Realisation of separoids and a Tverberg-type problem

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Abstract

A *separoid* is a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on pairs of disjoint subsets which is closed as a filter in the natural partial order (i.e., $A \dagger B \preceq C \dagger D \iff A \subseteq C$ and $B \subseteq D$). We discus the Geometric Representation Theorem for separoids: *every separoid* $(S, \dagger)$ *can be represented by a family of (convex) polytopes, and their Radon partitions, in the Euclidean space of dimension* $|S|-1$. Furthermore, we introduce a new kind of separoids' morphisms — called *chromomorphisms*— which allow us to study Tverberg's generalisation (1966) of Radon's theorem (1921) in the context of convex sets. In particular the following Tverberg-type theorem is proved: *Let* S *be a separoid of order* $|S| = (k-1)(\mathrm{d}(S)+1)+1$, *where* $\mathrm{d}(S)$ *denotes the (combinatorial) dimension of* S. *If there exists a monomorphism* $S \to P$ *into a separoid of points in general position in* $\mathbb{E}^d$, *then there exists a chromomorphism* $S \longrightarrow K_k$ *onto the complete separoid of order* k. This theorem is, in a sense, dual to the Hadwiger-type theorem proved by Arocha, Bracho, Montejano, Oliveros & Strausz [Disc. & Comp. Geom., **27**, 2002, 377-385].

Key words: separoids; abstract convexity; Tverberg's theorem; Hadwiger's theorem; chromomorphisms.

1 Introduction

Combinatorial convexity has its origins in the theorems of Helly (1923) [10] and Radon (1921) [14]; because of their simplicity, they have been applied and generalised in several directions (see [5]). The present exposition centres around one generalisation of each: Hadwiger's Transversal Theorem (1957) [9] and Tverberg's Theorem (1966) [19], which generalise Helly's and Radon's theorems, respectively (see [6, 8]).

In the one hand, Hadwiger generalises Helly's theorem in $\mathbb{E}^2$ changing the notion of "non-empty intersection" by the notion of "0-transversal"; then, he finds sufficient conditions, in the spirit of Helly's theorem, for a "1-transversal" to exist. However, he had to add additional conditions; in particular, he uses the notion of a "linear order" to which every "partial" transversal has to be consistent. This was further generalised by Katchalski (1980) [11], Goodman & Pollack (1988) [7] and Arocha, Bracho, Montejano, Oliveros & Strausz (2002) [1].

On the other hand, Tverberg generalises Radon's theorem giving the sufficient condition

$$|\mathcal{P}| \geq (k-1)(d+1)+1,$$

on a family of points $\mathcal{P} \subset I\!E^d$, for a k-partition to exist in such a way that the convex hull of the parts have a point in common. Tverberg's theorem is Radon's for $k = 2$. Observe that Tverberg's theorem is easily extendable to families of convex sets (instead of points, cf. Corollary 4). Furthermore, if we can guarantee the existence of a d-transversal, under suitable conditions, then we can guarantee the existence of a "Tverberg partition".

Using categoric notions of *separoids* [1, 3, 12, 13, 16, 17, 18] we find sufficient conditions —in the spirit of Hadwiger's and Tverberg's theorems— for a family of convex sets to have a k-partition such that the family of convex hulls have the same separation structure as if they have a common point (in [1] this is called a *virtual 0-transversal*).

Observe that the separation structure defined by hyperplanes, of a family of convex sets, is the same if every two have a common point than if all have a common point —no subset of the family can be separated from another. So, in terms of the combinatorial structure of separation —in terms of separoids— it is the same if we cannot separate any pair of singletons than if we have a "Tverberg partition".

The main theorem can be formulated as follows (cf. Theorem 5):

Theorem 1 *Let $\mathcal{S}$ be a family of $(k-1)(d+1)+1$ convex sets in $I\!E^n$ such that every $d+2$ convex sets admits a d-transversal. Suppose that in addition, there is an injective map $\varphi: \mathcal{S} \longrightarrow \mathcal{P}$ into a family of points $\mathcal{P} \subset I\!E^d$ in general position such that, if A is separated from B, subsets of $\mathcal{P}$, then $\varphi^{-1}(A)$ is separated from $\varphi^{-1}(B)$. Then, there exists a k-coloration of $\mathcal{S}$ in such a way that the convex hulls of each pair of colours, have a common point.*

2 Separoids

A *separoid* is a relation $\dagger \subset 2^S \times 2^S$ defined on the family of subsets of a set S with three simple properties: for every $A, B \subseteq S$

$$\begin{array}{cl} \circ & A \dagger B \Longrightarrow B \dagger A, \\ \circ\circ & A \dagger B \Longrightarrow A \cap B = \phi, \\ \circ\circ\circ & A \dagger B \text{ and } B \subset B' \ (\subseteq S \setminus A) \Longrightarrow A \dagger B'. \end{array}$$

The separoid is identified with the set S and is sometimes denoted as a pair $(S, \dagger)$. An element $A \dagger B$ is called a *Radon partition*. The *order* of the separoid is the cardinal $|S|$ and the *size* is half of the Radon partitions $\frac{1}{2}|\dagger|$. If the separoid is finite, the third axiom implies that it is enough to know the *minimal* Radon partitions to reconstruct the separoid; they encode all the structure.

A *separation* $A \mid B$ is a pair of disjoint sets that are not a Radon partition. When talking of a separation $A \mid B$ we often say *"A is separated from B"*. The separoid is *acyclic* if $\phi \mid S$ and the separations with the empty set are considered *trivial*.

The separation relation $|\subset 2^S \times 2^S$ satisfies the following properties

$$\begin{array}{rl} \circ & A \mid B \Longrightarrow B \mid A, \\ \circ\circ & A \mid A \Longrightarrow A = \phi, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \Longrightarrow A' \mid B. \end{array}$$

In other words the relation $|$ is a *symmetric, quasi-anti-reflexive* and *ideal* relation on the subsets of the given set S. It is clear that, in the finite case, it is enough to know *maximal* separations to reconstruct the separoid. The separoid is denoted sometimes as a pair $(S, |)$.

Clearly $\dagger$ and $|$ determine each other. They are related by the following equivalence

$$A \dagger B \iff A \nmid B \quad \text{and} \quad A \cap B = \phi.$$

Finite separoids have also an intrinsic notion of dimension. The *d-dimensional simploid* is the separoid of order $d+1$ and size 0 and it will be denoted by σ^d. Since there are no Radon partitions, every pair of disjoint subsets are separated and then, it can be represented with the vertex set of a simplex —hence the name. The *dimension* of a separoid S is the maximum dimension of its induced simploids

$$\mathrm{d}(S) = \max_{\sigma^d \hookrightarrow S} d.$$

S is called *complete* if for every $i, j \in S$ follows that $i \dagger j$. The complete separoid of order n is denoted by K_n.

Examples:

1. Consider a subset $X \subseteq \mathbb{E}^d$ of the d-dimensional Euclidean space and define the following relation

$$A \dagger B \iff \langle A \rangle \cap \langle B \rangle \neq \phi \quad \text{and} \quad A \cap B = \phi,$$

where $\langle A \rangle$ denotes de convex hull of A. The pair $\mathcal{P} = (X, \dagger)$ is a separoid and will be called a *point separoid*. In fact, the name of separoids arises as an abstraction of the fact that $A \mid B$ is a separation if and only if there exists a hyperplane strictly separating $\langle A \rangle$ from $\langle B \rangle$.

2. Consider a family $\mathcal{F}$ of convex sets in $\mathbb{E}^d$ and define the separoid $S(\mathcal{F})$ as above, that is, two subsets of the family $A, B \subset \mathcal{F}$ are separated $A \mid B$ if there exists a hyperplane that leaves all members of A on one side of it and those of B on the other. If $\mathcal{F}$ is finite and its members are compact, $S(\mathcal{F}) = (\mathcal{F}, |)$ is an acyclic separoid and will be called a *separoid of convex bodies* (see Theorem 2).

3. All acyclic separoids on three elements arise from one of the eight families of convex bodies in Figure 1. Those labelled **a**, **b**, **e** and **h** are the point separoids of order 3; in fact, they come from the four essentially different order types with three elements.

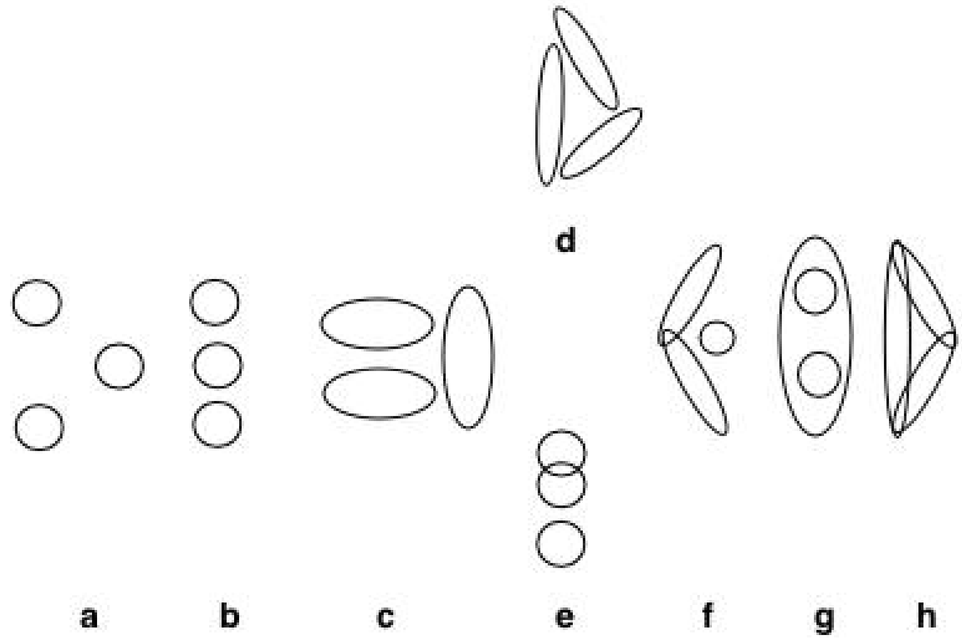

Figure 1. *The acyclic separoids of order 3*

Let S and T be two separoids. A *separoid morphism* is a function $\varphi\colon S \longrightarrow T$ with the property that for all $A, B \subseteq T$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

The *category of separoids* is defined with such morphisms between separoids. Other kind of "maps" —and therefore another category— will be defined in Section 4 (see also [13]). Two separoids are *isomorphic* $S \approx T$ if there is a bijective morphism between them whose inverse function is also a morphism.

More examples:

4. Consider a family of convex sets $\mathcal{F}$, choose a point in each of its elements to construct a point separoid $\mathcal{P}$ and define the obvious bijection $\varphi\colon \mathcal{P} \longrightarrow \mathcal{F}$. This is a morphism since every hyperplane that separates A from B, subsets of $\mathcal{F}$, also separates their respective points $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$. This construction is quite useful and we will say that $\varphi\colon \mathcal{P} \longrightarrow \mathcal{F}$ *is a choice on* $\mathcal{F}$.

5. Consider a family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ and let $\pi\colon \mathbb{R}^d \to \mathbb{R}^e$ be an affine projection ($e \leq d$). The obvious bijection $\hat{\pi}\colon \mathcal{F} \to \pi(\mathcal{F})$ is a morphism between their separoids $S(\mathcal{F}) \longrightarrow S(\pi(\mathcal{F}))$.

6. Consider a family of convex sets $\mathcal{F}$ and give them a coloration $\varsigma\colon \mathcal{F} \twoheadrightarrow \{c_1, \ldots, c_k\}$. If we denote by $\mathcal{F}' = \{\langle \varsigma^{-1}(c_i) \rangle\}$ the convex hulls of the colour classes' family, the obvious map $\mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism. This is a key ingredient in our study of Tverberg's theorem (Section 4).

7. In Figure 1, monomorphisms go from left to right between every pair of separoids. Observe that there is no monomorphism in either directions between the separoids labelled **d** and **e**.

3 The Geometric Representation Theorem

We show now how Example 2 is indeed the more general example; this is, every finite and acyclic separoid is a separoid of convex bodies (for the non-acyclic case, see [3]).

Theorem 2 *Every acyclic separoid of order n can be represented by a family of convex polytopes in the $(n-1)$-dimensional Euclidean space.*

Proof. Let $\dagger$ be an acyclic separoid on $S = \{1, \ldots, n\}$ (i.e., $A \dagger B \Longrightarrow |A||B| > 0$). To each element $i \in S$ and each (minimal Radon) partition $A \dagger B$ such that $i \in A$, we assign a point of $I\!R^n$

$$\rho^i_{A \dagger B} = \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

(where $\{\mathbf{e}_j\}$ denotes the canonical basis) and realise each element $i \in S$ as the convex hull of all such points

$$i \mapsto \mathcal{K}_i = \langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \rangle.$$

Observe that these convex sets are in the $(n-1)$-dimensional affine subspace spanned by the basis.

To see that this family of convex polytopes realises the separoid observe that, for each partition $A \dagger B$, the vertex set of the simplices $\langle \mathbf{e}_a : a \in A \rangle$ and $\langle \mathbf{e}_b : b \in B \rangle$ "moves" —half of the way each— to realise such a partition intersecting one to the other precisely in their baricenter. That is, let $A \dagger B$ be fixed; in order to prove that

$$\langle \mathcal{K}_a : a \in A \rangle \cap \langle \mathcal{K}_b : b \in B \rangle \neq \phi,$$

it is enough to prove that $\langle \rho^a_{A\dagger B} : a \in A \rangle \cap \langle \rho^b_{B\dagger A} : b \in B \rangle \neq \phi$ because $\rho^a_{A\dagger B} \in \mathcal{K}_a$ and therefore $\langle \rho^a_{A\dagger B} : a \in A \rangle \subset \langle \mathcal{K}_a : a \in A \rangle$ (analogously with B).

Now, if we let $\rho : I\!R^n \to I\!R^n$ be the translation

$$\rho(\mathbf{x}) = \mathbf{x} + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

we have that $\rho^a_{A\dagger B} = \rho(\mathbf{e}_a)$ and the baricenter of $\langle \rho^a_{A\dagger B} : a \in A \rangle$ is

$$\frac{1}{|A|}\sum_{a\in A}\rho^a_{A\dagger B} = \frac{1}{|A|}\sum_{a\in A}\rho(\mathbf{e}_a) = \rho\left(\frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right) = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

Analogously, using that $\rho^b_{B\dagger A} = -\rho(-\mathbf{e}_b)$, we have that

$$\frac{1}{|B|}\sum_{b\in B}\rho^b_{B\dagger A} = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

and therefore

$$\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi.$$

On the other hand, given a separation $\alpha \mid \beta$, define the affine extension of the equations

$$\psi_{\alpha|\beta}(\mathbf{e}_j) = \begin{cases} -1 & j \in \alpha \\ 1 & j \in \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \dots, n.$$

Now, it is enough to prove that for every $i \in \alpha$ (resp. β), we have that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ (resp. > 0). For this, observe that, if $i \in \alpha$ then

$$\psi_{\alpha|\beta}\left(\rho^i_{A\dagger B}\right) = \psi_{\alpha|\beta}\left(\mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\Big(\sum_{B\cap\alpha} \mathbf{e}_b + \sum_{B\cap\beta} \mathbf{e}_b\Big) - \frac{1}{|A|}\Big(\sum_{A\cap\alpha} \mathbf{e}_a + \sum_{A\cap\beta} \mathbf{e}_a\Big)\right]\right) =$$

$$= -1 + \frac{(|B\cap\beta| - |B\cap\alpha|)}{2|B|} + \frac{(|A\cap\alpha| - |A\cap\beta|)}{2|A|} \leq 0.$$

Equality holds if and only if $B \cap \beta = B$ and $A \cap \alpha = A$ which leads a contradiction. •

This result allow us to introduce an important invariant of separoids (see [1, 3]): the *geometric dimension* of a separoid S, denoted as $\mathrm{gd}(S)$, is the minimum dimension where it can be represented as a family of convex sets.

4 Tverberg's theorem

In the year of 1966, Tverberg published a generalisation of the classic Radon's theorem which can be written, in separoids terms, as follows

Theorem 3 (Tverberg [19]) *Let $\mathcal{P}$ be a point separoid in $\mathbb{E}^d$. If S is of order*

$$|\mathcal{P}| = (k-1)(d+1) + 1$$

then $\mathcal{P}$ can be divided in k disjoint subsets A_i, $i = 1, \dots, k$ such that there is a point in common into their convex hulls: $\bigcap_{i=1}^k \langle A_i\rangle \neq \phi$.

The proof of this result is far from trivial. The simplest one known by the author is based on Sarkaria's argument [15] and uses Bárány's generalisation [2] of Caratheodory's theorem [4].

Clearly, Theorem 3 can be extended to convex sets instead of points (just take a point in each of the convex sets, look for the partition using Theorem 3 and ... voilà!). This motivates the following discussion (and definition); Consider a family of convex bodies S and an effective 3-coloration of them $\varsigma\colon S \twoheadrightarrow \{R, G, B\}$ of them (recall Example 6). If we consider now the convex hull of each color class, $T = \{\langle\varsigma^{-1}(R)\rangle, \langle\varsigma^{-1}(G)\rangle, \langle\varsigma^{-1}(B)\rangle\}$, the resulting separoid most be one of the eight separoids depicted in Figure 1. More over, there is a canonic (natural) epimorphisms $\varphi\colon S \longrightarrow T$ between these separoids. Such an epimorphism satisfies an extra condition: for $A, B \subset T$,

$$(*) \qquad A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

An epimorphism which satisfies the extra condition $(*)$ will be called a *chromomorphism*. So, Theorem 2 implies the following

Corollary 4 *If S is a separoid of order $|S| = (k-1)(\mathrm{gd}(S)+1)+1$ then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

Proof. Consider S as a family of convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. Now, let $\varphi\colon P \longrightarrow S$ be a choice on S. Applying Theorem 3 to P we conclude that there exists a partition $P = A_1 \cup \ldots \cup A_k$ such that the convex hulls of the parts have a common point. If we denote by $K_k = \{1, \ldots, k\}$ (where $i \dagger j$ for all i and j) the elements of the complete separoid, clearly the function $\varsigma\colon S \to \{1, \ldots, k\}$ defined as $\varsigma(s) = i \iff \varphi^{-1}(s) \in A_i$ is a chromomorphism onto K_k. •

However, observe that the conclusion in the proof is a bit stronger than that of the statement; we had conclude that there is a point in common to *all* color classes' convex hulls. But in order to be isomorphic to a complete separoid it is enough that every pair of convex sets intersect (see the realisation of K_3 in Figure 1.**h**). So, it may be that the hypothesis of Corollary 4 can be weakened. This motivated our main result:

Theorem 5 *Let S be a separoid of order $|S| = (k-1)(\mathrm{d}(S)+1)+1$. Suppose that in addition, there exists a monomorphism $\mu\colon S \to P$ into a separoid of points in general position, and $\mathrm{d}(P) = \mathrm{d}(S)$. Then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

Proof. Let S be a d-dimensional separoid of order $(k-1)(d+1)+1$. Suppose there is a monomorphism $\mu\colon S \to P$ into a d-dimensional point separoid in general position. By Tverberg's theorem, there exists a chromomorphism $\tau\colon P \longrightarrow K_k$. We will show that $\varsigma = \tau \circ \mu$ is a chromomorphism.

For, let $i \dagger j$ be an edge of K_k (or a minimal Radon partition if you will). Since τ is chromomorphism, we have that $\tau^{-1}(i) \dagger \tau^{-1}(j)$. Then, there exist $A \subseteq \tau^{-1}(i)$ and $B \subseteq \tau^{-1}(j)$ such that $A \dagger B$ is minimal. Since P is in general position, $|A \cup B| = d+2$. Since μ is injective, $|\mu^{-1}(A \cup B)| = d+2$ and there exist $C \dagger D$ such that $C \cup D = \mu^{-1}(A \cup B)$. Therefore, since μ is a monomorphism, $\mu(C) \dagger \mu(D)$. Since P is a point separoid, it is a Radon separoid and we may suppose that $\mu(C) = A$ and $\mu(D) = B$. Finally, since $C \subseteq \varsigma^{-1}(i)$ and $D \subseteq \varsigma^{-1}(j)$, we have that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$, which concludes the proof. •

The Hadwiger-type hypothesis added to Theorem 5 still is "geometric" in nature... Is there a purely combinatorial Tverberg-type theorem?

That motivates the following

Problem. *Determine the minimum number $n(d, k)$ such that, if $|S| \geq n(\mathrm{d}(S), k)$ then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

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