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On the modular n-queen problem in higher dimensions

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Abstract

The modular n-queen problem in higher dimensions was introduced by Nudelman [6]. He showed that for a complete solution to exist in the d-dimensional modular n-chessboard, it is necessary that $\gcd(n,(2d-1)!)=1$, and that it is sufficient that $\gcd(n,(2^d-1)!)=1$. He conjectured that the last condition is also necessary and showed that this is indeed the case for the class of *linear* solutions. In this notes, we observe that the conjecture is true for the larger class of *polynomial* solutions, which are solutions we present as a natural generalization of the bidimensional solutions developed by Klöve [3]. We also generalize constructions of bidimensional solutions developed also by Klöve [4].

Mathematics Subject Classification: 05-02

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1 Introduction

Let $d \geq 1$ and $n \geq 1$ and from now on, let us denote by

$$\mathbb{Z}_n^d = \{x = (x_1, \cdots, x_d) : x_1, \cdots, x_d \in \mathbb{Z}_n\}$$

the *d-dimensional modular n-chessboard*. Let $\mathcal{O}_d := \{-1,0,1\}^d \setminus \{0\}^d$, and let us define the graph $G(n,d)$, the *attacking graph*, with vertex set $\mathbb{Z}_n^d$ and for every distinct $x, y \in \mathbb{Z}_n^d$, an edge $x \sim y$ iff x and y are contained in one of the n parallel hyperplanes orthogonal to some $\varepsilon \in \mathcal{O}_d$. That is, $x \sim y$ iff $x \neq y$ and there exists $\varepsilon \in \mathcal{O}_d$ such that $\varepsilon \cdot (x-y) \equiv 0 \mod n$; in this case we say that the "queens" at the "squares" x and y *attack each other*. The *d-dimensional modular n-queen problem* consists on finding

$$M(n,d) = \max\{|\mathcal{S}| : \mathcal{S} \subset \mathbb{Z}_n^d \text{ and } x \nsim y \text{ for every } x, y \in \mathcal{S}\},$$

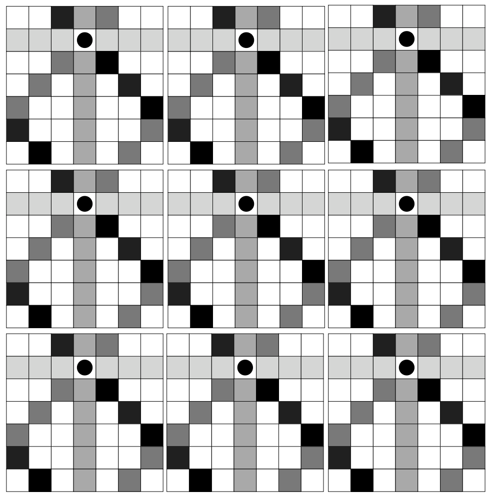

Figure 1: Domain of attack of a queen in the bidimensional $\mathbb{Z}_7$ lattice.

the maximum among the cardinalities of those subsets of vertices having no edges between them (for a graph G, such subsets of vertices are called *independent subsets*, and the maximum among the cardinalities of the independent subsets is called the *independence number of* G; hence the problem is to determine the independence number of $G(n,d)$).

In [1], the condition $\gcd(n,2)=1$ is proved to be necessary for $M(n,2)=n$. Then, in [3], the condition $\gcd(n,6)=1$ is proved to be necessary and sufficient for $M(n,2)=n$. The case $d=2$ is completed in [5] for the cases when $\gcd(n,6)>1$, with $M(n,2)=n-1$ if $\gcd(n,12)=2$ and $M(n,2)=n-2$ otherwise. In [6], the condition $\gcd(n,(2d-1)!)=1$ is proved to be necessary for $M(n,d)=n$ and the condition $\gcd(n,(2^d-1)!)=1$ is proved to be sufficient for $M(n,d)=n$ (if $d=2$, then $(2d-1)!=6=(2^d-1)!$ and hence $\gcd(n,6)=1$ is necessary and sufficient for $M(n,2)=n$).

2 Main Results

The *domain of attak* for $x\in\mathbb{Z}_n^d$, depicted in figure 2 for $d=2$ and $n=7$, consists of the union of $\frac{3^d-1}{2}=\frac{|\mathcal{O}_d|}{2}$ subspaces of codimension 1. For $M(n,d)=n$, it is necessary and sufficient the existence of a n-subset $\mathcal{S}\subset\mathbb{Z}_n^d$ such that the maps $x\mapsto\varepsilon\cdot x$ are injective for every $\varepsilon\in\mathcal{O}_d$, where $x\in\mathcal{S}$. In this case, the set $\mathcal{S}$ is an independent set of vertices of $G(n,d)$ of cardinality n and hence it constitutes a *complete* (n,d)*-solution*. In particular, a complete (n,d)-solution induces a complete (n,d')-solution for every $d'\leq d$. The following is a simple generalization to higher dimensions of a theorem in [3]; its proof is carried out similarly. We will denote, as usual, $[n]=\{1,\ldots,n\}$.

Theorem 2.1 *Let r, s be positive integers. Suppose that $n = \prod_{k=1}^{s} p_k^{\alpha_k}$, with $\alpha_k \geq 1$ for every $k \in [s]$, and let $P_n = \prod_{k=1}^{s} p_k$. For every $j = 0, \cdots, r$, let $a^{(j)} = (a_1^{(j)}, \cdots, a_d^{(j)}) \in \mathbb{Z}_n^d$ be such that*

1. $\gcd(\varepsilon \cdot a^{(1)}, n) = 1$ *for every* $\varepsilon \in \mathcal{O}_d$.
2. $a_i^{(j)} \equiv 0 \mod P_n$ *for every* $i \in [d]$ *and* $j \geq 2$.

For every $i \in [d]$, let $f_i(x) = \sum_{j=0}^{r} a_i^{(j)} x^j$ and let

$$\mathcal{S} = \{f(x) = (f_1(x), \cdots, f_d(x)) \in \mathbb{Z}_n^d : x \in \mathbb{Z}_n\}.$$

Then $\mathcal{S}$ is a complete (n, d)-solution.

Proof. Suppose that for $x, y \in \mathcal{S}$ and $\varepsilon \in \mathcal{O}_d$, $\varepsilon \cdot f(x) \equiv \varepsilon \cdot f(y) \mod n$. Then $\sum_{i=1}^{d} \varepsilon_i f_i(x) \equiv \sum_{i=1}^{d} \varepsilon_i f_i(y)$, which implies that

$$\sum_{i=1}^{d} \varepsilon_i \left(\sum_{j=0}^{r} a_i^{(j)} x^j \right) \equiv \sum_{i=1}^{d} \varepsilon_i \left(\sum_{j=0}^{r} a_i^{(j)} y^j \right) \mod n$$

and hence

$$\sum_{i=1}^{d} \varepsilon_i \left(\sum_{j=0}^{r} a_i^{(j)} (x^j - y^j) \right) \equiv 0 \mod n. \tag{1}$$

If $M_j = \sum_{j=2}^{r} a_i^{(j)} \left(\sum_{k=0}^{j-1} x^k y^{j-k-1} \right)$, then 1 implies

$$(x - y) \sum_{i=1}^{d} \varepsilon_i (a_i^{(1)} + M_i) \equiv 0 \mod n$$

which implies that

$$(x - y) \left(\varepsilon \cdot a^{(1)} + \sum_{i=1}^{d} \varepsilon_i M_i \right) \equiv 0 \mod n. \tag{2}$$

If p is a prime such that $p | \gcd \left(\varepsilon \cdot a^{(1)} + \sum_{i=1}^{d} \varepsilon_i M_i, n \right)$, then $p|n$, and hence $p|P_n$ and $P_n | M_i$ for every $i \in [d]$. Then $p| \sum_{i=1}^{d} \varepsilon_i M_i$ which implies that $p|\varepsilon \cdot a^{(1)}$, but this is imposible since $\gcd(\varepsilon \cdot a^{(1)}, n) = 1$. Then

$$\gcd \left(\varepsilon \cdot a^{(1)} + \sum_{i=1}^{d} \varepsilon_i M_i, n \right) = 1$$

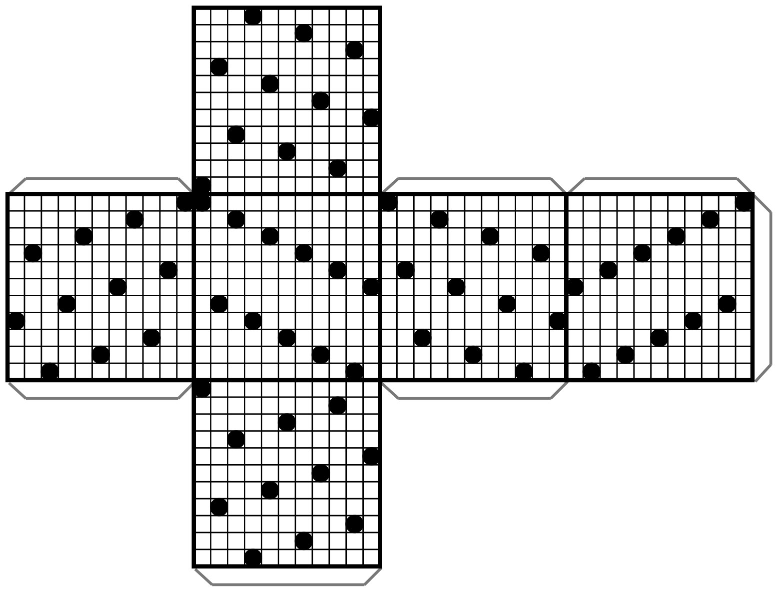

Figure 2: A linear solution in the 3-dimensional modular 11-chessboard.

and hence 2 implies that $x \equiv y \mod n$.

$\square$

Definition 2.2 *A complete (n, d)-solution $\mathcal{S} \subset \mathbb{Z}_n^d$ of the kind described in Theorem 2.1 is said to be a* polynomial *solution. If, in addition, $a_i^{(j)} \equiv 0 \mod n$ for every $j \geq 2$ and $i \in [d]$, then $\mathcal{S}$ is said to be a* linear *solution.*

A linear solution exists iff $\gcd((2^d - 1)!, n) = 1$ (see [6]). A polynomial solution exists iff there exists $a^{(1)} \in \mathbb{Z}^d$ such that $\gcd(\varepsilon \cdot a^{(1)}, n) = 1$ for every $\varepsilon \in \mathcal{O}_d$, that is, iff a linear solutions exists. Hence a polynomial solution exists iff $\gcd((2^d - 1)!, n) = 1$. It was shown in [3] that the number N_n of distinct $(n, 2)$-solutions which are polynomial is

$$N_n = n \prod_{p|n} (p-3) \prod_{k \geq 1} \frac{n^*}{\gcd(n^*, k!)},$$

where $n^* = n/P_n$. For example, $N_{13} = 130$. With an exhaustive search algorithm (see also [2]) we found that the number of all the complete $(13, 2)$-solutions is 4524, hence there exist non-polynomial solutions. For $n = 5, 7$ and 11, every $(n, 2)$-solutions are polynomial.

The following two propositions and their proofs are straightforward generalizations of those found in [4].

Proposition 2.3 *If $\{x^{(j)} = (x_1^{(j)}, \cdots, x_d^{(j)}) : j \in [n]\}$ is a complete (n, d)-solution and $k, \ell_1, \cdots, \ell_d \in \mathbb{Z}$, with $\gcd(k, n) = 1$, then*

$$\{y^{(j)} = (kx_1^{(j)} + \ell_1, \cdots, kx_d^{(j)} + \ell_d) : j \in [n]\}$$

is a complete (n,d)-solution.

Proof. If $\varepsilon \cdot y^{(j)} \equiv \varepsilon \cdot y^{(j')}$ for some $j, j' \in [n]$ and $\varepsilon \in \mathcal{O}_d$, then

$$\sum_{i=1}^{d} \varepsilon_i (k x_i^{(j)} + \ell_i) \equiv \sum_{i=1}^{d} \varepsilon_i (k x_i^{(j')} + \ell_i) \mod n,$$

which implies that

$$\sum_{i=1}^{d} \varepsilon_i k x_i^{(j)} \equiv \sum_{i=1}^{d} \varepsilon_i k x_i^{(j')} \mod n,$$

that is, $\varepsilon \cdot x^{(j)} \equiv \varepsilon \cdot x^{(j')} \mod n$, and hence $j = j'$.

$\square$

Proposition 2.4 *Let*

$$\mathcal{S}_1 = \{x^{(j)} = (x_1^{(j)}, \cdots, x_d^{(j)}) : j \in [n]\}$$

a complete (n,d)-solution and

$$\mathcal{S}_2 = \{y^{(k)} = (y_1^{(k)}, \cdots, y_d^{(k)}) : k \in [m]\}$$

a complete (m,d)-solution. For $\ell_r^{(j)} \in \mathbb{Z}$, with $j \in [n]$ and $r \in [d]$, let

$$z^{(j,k)} = \left(n(y_1^{(k)} + \ell_1^{(j)}) + x_1^{(j)}, \cdots, n(y_d^{(k)} + \ell_d^{(j)}) + x_d^{(j)} \right).$$

Then

$$\mathcal{S}_1 \times \mathcal{S}_2 = \{z^{(j,k)} : j \in [n] \text{ and } k \in [m]\}$$

is a complete (nm,d)-solution.

Proof. If $\varepsilon \cdot z^{(j,k)} \equiv \varepsilon \cdot z^{(j',k')} \mod mn$ for some $j, j' \in [n]$, $k, k' \in [m]$ and $\varepsilon \in \mathcal{O}_d$, then

$$\sum_{i=1}^{d} \varepsilon_i (n y_i^{(k)} + n \ell_i^{(j)} + x_i^{(j)}) \equiv \sum_{i=1}^{d} \varepsilon_i (n y_i^{(k')} + n \ell_i^{(j')} + x_i^{(j')}) \mod mn. \tag{3}$$

Then

$$n \sum_{i=1}^{d} \varepsilon_i (y_i^{(k)} + \ell_i^{(j)}) + \sum_{i=1}^{d} \varepsilon_i x_i^{(j)} \equiv n \sum_{i=1}^{d} \varepsilon_i (y_i^{(k')} + \ell_i^{(j')}) + \sum_{i=1}^{d} \varepsilon_i x_i^{(j')} \mod mn,$$

which implies that

$$\sum_{i=1}^{d} \varepsilon_i x_i^{(j)} \equiv \sum_{i=1}^{d} \varepsilon_i x_i^{(j')} \mod n,$$

and hence $\varepsilon \cdot x^{(j)} \equiv \varepsilon \cdot x^{(j')} \mod n$. Then $j = j'$, $x_i^{(j)} = x_i^{(j')}$ and $\ell_i^{(j)} = \ell_i^{(j')}$ for every $i \in [d]$. Substituting in 3, cancellation yields

$$n \sum_{i=1}^{d} \varepsilon_i y_i^{(k)} \equiv n \sum_{i=1}^{d} \varepsilon_i y_i^{(k')} \mod mn,$$

and hence $\varepsilon \cdot y^{(k)} \equiv \varepsilon \cdot y^{(k')} \mod m$, implying $k = k'$, and hence we get $z^{(j,k)} = z^{(j',k')}$.

□

3 Remarks

First, let us restate the Nudelman's conjecture.

Conjecture 3.1 (Nudelman) *Let $d \geq 2$ and $n \geq 1$. Then $M(n, d) = n$ if and only if $\gcd(n, (2^d - 1)!) = 1$.*

The conjecture is true when $d = 2$, as said before. Following [3], we can prove this by solving a system of congruences, roughly as follows. Suppose that $f \colon \mathbb{Z}_n \to \mathbb{Z}_n$ constitutes a $(n, 2)$-solution. Let

$$S_1 = 1 + \cdots + n = \frac{n(n+1)}{2} \tag{4}$$

and

$$S_2 = 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{5}$$

Since both $\{x + f(x) : x \in [n]\}$ and $\{x - f(x) : x \in [n]\}$ form a complete set of residues $\mod n$, then

$$S_1 \equiv \sum_{x=1}^{n} (x + f(x)) \equiv \sum_{x=1}^{n} (x - f(x)) \mod n,$$

yielding $S_1 \equiv 0 \mod n$, which is possible only if $\gcd(n, 2) = 1$. Assuming $\gcd(n, 2) = 1$, let

$$T \equiv \sum_{x=1}^{n} x f(x) \mod n.$$

By hypothesis,

$$S_2 \equiv \sum_{x=1}^{n}(x+f(x))^2 \equiv \sum_{x=1}^{n}(x-f(x))^2 \mod n,$$

that is, we have $S_2 \equiv 2S_2 + 2T \equiv 2S_2 - 2T \mod n$, yielding $2S_2 \equiv 0 \mod n$, which is possible only if $\gcd(n,3) = 1$.

The same argument works for S_4 and leads the conclusion that $\gcd(n,5) = 1$ is necessary for $M(n,3) = n$, as it was already known. However, when trying to carry out this procedure to the next prime number, the crossed terms T's become rather complicated when raising to powers higher than two and cancellation is no longer possible. This obstruction, that we call the *Bernoulli obstruction*, makes us believed that the conjecture could be false (observe that the denominators in 4 and 5 are the first two Bernoulli, hence the name). But, if a counterexample exists, in the light of Theorem 2.1, new techniques would be required to find it.

On the other hand, trying to solve Nudelman's Conjecture in the positive direction, we arrived to the following

Conjecture 3.2 *If $p|n$ and $M(n,d) = n$ then $M(p,d) > 1$.*

Remark 3.3 *If conjecture 3.2 is true, as pointed out by Nudelman, conjecture 3.1 follows from the fact that a single queen attacks any other position if the chessboard is small enough, viz. $n \leq 2^d - 1$.*

Finally, besides the previous two obstructions, we found out a combinatorial one.

Remark 3.4 (Hyperplane complexity). *If $\gcd(n,6) = 1$ and $d \geq 3$, then $Aut(\mathbb{Z}_n^d)$ does not act transitively on the hyperplanes determined by $\mathcal{O}_d$, that is to say, $\mathcal{O}_d$ determines different types of hyperplanes in $\mathbb{Z}_n^d$.*

So, for example, in the 3-dimensional cheesboard of size 7, there are positions which attack each other by exactly $1, 2, 3$ or 4 hyperplanes. This makes impossible to use a simple combinatorial approach to solve the problem.

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