Open problems of separoids

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A *separoid* is a set S endowed with a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on its pairs of disjoint subsets which is closed as a filter in the natural partial order induced by the inclusion. That is, if $A, B \subseteq S$ then

- $A \dagger B \implies A \cap B = \emptyset$,
- •• $A \dagger B$ & $B \subset B'$ $(\subseteq S \setminus A) \implies A \dagger B'$.

The pair $A \dagger B$ is a *Radon partition*, each part (A and B) is a *component* and the union $A \cup B$ is the *support* of the partition. The separoid is *acyclic* if $A \dagger B \implies |A||B| > 0$.

Theorem 1 (Arocha et al. [1] and Strausz [7, 8]) *Every finite separoid can be represented by a family of convex sets in some Euclidean space; that is, for every separoid S there exists a family of convex sets $\mathcal{F} = \{C_i \subseteq \mathbb{E}^d : i \in S\}$ such that, for $A, B \subseteq S$*

$$(*) \qquad A \dagger B \iff \langle C_a \in \mathcal{F} : a \in A\rangle \cap \langle C_b \in \mathcal{F} : b \in B\rangle \neq \emptyset \quad (\textit{and} \quad A \cap B = \emptyset),$$

where $\langle\cdot\rangle$ denotes the convex hull. Furthermore, if the separoid is acyclic, then such a representation can be done in the $(|S|-1)$-dimensional Euclidean space.

For the acyclic case, let S be identified with the set $\{1, \ldots, n\}$. For each element $i \in S$ and each pair $A \dagger B$ such that $i \in A$, let $\rho^i_{A\dagger B}$ be the point of $\mathbb{R}^n$ defined by

$$\rho^i_{A\dagger B} = e_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B} e_b - \frac{1}{|A|}\sum_{a\in A} e_a\right],$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Observe that all such points are in the affine hyperplane spanned by the basis. Now, each element $i \in S$ is represented by the convex hull of all such points:

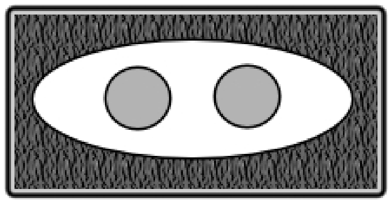
$$i \mapsto C_i = \left\langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \right\rangle.$$

From here, it is easy to prove $(*)$. •

Question 1 *¿Is there a geometric representation theorem for infinite separoids? Guess: YES.*

The minimum dimension where the separoid S can be represented is the *geometric dimension* of S and it is denoted by $\mathrm{gd}(S)$. The (*combinatorial*) *dimension* $\mathrm{d}(S)$ is the minimum d such that every subset with $d+2$ elements is the support of a Radon partition. The separoid S is in *general position* if no $\mathrm{d}(S)+1$ elements are the support of a Radon partition. The seaporid S is a *point separoid* if it can be represented by a family of points $P \subset \mathbb{E}^d$ in some Euclidean space (cf. [4]).

Theorem 2 (Bracho & Strausz [2]) *Let S be a separoid in general position. S is a point separoid if and only if $\mathrm{d}(S) = \mathrm{gd}(S)$.*



Marcos: *The smallest non-point separoid S such that $\mathrm{d}(S) = \mathrm{gd}(S)$.*

Question 2 *¿Is there a polynomial-verifiable property of separoids Ψ which makes the following statement true: S is a point separoid if and only if $\Psi(S)$ and $\mathrm{d}(S) = \mathrm{gd}(S)$? Guess: YES.*

Given two separoids S and T, a mapping $h\colon S \to T$ is a *homomorphism* if the image of minimal Radon partitions are minimal Radon partitions; that is, for $A, B \subseteq S$,

$$A \dagger B \text{ minimal} \implies h(A) \dagger h(B) \text{ minimal.}$$

Theorem 3 (Nešetřil & Strausz [6]) *The category of separoids endowed with homomorphisms is a universal category. That is, every category can be represented by an induced subcategory of separoids' homomorphisms.*

Question 3 *¿Is the subcategory of point separoids universal? Guess: NO.*

The *homomorphisms order* of separoids is defined with the relation

$$S \preceq T \iff \exists h\colon S \longrightarrow T,$$

and identifying those separoids S and S' which $S \preceq S'$ and $S' \preceq S$. A partially ordered set $(X, \leq)$ is *fractal* if for each interval $[x, y] := \{z \in X : x \leq z \leq y\}$ there exists a monotone and injective function $\iota\colon X \hookrightarrow [x, y]$.

Theorem 4 (Nešetřil [5]) *The homomorphisms order of graphs is fractal.*

Question 4 *¿Is the homomorphisms order of separoids fractal? Guess: YES.*

A pair of disjoint subsets $A \cap B = \emptyset$ which are not a Radon partition, are said to be *separated* and denoted $A \mid B$. A mapping $\mu\colon S \to T$ is a *morphism* if the preimage of separations are separations; that is, for $C, D \subseteq T$,

$$C \mid D \implies \mu^{-1}(C) \mid \mu^{-1}(D).$$

A morphism is a *monomorphism* if it is injective.

Theorem 5 (Strausz [9]) *Let S be a d-dimensional separoid of order $|S| = (k-1)(d+1)+1$. Suppose that in addition there exists a monomorphism $\mu\colon S \longrightarrow P$ into a d-dimensional point separoid P in general position. Then there exists a k-colouring $\varsigma\colon S \to \{1,\dots,k\}$ such that each pair of chromatic classes are the components of a Radon partition; that is,*

$$1 \le i < j \le k \implies \varsigma^{-1}(i) \dagger \varsigma^{-1}(j).$$

Question 5 *¿How far the additional hypothesis can be weakened —the hypothesis "$\exists\mu\dots$"— while maintaining the conclusion of the previous Theorem? ¿Is there a purely combinatorial Tverberg-type theorem for separoids? Guess: Not much and…I do not know.*

A morphism is an *epimorphism* if it is surjective. An epimorphism $\varsigma\colon S \to T$ is a *chromomorphism* if the preimage of minimal Radon partitions are Radon partitions; that is, for $C, D \subseteq T$,

$$C \dagger D \text{ minimal} \implies \varsigma^{-1}(C) \dagger \varsigma^{-1}(D).$$

The *complete* separoid K_k is the separoid of k elements such that $i \dagger j$, for all $i, j \in K_k$. The (k,d)-*Tverberg number* $\vartheta(k,d)$ is the minimum n such that all d-dimensional separoids of order at least n maps onto K_k with a chromomorphism; that is, $\vartheta(k,d)$ is minimal with the property

$$|S| \ge \vartheta(k, \mathrm{d}(S)) \implies \exists\varsigma\colon S \longrightarrow K_k.$$

Theorem 6 (Montellano et al. [3]) *For all $d > 0$ and $k > 2$ it follows that*

$$(k-1)(d+1)+1 < \vartheta(k,d) < \binom{k}{2}(d+1)+1.$$

Furthermore, for each $d > 0$ there exists a constant C_d such that for all $k \ge d+2$,

$$\vartheta(k,d) \le C_d k \log k.$$

Question 6 *¿Is it true that $\vartheta(k,d) \in O(kd)$? ¿Can $\vartheta(k,d)$ be precisely determined? Guess: YES and…I do not know, I am working on that.*

Metaproblem. *Choose your favourite problem from graph theory and/or discrete geometry and generalise it to include all separoids. Now, solve it!*

For instance:
¿Does the *perfect separoids* conjecture holds?
¿Does *Erdős-Szekeres* theorem holds for separoids?
¿Does *visibility separoids* have chromatic number bounded by a function of their clique number?

References

[1] J. L. Arocha, J. Bracho, L. Montejano, D. Oliveros, and R. Strausz, *Separoids, their categories and a Hadwiger-type theorem*, Discrete & Computational Geometry, 27 (2002), pp. 377–385.

[2] J. Bracho and R. Strausz, *Separoids and a characterisation of linear uniform oriented matroids*, Tech. Rep. 2002-566, KAM-DIMATIA Series, Charles University at Prague, 2002. (Submitted to JCTA as a note with the title 'Separoids of points in general position').

[3] J. J. Montellano-Ballesteros, A. Pór, and R. Strausz, *Tverberg-type theorems for separoids*. (Submitted to DCG), 2004.

[4] J. J. Montellano-Ballesteros and R. Strausz, *Counting polytopes via the Radon complex*, Journal of Combinatorial Theory, Series A, 106 (2004), pp. 109–121.

[5] J. Nešetřil, *The homomorphism structure of classes of graphs*, in Recent Trends in Combinatorics, E. Győry and V. T. Sós, eds., Cambridge University Press, 2001, pp. 177–184.

[6] J. Nešetřil and R. Strausz, *Universality of Separoids*. (To be submitted... soon!), 2002.

[7] R. Strausz, *On Radon's theorem and representations of separoids*, Tech. Rep. 2003-118, ITI Series, Charles University at Prague, 2003. (Submitted to Graphs and Combinatorics for the special volume in honour to Victor Neumann-Lara).

[8] ——, *On Separoids*, PhD thesis, Universidad Nacional Autónoma de México, 2004.

[9] ——, *Representations of separoids and a Tverberg-type problem*, in IX. Midsummer Combinatorial Workshop, Prague 2002, M. Mareš, ed., KAM-DIMATIA Series, Charles University at Prague, 2004, pp. 42–51. (Volume 2004-686).

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