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Counting polytopes via the Radon complex

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Abstract

A convex polytope is the convex hull of a finite set of points. We introduce the Radon complex of a polytope—a subcomplex of an appropriate hypercube which encodes all Radon partitions of the polytope's vertex set. By proving that such a complex, when the vertices of the polytope are in general position, is homeomorphic to a sphere, we find an explicit formula to count the number of d-dimensional polytope types with $d+3$ vertices in general position.

Keywords: Convex polytopes; Polytopes types; Radon's theorem; Radon complex; Separoids

1. Introduction

In the sequel, a (convex) *polytope* $P = \langle \mathscr{P} \rangle$ will be the convex hull of a finite set $\mathscr{P} \subset \mathbb{E}^d$ for which each point is a vertex, i.e., each point $\mathbf{p} \in \mathscr{P}$ can be separated by a hyperplane from the rest of the set. Moreover, we will assume that $\mathscr{P}$ spans $\mathbb{E}^d$ affinely. Its *order* and *dimension* are $n = |\mathscr{P}|$ and d, respectively. Also, we will identify the polytope P and the set $\mathscr{P}$ of its vertices.

It is possible to assign to each polytope $\mathscr{P}$ a poset—its face lattice—and a complex $\mathscr{R} = \mathscr{R}(\mathscr{P})$ embedded in the n-dimensional cube—its Radon complex—which encodes the Radon partitions of its vertices. We will say that two polytopes have the same *combinatorial type* iff their face lattices, and therefore their Radon complexes, are isomorphic. We use the "Radon complexes' types" to count combinatorial types of polytopes.

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In [8] Grünbaum proved that the number of combinatorial types of d-dimensional convex polytopes with $d+2$ points is $\lfloor \frac{1}{4}d^2 \rfloor$—we will exhibit a new proof of this fact—and the number of combinatorial types of d-dimensional convex polytopes of $d+3$ vertices is determined by Lloyd in [9].

In this paper we focus our attention on the case of d-dimensional convex polytopes whose $d+3$ vertices are in *general position* (i.e. no $r+1$ vertices of the polytope are contained in an r-dimensional subspace, for $r<d$). It is proved that

Theorem 1. *There are exactly $v(2n) - \lceil \frac{n}{2} \rceil$ combinatorial types of (convex) polytopes of order n in dimension $n-3$ whose vertices are in general position, where $v(2n)$ denotes the number of antipodal bicolored necklaces of length $2n$.*

Here, by an *antipodal bicolored necklace* we mean a dihedral arrangement of two colors (say 0 and 1) in which every element receives the opposite color of its farthest one (i.e., if an element receives color 0, its antipode receives color 1, and vice versa). Explicit formulae to calculate $v(2n)$ are given.

2. Preliminaries

2.1. Separoids

A *separoid* is a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on the family of subsets of a set S with two simple properties (cf. [1,3,11,14]): for every $A, B \subseteq S$

- ◦ $A \dagger B \Rightarrow A \cap B = \phi$,
- ◦◦ $A \dagger B$ and $C \subseteq S \setminus A \Rightarrow A \dagger (B \cup C)$.

If a pair of disjoint subsets $A, B \subseteq S$ are not related, we say that they are *separated*—hence the name of the structure. The separoid is identified with the set S.

Given a point configuration $\mathscr{P}$, its separoid $S = S(\mathscr{P})$ (also known as a linear or stretchable oriented matroid [2] or as an order type [6]) is given by the relation: if $A, B \subseteq \mathscr{P}$ then

$$A \dagger B \Leftrightarrow A \cap B = \phi \quad \text{and} \quad \langle A \rangle \cap \langle B \rangle \neq \phi$$

(where $\langle A \rangle := \{\sum \lambda_a \mathbf{a} : \sum \lambda_a = 1 \text{ and } \lambda_a \geqslant 0\}$ denotes the *convex hull* of A). The classic Radon's theorem (see e.g. [4,5]) guarantees that the relation is non-empty if there are enough points with respect to the dimension, viz. if $n = |\mathscr{P}| \geqslant d+2$. Therefore we call a related pair $A \dagger B$ a *Radon partition* and each of the parts (A and B) a *Radon component*. However, we sometimes omit the adjective "Radon" and use simply the terms *partition* and *component*, respectively. Clearly, it is enough to know the *minimal* Radon partitions to reconstruct all of them (where by minimal we mean with respect to the order $A \dagger B \preceq C \dagger D \Leftrightarrow A \subseteq C$ and $B \subseteq D$). When a pair of disjoint subsets $A, B \subseteq \mathscr{P}$ are not a Radon partition, we say that "A is *separated* from B" and denote the fact by $A|B$. As before, it is enough to know *maximal separations* to reconstruct all the structure. To emphasizes the combinatorial structure of $\mathscr{P}$, we sometimes say that $\mathscr{P}$ is a *point separoid*.

Now, consider the family of subsets 2^S of a given n-set S. The natural order $\subseteq$ imposes on this family the structure of an n-cube. More precisely, the faces of the n-cube are given by intervals of the form

$$[A, B] := \{C \subseteq S : A \subseteq C \subseteq B\}.$$

We are going to think of this hypercube as an $(n-1)$-sphere (so the facet $[\phi, S]$ is dropped out) and denote it by $\mathscr{Q}_n$.

The *Radon complex* $\mathscr{R}(S)$ of a separoid S is simply the subcomplex of $\mathscr{Q}_n$ induced by all the components of its separoid. That is, an interval $[A, B]$ will be in the (Radon) complex iff all of its vertices $C \in [A, B]$ are components of S. In other words, a vertex $C \in \mathscr{Q}_n$ is in the complex iff there exists a disjoint subset $D \subset S$ such that $C \dagger D$ (and therefore $C \dagger \overline{C}$, where $\overline{C} = S \backslash C$ denotes the complement), and a face of $\mathscr{Q}_n$ is in the complex iff all of its vertices are in the complex. (See Fig. 1; in it, the points are represented by "little" convex sets. Observe that the separoid structure is preserved.)

Lemma 1. *Let $\mathscr{P} \subset \mathbb{E}^d$ be a point separoid. Then*

$$A \dagger B \Leftrightarrow [A, \overline{B}] \subset \mathscr{R}(\mathscr{P}).$$

Moreover, $A \dagger B$ is a minimal Radon partition if and only if $[A, \overline{B}]$ is a facet of $\mathscr{R}(\mathscr{P})$.

Proof. For the necessity, let $A \dagger B$ be a Radon partition of a separoid $\mathscr{P}$. It is clear that for all $C \subseteq \overline{B}$ we have that $(A \cup C) \dagger B$, therefore every vertex of $[A, \overline{B}]$ is a component of the given separoid $\mathscr{P}$.

For the sufficiency, let $[A, \overline{B}]$ be a face of $\mathscr{R}(\mathscr{P})$ and denote by $C := \overline{B} \backslash A$ the difference of those subsets. Clearly every vertex of such a face is of the form $A \cup C'$,

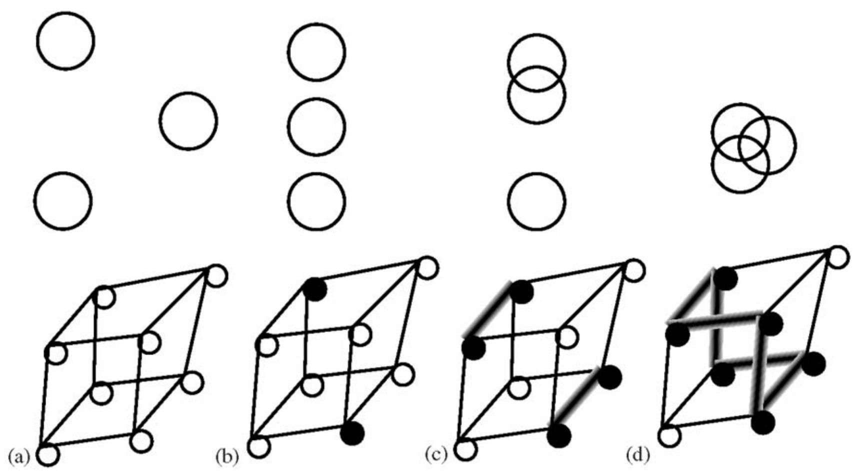

Fig. 1. The point separoids of order 3 and their Radon complexes.

for some $C' \subseteq C$. Then, since they are vertices of the complex, for all $C' \subseteq C$ we have that $A \cup C' \dagger \overline{A \cup C'}$, that is, the set $\{A \cup C' : C' \subseteq C\}$ is a subset of the components of $\mathscr{P}$.

Now, suppose that $A|B$. It is easy to see that in a configuration of points, every separation can be extended to a maximum one. Denote by C_a, respectively C_b, those elements of C which are on the same side of A, respectively B, so $C = C_a \cup C_b$. That is, $A \cup C_a | B \cup C_b$ but, as previously settled, $A \cup C_a \dagger \overline{A \cup C_a}$ which is a contradiction. □

We mention one more lemma which will allow us to prove Theorem 2. However its proof is technical and adds nothing to the present context, so the reader is referred to [13] for the details. In it, $\mathscr{K} \cap \ell$ denotes the intersection of a polytopal complex $\mathscr{K}$ with an affine subspace ℓ in the usual sense—thought of as a subcomplex of $\mathscr{K}'$, the first barycentric subdivision of $\mathscr{K}$—and $\mathscr{K} \sqcap \ell$ denotes the *fat intersection*—the subcomplex (of the barycentric subdivision) induced by all closed faces of $\mathscr{K}$ that "touch" ℓ (not necessarily in their interior). See Fig. 2.

Lemma 2. *Let $\mathscr{K}$ be a polytopal complex, and ℓ an affine subspace that intersects it in the interior. Then there exists a strong homotopical retraction $\rho : |\mathscr{K} \sqcap \ell| \searrow |\mathscr{K} \cap \ell|$.*

The following generalizes Radon's theorem.

Theorem 2. *Let $\mathscr{P} \subset \mathbb{E}^d$ be a point separoid of order $n \geqslant d+2$. If $\mathscr{R} = \mathscr{R}(\mathscr{P})$ denotes its Radon complex, then $\mathscr{R}$ is homotopic to the $(n-d-2)$-sphere. Moreover, if $\mathscr{P}$ is in general position, then $\mathscr{R}$ is homeomorphic to such a sphere.*

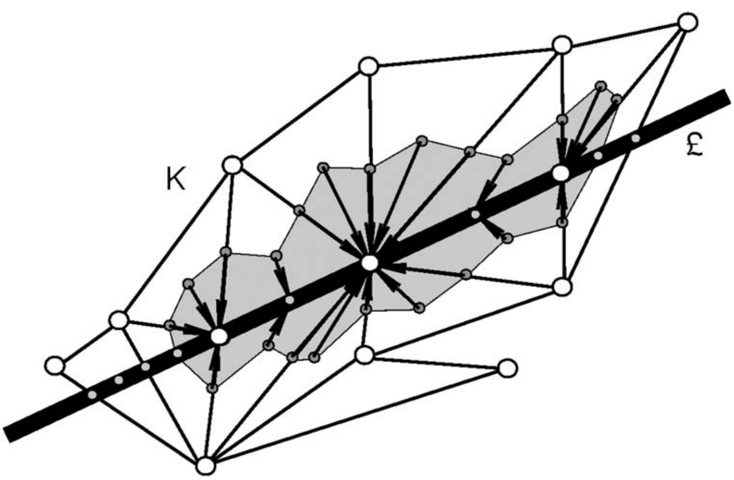

Fig. 2. The fat intersection.

Proof. Let $\mathscr{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{R}^d)^n$ be a configuration of points, $S = S(\mathscr{P})$ its separoid and $\mathscr{R} = \mathscr{R}(S)$ its Radon complex. We will identify the configuration with the intersection of the kernel $K = \varphi^{-1}(\mathbf{0})$ of its linear function $\varphi : \mathbb{R}^n \to \mathbb{R}^d$ (where $\varphi(\mathbf{e}_i) = \mathbf{p}_i$), and the hyperplane

$$\Pi = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum x_i = 0 \right\}.$$

This $(n-d-1)$-subspace of Π will be denoted by $\ell = K \cap \Pi$. A straightforward argument shows that this assignment is well defined and, modulo affine transformations, is one-to-one.

Give to $\mathbb{R}^n$ the structure of a (Manhattan) normed space and denote by

$$|\mathcal{O}| = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum |x_i| = 2 \right\},$$

the sphere of radius 2 centered at the origin. Recall the definition of the fat intersection

$$\mathcal{O} \sqcap \ell := \mathcal{O}'[\sigma \in \mathcal{O} : \sigma \cap \ell \neq \phi]$$

and define the complex of its dual faces

$$\mathfrak{R} := \{ \delta(\sigma) \in \mathcal{Q}_n : \sigma \in \mathcal{O} \text{ and } \sigma \cap \ell \neq \phi \},$$

where $\delta : \mathcal{O} \to \mathcal{Q}_n$ is the obvious duality function from the n-crosspolytope $\mathcal{O}$ onto the n-cube (see Fig. 3).

Clearly $\mathfrak{R}' = \mathcal{O} \sqcap \ell$. Observe also that, since ℓ is a subspace of dimension $n-d-1$, then $\mathcal{O} \cap \ell$ is a sphere of dimension $n-d-2$. Now, due to the previous lemma, $\mathcal{O} \cap \ell$ is a strong retract of $\mathcal{O} \sqcap \ell$ and therefore $\mathfrak{R}$ has the homotopy type of the $(n-d-2)$-sphere

$$\mathfrak{R} \searrow \mathbb{S}^{n-d-2}.$$

Claim. $\mathfrak{R}$ is equal to $\mathscr{R}$.

Proof. Due to Lemma 1, it is enough to prove that $[A, \overline{B}]$ is a face of $\mathfrak{R}$ if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. For this, let $\sigma \in \mathcal{O}$ be a face of the n-crosspolytope and $(z_i) \in \{-1, 0, 1\}^n$ its corresponding signed vector (i.e., $|\sigma| = \langle 2 z_i \mathbf{e}_i : z_i \neq 0 \rangle$). Then

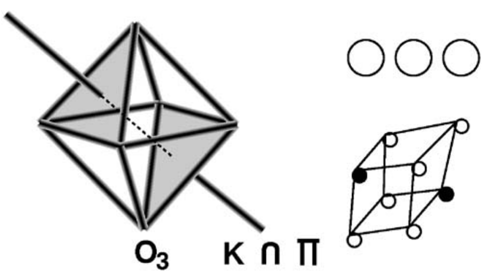

Fig. 3. The proof of Theorem 2.

σ has associated the 3-partition of $\mathscr{P}$ given by $A = \{\mathbf{p}_i \in \mathscr{P} : z_i = 1\}$, $B = \{\mathbf{p}_i \in \mathscr{P} : z_i = -1\}$ and $C = \overline{A \cup B} = \{\mathbf{p}_i \in \mathscr{P} : z_i = 0\}$ and, by the definition of δ, we have that $\delta(\sigma) = [A, A \cup C]$. Therefore, it is enough to prove that

$$\sigma \cap \ell \neq \phi \Leftrightarrow \langle A \rangle \cap \langle B \rangle \neq \phi.$$

For, let $\mathbf{x} \in \sigma \cap \ell$, then

$$\sum x_i \mathbf{p}_i = \mathbf{0}, \quad \sum x_i = 0 \quad \text{and} \quad \sum |x_i| = 2.$$

The first equation is due to $\mathbf{x} \in K$, the second because $\mathbf{x} \in \Pi$ (all these since $\mathbf{x} \in \ell = K \cap \Pi$) and the third one because $\mathbf{x} \in \mathcal{O}$.

Moreover, since $\mathbf{x} \in \sigma$, we are allowed to write

$$\frac{1}{2}\mathbf{x} = \sum \lambda_i (z_i \mathbf{e}_i)$$

as a convex combination ($\sum \lambda_i = 1$ and $\lambda_i \geqslant 0$) of some canonical vectors or their corresponding negatives. Combining these ($x_i = 2 z_i \lambda_i$) we have that

$$\sum_{\mathbf{p}_i \in A} \lambda_i \mathbf{p}_i = \sum_{\mathbf{p}_i \in B} \lambda_i \mathbf{p}_i$$

and

$$\sum_{\mathbf{p}_i \in A} \lambda_i = \sum_{\mathbf{p}_i \in B} \lambda_i = 1.$$

This last happens if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. Since all previous steps can be followed the other way around, we have concluded the proof of the claim, and therefore $\mathscr{R} \searrow \mathbb{S}^{n-d-2}$. $\square$

For the case of general position, observe that $\mathscr{R}$ has a face $[A, \overline{B}]$ of dimension greater than $n-d-2$ if and only if $|\overline{B} \backslash A| > n-d-2$ and this is equivalent to the existence of a partition $A \dagger B$ where $|A \cup B| < d+2$. If the separoid S is in general position this last is impossible, and then, since all facets have dimension $n-d-2$, we have that

$$|\mathcal{O} \sqcap \ell| = |\mathcal{O} \cap \ell|.$$

Therefore $\mathscr{R}$ is homeomorphic to the $(n-d-2)$-sphere $\mathscr{R} \cong \mathbb{S}^{n-d-2}$ and we are done. $\square$

Observe that a separoid S, thought of as an antipodal ideal in the face lattice of the n-octahedron, is a point separoid if and only if there exists a plane K such that $S = K \cap \Pi \cap \mathcal{O}_n$.

2.2. The case $n = d + 2$

Let us see now how this result allow us to count "polytope types" with few vertices. A (*convex*) *polytope* $\mathscr{P}$ is a (finite) point separoid, where every singleton is separated from its complement (i.e., for all $i \in \mathscr{P}$ it follows that $i|\bar{i}$), with the combinatorial structure (the poset) of its faces. The separoid encodes the faces as follows (cf. [2]): each face $\tau = \langle T \rangle$ is the convex hull of a subset of points $T \subset \mathscr{P}$ which are separated from its complement $(T|\bar{T})$ and, for each minimal Radon partition $A \dagger B$, if $A \subset T$ then $B \subset T$. Two polytopes are said to have (to share) the same *combinatorial type* iff their face-posets are isomorphic. Clearly, two polytopes $\mathscr{P}$ and $\mathscr{P}'$ have the same type if and only if their separoids are isomorphic, i.e. if and only if there exists a bijection $\varphi : \mathscr{P} \to \mathscr{P}'$ such that $A \dagger B \Leftrightarrow \varphi(A) \dagger \varphi(B)$. Therefore, due to Lemma 1, two polytopes have the same type if and only if, after renaming the vertices of one of them, they have the same Radon complex.

We are going to say that two Radon complexes $\mathscr{R}, \mathscr{R}' \subset \mathscr{Q}_n$ *share the same type*, and denote it by $\mathscr{R} \approx \mathscr{R}'$, iff there is a permutation on the singletons of $\mathscr{Q}_n$ which sends one Radon complex onto the other. Then, two polytopes have the same type if and only if their Radon complexes share the same type.

We can now prove a classic result in combinatorial theory of convex polytopes.

Theorem 3 (Grünbaum [8]). *There are precisely* $\left\lfloor \frac{1}{4} d^2 \right\rfloor$ *combinatorial types of* d*-polytopes with* $d+2$ *vertices.*

Proof. Let $\mathscr{P} \in \mathbb{E}^d$ be a polytope of order $|\mathscr{P}| = d+2$. By Theorem 2, its Radon complex $\mathscr{R} = \mathscr{R}(\mathscr{P})$ is homotopically equivalent to the 0-sphere and therefore is the union of two intervals $[A, B]$ and $[\bar{A}, \overline{B}]$. Since every element is separated from its complement, neither of these intervals contains a singleton nor a subset of cardinality $d+1$. With this extra condition we have that

$$\mathscr{R} \hookrightarrow \mathscr{Q}_{d+2} \backslash \left\{ \phi, \binom{\mathscr{P}}{1}, \binom{\mathscr{P}}{d+1}, \mathscr{P} \right\},$$

where $\mathscr{P} = \bar{\phi}$ is identified with the base set and $\binom{\mathscr{P}}{k} = \{A \subset \mathscr{P} : |A| = k\}$ denotes the family of k-subsets of it. If $[A, B] \cong \mathscr{Q}_0$, there are $\left\lceil \frac{d-1}{2} \right\rceil$ essentially different ways to embed $\mathscr{R}$; if $[A, B] \cong \mathscr{Q}_1$, there are $\left\lceil \frac{d-2}{2} \right\rceil$; if $[A, B] \cong \mathscr{Q}_2$, there are $\left\lceil \frac{d-3}{2} \right\rceil$; …; if $[A, B] \cong \mathscr{Q}_{d-1}$, there is one $(= \left\lceil \frac{1}{2} \right\rceil)$ way to embed $\mathscr{R}$. Therefore we have the sum

$$\left\lfloor \frac{d^2}{4} \right\rfloor = \left\lceil \frac{d-1}{2} \right\rceil + \left\lceil \frac{d-2}{2} \right\rceil + \cdots + \left\lceil \frac{1}{2} \right\rceil$$

and we are done. □

2.3. Necklaces

By a *bicolored necklace* we mean a dihedral arrangement of 0's and 1's. Given a necklace $N = [x_1, \ldots, x_m]$, two elements x_i, x_j (with $i, j \in \mathbb{N}$) will be considered the same iff $i \equiv j \pmod m$. We will say that N is *antipodal* iff $x_i = 1 - x_{i+n}$, for all $i = 1, \ldots, m = 2n$. Observe that the antipodality implies that m is even.

An automorphism $\Omega \in \mathrm{Aut}(N)$ (an element of the dihedral group $\mathbb{D}_m$) will be called a *specularity* iff there exists an $\omega \in \mathbb{N}$ such that $\Omega(x_{\omega-i}) = x_{\omega+1+i}$ for all i, or $\Omega(x_{\omega-i}) = x_{\omega+i}$ for all i. Such an ω will be called the specularity *axis*. The axis is called an *edge* or a *vertex* axis, respectively. Observe that if ω is an specularity axis, its antipode $(\omega + n)$ is so; we will consider these as one specularity axis.

The *period* of N is the minimum $k \in \mathbb{N}$ such that $x_i = x_{i+k}$, for all i. In particular, this means that $\mathbb{Z}_{\frac{m}{k}} < \mathrm{Aut}(N)$. If the necklace N is of period k, we construct the necklace $N^k := [x_1, \ldots, x_k]$. Observe that

$$\mathrm{Aut}(N^k) = \begin{cases} \mathbb{D}_0 \cong \mathbb{Z}_1 & \text{or} \\ \mathbb{D}_1 \cong \mathbb{Z}_2 \end{cases}$$

depending on the existence of a specularity axis.

In the sequel we will consider $[0, 1]$ as an antipodal necklace with a vertex axis.

Lemma 3. *Let $N = [x_1, \ldots, x_{2n}]$ be an antipodal bicolored necklace of period k. Then*

(a) *$\frac{2n}{k}$ is an odd integer and k is even.*
(a) *If N has an edge axis, then n is even; and if N has a vertex axis then n is odd.*
(c) *N^k is an antipodal necklace of period k with at most one specularity axis. Moreover, ω is an edge (resp. a vertex) axis of N if and only if ω is an edge (resp. vertex) axis of N^k.*

Proof. Clearly k divides $2n$. If $\frac{2n}{k}$ is even, since $x_1 = x_{1+rk}$ for all r and $n = \frac{2n}{2k}k$, then $x_1 = x_{1+n}$, which contradicts the antipodality. Thus $\frac{2n}{k}$ is odd, which implies that k is even, and (a) follows. Let $\omega = 1$ be an edge axis of N. Then $x_{1-i} = x_{2+i}$ for all i and, by the antipodality, $1 - x_{n+1-i} = x_{1-i}$. So for all i, $1 - x_{n+1-i} = x_{2+i}$ which implies there is no i such that $n + 1 - i = 2 + i$ and so n is even. By an analogous argument we see that if $\omega = 1$ is a vertex axis, n most be odd and (b) follows. For (c), it is not hard to see that if N^k has period $t < k$, then N has period at most t, which is a contradiction. So N^k has period k and it has at most one specularity axis. Since k is even and $\frac{2n}{k}$ is odd then, for $r = \frac{\frac{2n}{k}-1}{2}$ we have that $n = \frac{k}{2} + rk$ and therefore $x_{\frac{k}{2}+i} = x_{n+i}$ for all i. By the antipodality of N, $x_i = 1 - x_{i+n}$ for all i, then $x_i = 1 - x_{\frac{k}{2}+i}$ for all i and N^k is antipodal. Finally let us suppose ω is an axis of N and without losing generality let $\omega = \frac{k}{2}$. It is easy to see that ω is also an axis of N^k. Conversely, if $\omega = \frac{k}{2}$ is an axis of $N^k = [x_1, \ldots, x_k]$, by "gluing" as many copies of the sequence $(x_1, \ldots, x_k)$ necessary to reconstruct N, we can see that N will have ω as an axis (see Fig. 4). □

For each even integer $m = 2n$, let $\mathscr{K}(m) := \{k \in \mathbb{N} : \frac{m}{k}$ is an odd integer and k is even$\}$ (i.e. $\mathscr{K}(m)$ is the set of possible periods of an antipodal bicolored necklace of order m); and let $v_r(m, k)$ be the number of antipodal bicolored necklaces of length m, period k and r specularity axes. Recall that $v(m)$ denotes the number of antipodal bicolored necklaces and observe that $v(m) = \sum_{k \in \mathscr{K}(m)} \sum_{r \geqslant 0} v_r(m, k)$.

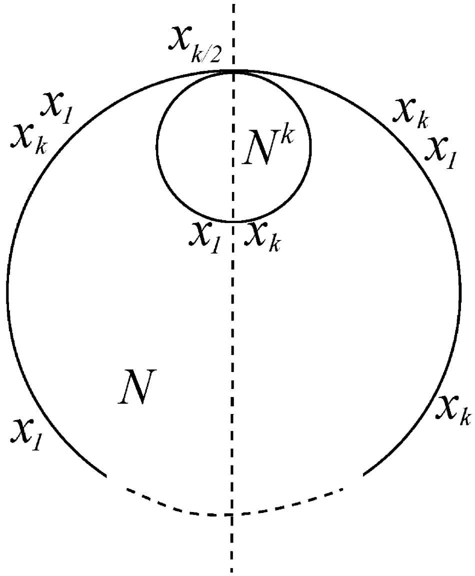

Fig. 4. An inherited axis.

For the rest of this section, given a 'color' $x \in \mathbb{Z}_2$ we will denote by $\bar{x} := 1 - x$ the other color, and given a sequence of colors $s = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$, $\bar{s}$ will denote the sequence $(\bar{x}_1, \ldots, \bar{x}_n)$. Observe that, given any sequence of colors s, $s \cup \bar{s} := [x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n]$ is an antipodal necklace.

Theorem 4. *For each even integer $m = 2n$,*

$$v(m) = \sum_{k \in \mathscr{K}(m)} \frac{1}{2k} \left(2^{\frac{k}{2}} - \sum_{i \in \mathscr{K}(k) \setminus \{k\}} 2iv_0(i,i) + iv_1(i,i) \right) + 2^{\lceil \frac{n}{2} \rceil - 2}.$$

Proof. Let $N = [x_1, \ldots, x_m]$ be an antipodal bicolored necklace of length $m = 2n$ and period k, and let $f(N) = |\{ s \in \mathbb{Z}_2^n : s \cup \bar{s} = N \}|$. Since

$$f(N) = \begin{cases} 2k & \text{if } \operatorname{Aut}(N) = \mathbb{Z}_{\frac{m}{k}}, \\ k & \text{if } \operatorname{Aut}(N) = \mathbb{D}_{\frac{m}{k}}, \end{cases}$$

then, due to Lemma 3(c), $2^n = \sum_{k \in \mathscr{K}(m)} (kv_{\frac{m}{k}}(m,k) + 2kv_0(m,k))$. So, for each even integer m, it follows that

$$v_1(m,m) + v_0(m,m) = \frac{1}{2m} \left(2^n - \sum_{k \in \mathscr{K} \setminus m} (kv_{\frac{m}{k}}(m,k) + 2kv_0(m,k)) \right) + \frac{v_1(m,m)}{2}.$$

Since for each $k \in \mathcal{K}(m)$, again by Lemma 3(c), we have that $v_{\frac{m}{k}}(m,k) = v_1(k,k)$ and $v_0(m,k) = v_0(k,k)$, it follows that

$$v(m) = \sum_{k \in \mathcal{K}(m)} \frac{1}{2k}\left(2^{\frac{k}{2}} - \sum_{i \in \mathcal{K}(k)\backslash k} (iv_1(i,i) + 2iv_0(i,i))\right) + \frac{1}{2}\sum_{k \in \mathcal{K}(m)} v_{\frac{m}{k}}(m,k).$$

Finally, for each $s = (x_1, \ldots, x_{\lceil \frac{n}{2} \rceil})$ we can construct an antipodal bicolored necklace (of length m) of the form

$$N(s) = [x_1, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}}, \ldots, x_1, \bar{x}_1, \ldots, \bar{x}_{\frac{n}{2}}, \bar{x}_{\frac{n}{2}}, \ldots, \bar{x}_1]$$

or

$$N(s) = [x_1, \ldots, x_{\lceil \frac{n}{2} \rceil}, x_{\lfloor \frac{n}{2} \rfloor}, \ldots, x_1, \bar{x}_1, \ldots, \overline{x_{\lceil \frac{n}{2} \rceil}}, \overline{x_{\lfloor \frac{n}{2} \rfloor}}, \ldots, \bar{x}_1],$$

depending on the parity of n; see Lemma 3(b). Since, again by Lemma 3(c) (given two specularity axes, there is an automorphism which sends one onto the other), $N(s) = N(s') \Leftrightarrow s' = s$ or $s' = \bar{s}$, we have that $\sum_{k \in \mathcal{K}(m)} v_{\frac{m}{k}}(m,k) = 2^{\lceil \frac{n}{2} \rceil - 1}$. The result follows. □

From here, it follows easily that

Corollary 5. *For each even integer $m = 2n$,*

$$v(m) = \sum_{k \in \mathcal{K}(m)} \frac{1}{2k}\left(2^{\frac{k}{2}} - \sum_{i \in \mathcal{K}(k)\backslash\{k\}} i(2v(i) - v_1(i,i))\right) + 2^{\lceil \frac{n}{2} \rceil - 2}$$

and

$$v_1(i,i) = 2^{\lceil \frac{i}{4} \rceil - 1} - \sum_{j \in \mathcal{K}(i)\backslash i} v_1(j,j).$$

Recall that $[0,1]$ is an antipodal necklace with a vertex axis, therefore $v(2) = v_1(2,2) = 1$.

3. Main result

As Goodman and Pollack [6] showed a configuration $\mathcal{P} \subset \mathbb{E}^d$ can be recovered from the combinatorics of its separoid. Moreover, if $n = |\mathcal{P}|$ denotes the order and $d = \mathrm{d}(\mathcal{P})$ the dimension, this can be done in time $n^d \log n$. On the other hand, even for dimension $d = 2$, to decide if a relation (a separoid) comes from a configuration of points is NP-hard (see [12]). Theorem 2, in particular, implies that

Lemma 4. *The Radon complex of a point separoid $\mathscr{P} \subset \mathbb{E}^d$ of order $n = d+3$ in general position is a cycle of length $2n$.*

Proof. Since point separoids satisfies the *interchange axiom* (i.e, if $A \dagger B$ is minimal then for all $i \notin (A \cup B)$ there exists a unique $j \in (A \cup B)$ such that $(A \setminus j) \dagger (B \setminus j \cup i)$ is minimal. See [7]), then for each vertex of the complex A there are exactly two neighbors of the form $A \cup i$ or $A \setminus j$. That is, the Radon complex is a 2-regular graph. Moreover, since it is a one-dimensional sphere (Theorem 2), then it is a cycle. The order is a straightforward consequence of the interchange axiom. □

Observe that, since $\mathscr{R}(\mathscr{P})$ is in fact an antipodal cycle (which means that if $A \in V(\mathscr{R})$ then $\bar{A} \in V(\mathscr{R})$), then its embedding in $\mathscr{Q}_n$ is isometric (cf. [10]).

Proof of Theorem 1. Let $\mathscr{P} \subset \mathbb{E}^d$ be a polytope of order $n = d+3$. By Lemma 4, its Radon complex $\mathscr{R} = \mathscr{R}(\mathscr{P})$ is an antipodal cycle of length $2n$ inside $\mathscr{Q}_n$. Assign one of the two possible orientations to this cycle. Since two vertices A, A' of $\mathscr{R} \subset \mathscr{Q}_n$ are adjacent if and only if they differ in exactly one element ($|A \triangle A'| = 1$), by walking from A to A' in the given orientation, we can assign to such an edge a 0 if we drop the element $A \triangle A'$ or a 1 if we added it. It is easy to see that we had constructed an antipodal bicolored necklace whose vertices are the edges of $\mathscr{R}$. Let us denote it by $N = N(\mathscr{R})$.

Now, given an antipodal bicolored necklace N', we can reconstruct the Radon complex of a point separoid (with the desired properties) modulo the name of the singletons. That is, we can construct an antipodal cycle $\mathscr{R}'$ of length $2n$ inside $\mathscr{Q}_n$ such that $N' = N(\mathscr{R}')$ and $\mathscr{R}' \approx \mathscr{R}$. The construction is as follows; since N' is antipodal, there is an edge e_1 (and its antipode e_{n+1}) such that if we cut the cycle N in these edges, there are $\lfloor \frac{n}{2} \rfloor$ 0s in one side and $\lceil \frac{n}{2} \rceil$ 0s on the other. Let us assign to e_1 the set $A_1 = \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$. Now, walking in the direction of the half that has $\lceil \frac{n}{2} \rceil$ zeros, we assign to each edge e_i, recursively, the set

$$A_i := \begin{cases} A_{i-1} \cup a & \text{if } e_{i-1} \cap e_i = 1, \\ A_{i-1} \backslash b & \text{if } e_{i-1} \cap e_i = 0, \end{cases}$$

where a denotes the biggest element of $\overline{A_{i-1}}$ and b denotes the smallest element of A_{i-1}. Observe that after n steps of this procedure we arrive to the antipode of A_1 (i.e., $A_{n+1} = \bar{A}_1$) and therefore we can close the cycle by adding the antipode of such a path to end with an antipodal cycle $\mathscr{R}'$ of length $2n$ inside $\mathscr{Q}_n$. It is easy to see that $\mathscr{R}$ and $\mathscr{R}'$ share the same type.

Therefore, we had constructed a one-to-one correspondence between the antipodal cycles of length $2n$ inside $\mathscr{Q}_n$ (modulo $\approx$) and the antipodal bicolored necklaces of length $2n$.

Finally, since $\mathscr{P}$ is a polytope (every singleton is separated from its complement), then $\mathscr{R}$ does not contain any singleton (or subset of size $n-1$) and therefore it contains neither the empty set nor the total one. Since there are exactly $\lceil \frac{n}{2} \rceil$

antipodal cycles of length $2n$ in $\mathcal{Q}_n$ which contain these sets, we have concluded the proof. □

4. Remarks and open problems

Even though the formula of Theorem 4 seems to be quite difficult to calculate, in practice it is not so. It always reduces to knowing the number of antipodal bicolored necklaces for twice an odd prime number and for powers of 2. Moreover, in such cases the formula reduces to

Corollary 6.

- *If* $n = 2^{\alpha}$ *then there are exactly*

  $$2^{2^{\alpha}-\alpha-2} + 2^{2^{\alpha-1}-2} - 2^{\alpha-1}$$

  polytope types of order n *in dimension* $n-3$ *whose vertices are in general position.*
- *If* p *is an odd prime number, there are exactly*

  $$\frac{2^{p-1}+p-1}{2p} + 2^{\frac{p-3}{2}} - \frac{p+1}{2}$$

  polytope types of order p *in dimension* $p-3$ *whose vertices are in general position.*

A result which surprisingly follows from this last corollary, is the well-known theorem due to Fermat which asserts that, if p is an odd prime number, $2^{p-1} \equiv 1 (\mathrm{mod}\, p)$.

Observe that, with the same lines as above, we can try to count polytopes with $n = d+4$ vertices by counting the antipodal planar graphs inside $\mathcal{Q}_{d+4}$. However, there will appear some of them which do not arise from point configurations, but some non-realizable oriented matroids (cf. [10]). So, in order to use the previous techniques in such a case, more ingredients are needed: How many antipodal planar graphs are embedded in $\mathcal{Q}_n$ in such a way that they induce a 2-sphere? Is there a recursive formula? How many of them are "realizable"?

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