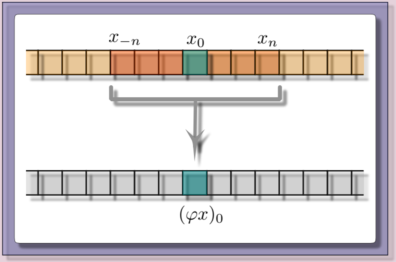

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Finitary

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The *expected coding time* of φ is

$$\int n(x) d\mu(x)$$

The classification of finitary isomorphisms with *finite* expected coding time is *open* (see [P-77]) ▸ POPEY

Classification of magic word isomorphisms

Rough idea of the proof

- Reduce to loop shifts

$$\varphi\colon \sigma_f \longrightarrow \sigma_g$$

- Find suitable magic words

- Decompose g in a *sandwich*-fashion

$$g = l + r + b + h$$

- Find factorization of *basic moves*

$$f = (b + l\,(l+h)^*\,(b+r))\,(r + h\,(l+h)^*\,(b+r))^*$$

END

Entropy-conjugacy

Entropy-negligible sets

Measurable system $S\colon X \to X$

$h(S) = \sup\{h(S,\mu) \mid \mu$ is a S-invariant Borel probability measure$\}$

$N \subset X$ is *entropy negligible* if there is $h < h(S)$ such that

$\mu(N) = 0$ for all ergodic μ for which $h(S,\mu) > h$

Entropy conjugacy

Two measurable systems $S\colon X \to X$ and $T\colon Y \to Y$

Entropy negligible subsets $X_0 \subset X$ and $Y_0 \subset Y$

Bimeasurable bijection $\gamma\colon X \setminus X_0 \to Y \setminus Y_0$

$T \circ \gamma = \gamma \circ S$ for all $x \in X \setminus X_0$

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