On Radon's theorem and representations of separoids

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Abstract

Separoids —a natural generalization of oriented matroids— are symetric relations $\dagger \subset \binom{2^{\mathcal{F}}}{2}$ defined on pairs of disjoint subsets which are closed, as filters, by the natural partial order ($A \dagger B \preceq C \dagger D$ iff $A \subseteq C$ and $B \subseteq D$); they encode the separation structure of the families $\mathcal{F}$ of convex sets as follows: a pair of disjoint subsets $A, B \subseteq \mathcal{F}$ is a *Radon partition*, denoted as $A \dagger B$, iff their convex hulls intersect; otherwise, we say that they are *separated* —hence the name of the structure. We will show here that *every separoid S can be represented (realized) with a family of convex sets $\mathcal{F}$;* that is, given an abstract (combinatorial) separoid S we will construct a family of convex sets whose separation structure (whose Radon partitions) are exactly those of the separoid S.

This paper surveys and introduces new results related to the structural aspects of Radon's theorem (1921) and some of its generalizations. The aim is to demonstrate that these forms today a compact collection of results and methods which perhaps deserve the name: *Separoids*. Due to space limitations I have to concentrate on a few sample areas only: a new Geometric Representation Theorem for separoids, a dual version of the Basic Sphericity Theorem for oriented matroids (Folkman & Lawrence 1978), a characterization of "bipartite" separoids, and Tverberg's generalization of Radon's theorem.

At the end, in the spirit of Tverberg's theorem (1966), a generalization of separoids is introduced —they will be named *hyperseparoids*— and some of its basic properties are settled: *hyperseparoids are to separoids as Tverberg's theorem is to Radon's theorem.* In particular, it is shown how hyperseparoids capture some geometric information which separoids do not, and the Geometric Representation Theorem is generalized for hyperseparoids.

Key words: Radon's theorem, Tverberg's theorem, convex sets, configurations of points, order types, oriented matroids, separoids, abstract convexity, lattices, homomorphisms, categories.

As suggested by Danzer et al. [], the inter-relation between Radon's, Helly's and Carathéodory's theorems "could best be understood by formulating various axiomatic settings for the theory of convexity". *Separoid Theory* can be seen as a new attempt in this direction.

Separoids —antipodal filters in the face lattice of some hypercube— have their origin (Strausz 1998) in the study of *Geometric Transversal Theory* (see also, Goodman, Pollack & Wenger 1993). They generalize *oriented matroids*, as introduced by Folkman & Lawrence and Bland & Las Vergnas (1978), and therefore the notion of *order type*, as introduced by Goodman & Pollack (1983). Separoids was first used to describe the space of hyperplanes transversal to a family of convex sets (see Arocha, Bracho, Montejano, Oliveros & Strausz 2002). Also, they had been useful to solve some other problems of oriented matroid theory (cf. Bracho & Strausz 2000 and Montellano-Ballesteros

& Strausz 2001), and they are finding their own place in the study of combinatorial geometry of convex sets. They are the objects of the most general categories where Radon's theorem holds; they capture the combinatorial essence of it (see also Nešetřil & Strausz 2002).

Radon's theorem can be formulated as follows:

Theorem 1 (Radon 1921) *Let $S \subseteq \mathbb{E}^d$ be a set in the d-dimensional Euclidean space. If a subset $X \subseteq S$ consists of $d+2$ or more points, then it contains two disjoint subsets $A, B \subset X$ whose convex hulls intersect:*

$$A \cap B = \phi \quad \textit{and} \quad \langle A \rangle \cap \langle B \rangle \neq \phi.$$

We will encode such pairs with a relation $\dagger \subset 2^S \times 2^S$:

$$A \dagger B \iff A \cap B = \phi \quad \text{and} \quad \langle A \rangle \cap \langle B \rangle \neq \phi;$$

and the obvious properties of this realtion will be the axioms of a separoid.

The aim of this paper is to survey "old" and present new results of separoids. However, due to space limitations it will concentrate on a few sample areas only.

In Section 1 separoids are formally defined and their basic properties are settled there: a simple characterization of Radon's theorem, via the combinatorial notion of dimension $\mathrm{d}(S)$ (Lemma 1); the definition of the geometric dimension $\mathrm{gd}(S)$, via the new Geometric Representation Theorem (Theorem 2); and a characterization of *uniform point separoids* —finite subsets of the Euclidean space in general position— via these invariants (Theorem 5), viz., *a uniform separoid S is a point separoid if and only if $\mathrm{d}(S) = \mathrm{gd}(S)$.*

In Section 2 the *Radon complex* of a separoid —a subcomplex of the hypercube's face lattice— is introduced. Radon's theorem is used (and generalized) there to prove that (Theorem 6), *for point separoids, such a complex is homotopically equivalent to the sphere of dimension $|S|-\mathrm{d}(S)-2$. More over, in the uniform case such an equivalence is shown to be an homeomorphism.* A generalization of this result, to include oriented matroids, leads to the characterization of cocircuit graphs of oriented matroids (Theorem 7).

Section 3 is devoted to introduce several kind of morphisms and operations with separoids. It is shown how the simplest form of Radon's theorem and the characterization of *strongly 2-colorable separoids* —separoids which can be "mapped" onto the separoid K_2 (to be defined)— are equivalent (Theorem 9).

In Section 4, Tverberg's generalization (1966) of Radon's theorem is studied. A simple application is settled (Corollary 13), viz., *every separoid S of order $|S| = (k-1)(\mathrm{gd}(S)+1)+1$, for some $k \in \mathbb{N}$, can be mapped onto the separoid K_k.* However this result is far to imply Tverberg's theorem, even for uniform point separoids. It is shown, via several examples, that such a result is essencially optimal inside the theory of separoids. Therefore a new (and "bigger") category is introduced; the objects will be called *hyperseparoids* and the Geometric Representation Theorem is generalized (Theorem 14 and Corollary 15).

We will close on Section 5 with some remarks and open problems.

1 Separoids: the geometric dimension

1.1 The objects

A *separoid* is a relation $\dagger \subset 2^S \times 2^S$ defined on the family of subsets of a set S with three simple properties: for every $A, B \subseteq S$

$$\begin{array}{cc} \circ & A \dagger B \Longrightarrow B \dagger A, \\ \circ\circ & A \dagger B \Longrightarrow A \cap B = \phi, \\ \circ\circ\circ & A \dagger B \text{ and } C \subseteq S \setminus A \Longrightarrow A \dagger B \cup C. \end{array}$$

The separoid is identified with the set S and is sometimes denoted as a pair $(S, \dagger)$. An element $A \dagger B$ is called a *Radon partition*. Each part (A and B) is known as a *Radon component* and the union of its parts $A \cup B$ is called the *support* of the partition. The *order* of the separoid is the cardinal $|S|$ and the *size* is half of the Radon partitions $\frac{1}{2}|\dagger|$. Observe how, if the separoid is finite, the third axiom implies that it is enough to know the *minimal* Radon partitions to reconstruct the separoid; they encode all the structure.

A *separation* $A \mid B$ is a pair of disjoint sets that are not a Radon partition. When talking of a separation $A \mid B$ we often say *"A is separated from B"*. The separoid is *acyclic* if $\phi \mid S$ and the separations with the empty set are considered *trivial*.

The separation relation $| \subset 2^S \times 2^S$ satisfies the following properties

$$\begin{array}{cc} \circ & A \mid B \Longrightarrow B \mid A, \\ \circ\circ & A \mid A \Longrightarrow A = \phi, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \Longrightarrow A' \mid B. \end{array}$$

Therefore, it is said that the separation relation is a *symmetric, quasi-antireflexive* and *ideal* relation on the subsets of the given set S. It is clear that, in the finite case, it is enough to know *maximal* separations to reconstruct the separoid. The separoid is denoted sometimes as a pair $(S, |)$.

Clearly $\dagger$ and $|$ determines each other: $A \dagger B \iff A \nmid B$ and $A \cap B = \phi$.

Examples:

1. Consider a subset $P \subseteq \mathbb{E}^d$ of the d-dimensional Euclidean space and define the following relation

$$A \dagger B \iff \langle A \rangle \cap \langle B \rangle \neq \phi \quad \text{and} \quad A \cap B = \phi,$$

where $\langle A \rangle$ denotes de convex hull of A. If P is finite, the pair $S(P) = (P, \dagger)$ is an acyclic separoid and will be called a *point separoid*. In fact, the name of separoids arises as an abstraction of the fact that $A \mid B$ is a non-trivial separation if and only if there exists a hyperplane strictly separating $\langle A \rangle$ from $\langle B \rangle$. Theorems 5 and 9 characterize important classes of point separoids.

2. Consider a family $\mathcal{F}$ of convex sets in $\mathbb{E}^d$ and define the separoid $S(\mathcal{F})$ as above, that is, two subsets of the family $A, B \subseteq \mathcal{F}$ are separated if there exists a hyperplane that leaves all members of A on one side of it and those of B on the other. If $\mathcal{F}$ is finite and the elements of $\mathcal{F}$ are compact, then $S(\mathcal{F}) = (\mathcal{F}, |)$ is an acyclic separoid and will be called a *separoid of convex sets*. The Geometric Representation Theorem (Theorem 2) proves that every finite acyclic separoid S is isomorphic to a separoid of convex sets; more over S can be realized in the Euclidean space of dimension $|S| - 1$.

3. Consider a (simple) graph $G = (V, E)$ and define two elements $u, v \in V$ of the vertex set to be a minimal Radon partition $u \dagger v$ if and only if the pair is an edge $uv \in E$. Then $S(G) = (V, \dagger)$ is also a separoid. In fact, this is an embedding of the category of graphs into that of separoids endowed with homomorphisms (see Section 3).

4. Consider an oriented matroid $\mathcal{M} = (E, \mathcal{L})$ and identify it with the subset $\mathcal{L} \subseteq \{-, 0, +\}^E$ of its *covectors* in the usual way (see Björner et al.). Let $\mathcal{T} = \mathcal{T}(\mathcal{L})$ be the set of *topes* (maximal covectors) and define the following relation $| \subset 2^E \times 2^E$ on the subsets of E: $A, B \subset E$ are separated, $A \mid B$, if and only if there exist a tope $T \in \mathcal{T}$ such that $A \subseteq T^+ := \{e \in E : T_e = +\}$, and $B \subseteq T^- := \{e \in E : T_e = -\}$. The pair $S(\mathcal{M}) = (E, |)$ is a separoid. In Section 2 this example will be used implicitly with a different approach, and the Basic Sphericity Theorem will be related to Radon's theorem.

5. As a special case of oriented matroids, from a digraph $D = (V, E)$ a separoid can be defined. Let the set of arrows be the base set and define two subsets $A, B \subset E$ to be separated $A \mid B$ iff for every circuit of the underling graph in which the arrows in one direction are contained in A, the arrows in the other direction are not contained in B. $S(D) = (E, |)$ is a separoid, and it is acyclic if and only if D is so —hence the name.

6. Consider a topological space $T = (X, \tau)$ and define two subsets $A, B \subset X$ to be separated if and only if there exist disjoint neighborhoods of them, i.e. there exist $\alpha, \beta \in \tau$ such that $A \subseteq \alpha, B \subseteq \beta$ and $\alpha \cap \beta = \phi$. This is clearly an acyclic separoid. For more on infinite separoids, look at Nešetřil & Strausz 2002.

7. All acyclic separoids on three elements arise from one of the eight families of convex bodies in Figure 1. Those labeled **a**, **b**, **e** and **h** are the point separoids of order 3; in fact, they come from the four essentially different order types with three elements.

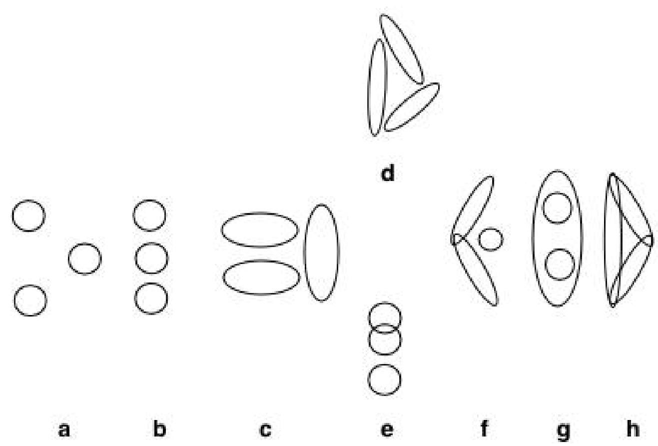

Figure 1. *The acyclic separoids of order 3*

1.2 The morphisms

Let S and T be two separoids. A *separoid morphism* is a function $\varphi: S \longrightarrow T$ with the property that for all $A, B \subseteq T$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

The *category of separoids* is defined with such morphisms between separoids. Other kinds of "maps" —and therefore other categories— will be defined in Section 3.

Two separoids are *isomorphic* $S \approx T$ if there is a bijective morphism between them whose inverse function is also a morphism. If $S \subseteq T$ is a subset of a separoid $\dagger \subset 2^T \times 2^T$, the *induced* separoid $T[S]$ is the restriction $\dagger \subset 2^S \times 2^S$ and an *embedding* $S \hookrightarrow T$ is an injective morphism that is an isomorphism between the domain and the induced separoid of its image.

Examples:

8. Consider a family of convex sets $\mathcal{F}$, choose a point in each of its elements to construct a point separoid $\mathcal{P}$ and define the obvious bijection $\varphi: \mathcal{P} \to \mathcal{F}$. This is a morphism since every hyperplane that separates A from B, subsets of $\mathcal{F}$, also separates their respective points $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$. This construction is very useful and it will be said that $\mathcal{P}$ *is a choice on* $\mathcal{F}$.

9. Consider a family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ and let $\pi: \mathbb{R}^d \to \mathbb{R}^e$ be an affine projection ($e \leq d$). The obvious bijection $\hat{\pi}: \mathcal{F} \to \pi(\mathcal{F})$ is a morphism between their separoids $S(\mathcal{F}) \longrightarrow S(\pi(\mathcal{F}))$.

10. Consider the embedding $G \mapsto S(G)$ sugested by Example 3. If $\varphi: G \longrightarrow H$ is a graph homomorphism, the same map $\varphi: S(G) \to S(H)$ is a morphism of separoids.

11. Consider a family of convex sets $\mathcal{F}$ and give them a coloration $\varsigma: \mathcal{F} \twoheadrightarrow \{c_1, \ldots, c_k\}$. If we denote by $\mathcal{F}' = \{\langle \varsigma^{-1}(c_i) \rangle\}$ the convex hulls of the color classes' family, the obvious map $\mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism. This is a key ingridient in our study of Tverberg's theorem (Section 4).

12. *Strong* and *weak maps* of oriented matroids are both examples of morphisms between their respective separoids (cf. Björner et al. 1993 sec. **7.7**).

13. In Figure 1, monomorphisms go from left to right between every pair of separoids. Observe that there is no monomorphism in either directions between the separoids labeled **d** and **e** (cf. Diagram 1).

1.3 Basic notions

Separoids have an intrinsic notion of dimension which is easy to determine. The *d-dimensional simploid* is the separoid of order $d+1$ and size 0; it will be denoted by σ^d and can be represented with the vertex set of a simplex —hence the name. The *dimension* of a separoid S is the maximum dimension of its induced simploids (Figure 1.**a** represents σ^2)

$$\mathrm{d}(S) = \max_{\sigma^d \hookrightarrow S} d.$$

S is called *complete* if for every $i, j \in S$ follows that $i \dagger j$. The complete separoid of order n is denoted by K_n (Figure 1.**h** represents K_3).

With this definition of dimension at hand, it is quite easy to translate to separoid terms the classic Radon's theorem; they capture the combinatorial essence of it (cf. Theorem 1).

Lemma 1 (Radon) *Let S be a d-dimensional separoid, then every subset $X \subseteq S$ of cardinality at least $d+2$ contains two disjoint subsets $A, B \subset X$ which are not separated: $A \dagger B$.*

Proof. Follows immediately from the fact that X is not a simploid. •

There have been many authors that observe that Radon's theorem can be settled in a more precise way (cf. Eckhoff 1993): *Let $P \subset \mathbb{E}^d$ be a set of $d+2$ points in general position. Then*

P contains a unique partition in two disjoint subsets whose convex hulls have a common point. Moreover, this point is also unique. This motivates the next definitions.

A separoid S is called a *Radon separoid* if for every pair of minimal Radon partitions $A \dagger B$ and $C \dagger D$ it follows that

$$A \cup B \subseteq C \cup D \Longrightarrow \{A, B\} = \{C, D\}.$$

It is said that S is in *general position* if every subset with $\mathrm{d}(S) + 1$ elements induces a simploid.

In Figure 1, only those separoids labeled **c** and **d** are not Radon separoids, and only those labeled **e**, **f** and **g** are not in general position.

Lemma 2 *Let S be a d-dimensional separoid in general position. If $A \dagger B$ is a minimal Radon partition, then the cardinality of the support $|A \cup B|$ is at least $d + 2$.*

Proof. The cardinality of the support cannot be smaller because every subset $\sigma \subset S$ of cardinality $d + 1$ or less is an induced simploid. •

1.4 The Geometric Representation Theorem

Since the beginning of separoids' theory, we faced the problem of *realization;* does there exist geometric objects to model each (combinatorial) separoid? In the finite case, the answer is affirmative [BS]: *every finite separoid can be represented by a family of convex sets in some Euclidean space and their separations by hyperplanes.* This result allow us to introduce the *geometric dimension* of a separoid as the minimum dimension where such a realization can be done. Such an invariant leads the topological description of the space of all *transversal hyperplanes* of a family of convex bodies (cf. Theorem 3) and a characterization of those uniform oriented matroids which arise from an order type (cf. Theorem 4).

However, the realization we first found (see Arocha et al. 2002), gives a quite bad bound of this invariant; it was in terms of the number of separations, which grows exponentially with respect to the order. Here, I will exhibit a better bound —and easier construction— for the acyclic case.

Theorem 2 *Every acyclic separoid of order n can be represented by a family of convex polytopes in the $(n-1)$-dimensional affine space.*

Proof. Let $(S, \dagger)$ be an acyclic separoid (i.e., $A \dagger B \Longrightarrow |A||B| > 0$). To each element $i \in S$ and each (minimal Radon) partition $A \dagger B$ such that $i \in A$, we assign a point of $\mathbb{R}^n$ (where $n = |S|$)

$$\rho^i_{A \dagger B} = \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum \mathbf{e}_b - \frac{1}{|A|}\sum \mathbf{e}_a\right],$$

(where $\{\mathbf{e}_j\}$ denotes the canonical basis, $b \in B$ and $a \in A$) and realize each element $i \in S$ as the convex hull of all such points

$$i \mapsto \langle \rho^i_{A \dagger B} : i \in A \text{ and } A \dagger B \rangle.$$

Observe that these convex sets are in the $(n-1)$-dimensional affine subspace spanned by the basis.

To see that this family of convex polytopes realizes the separoid observe that, for each partition $A \dagger B$, the vertex set of the simplices $\langle \mathbf{e}_a : a \in A \rangle$ and $\langle \mathbf{e}_b : b \in B \rangle$ "moves" —half of the way each— to realize such a partition intersecting one to the other precisely in their baricenter, therefore

$$\langle \rho^a_{A \dagger B} \rangle \cap \langle \rho^b_{B \dagger A} \rangle \neq \phi.$$

On the other hand, given a separation $\alpha \mid \beta$, define the affine extension of the equations

$$\psi_{\alpha|\beta}(\mathbf{e}_j) = \begin{cases} -1 & j \in \alpha \\ 1 & j \in \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, n.$$

Now, it is enough to prove that for every $i \in \alpha$ (resp. β), we have that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ (resp. > 0). For this, observe that, if $i \in \alpha$ then

$$\psi_{\alpha|\beta}\left(\rho^i_{A\dagger B}\right) = \psi_{\alpha|\beta}\left(\mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\Big(\sum_{B\cap\alpha} \mathbf{e}_b + \sum_{B\cap\beta} \mathbf{e}_b\Big) - \frac{1}{|A|}\Big(\sum_{A\cap\alpha} \mathbf{e}_a + \sum_{A\cap\beta} \mathbf{e}_a\Big)\right]\right) =$$

$$= -1 + \frac{(|B\cap\beta| - |B\cap\alpha|)}{2|B|} + \frac{(|A\cap\alpha| - |A\cap\beta|)}{2|A|} \leq 0.$$

Equality holds if and only if $B \cap \beta = B$ and $A \cap \alpha = A$ which leads a contradiction. •

In the infinite case (eventhough if the dimension remains finite) the question of realizability is open so, from now on, we assume that the separoid is finite: $|S| \in \mathbb{N}$.

The *geometric dimension* of a separoid can be defined as the minimum dimension of the Euclidean space where the given separoid S can be realized as a separoid of convex sets; we denote it by $\mathrm{gd}(S)$. We can reformulate Theorem 2 in the following

Corollary 3 *The geometric dimension of an acyclic separoid is bounded by its order, viz.,*

$$\mathrm{gd}(S) \leq |S| - 1.$$

This bound is best possible as the simploid shows (see also Figure 1). An algorithm to calculate this invariant is not known, and that it is an NP-hard problem is conjectured.

On the other side, it is easy to see how Radon's theorem implies that the (combinatorial) dimension bounds the geometric dimension.

Lemma 3 *For any separoid S, its dimension is not greater than its geometric dimension, i.e.,*

$$\mathrm{d}(S) \leq \mathrm{gd}(S).$$

Proof. Let S be d-dimensional with geometric dimension $g = \mathrm{gd}(S)$, and suppose that $g < d$. Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^g$ such that $S \approx S(\mathcal{F})$ (cf. Example 2). Since S is d-dimensional, the family $\mathcal{F}$ contains a simploid σ^d of order $d+1$. Let $\mathcal{P}$ be a choice on $\mathcal{F}[\sigma^d]$ (recall Example 8). Then, $\mathcal{P}$ consists of $g+2$ or more points in $\mathbb{R}^g$ and, due to Theorem 1, there exists a partition of them in two subsets whose convex hulls intersect. Therefore they are not separated. This contradicts the fact that σ^d was a simploid. •

1.5 Some applications

Theorem 2 had find many applications (see Strausz 2001s). Let me mention just a couple, may be the most representatives.

The proof of the following result uses a Borsuk-Ulam type theorem in the category of *comorphisms* —maps that, instead of "pulling" the separations relation, "pushes" it— but it is beyond the scope of this paper. In it, by an *essential subspace* we meant a topological subspace which cannot be contracted (which is not homotopically equivalent) to a point. See Arocha et al. (2002) for the details and pertinent definitions.

Theorem 4 *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{E}^{d+1}$ and $\mathcal{T} = \mathcal{T}(\mathcal{F})$ the space of hyperplanes transversal to all elements of the family. If $\mathrm{gd}(S(\mathcal{F})) < d$ then $\mathcal{T}$ is an essential subspace of $\mathbb{P}^d$.* •

The next one appears in Bracho & Strausz 2000, however the proof is reproduced here because it is a strong application of Radon's theorem and it is both, short and elegant. For it, we will use a lemma which is just mentioned here but it is intuitively clear.

Lemma 4 *Let S be a general position separoid. If $\mathrm{gd}(S) = \mathrm{d}(S)$ then S is a Radon separoid.* •

Theorem 5 *A general position separoid is a point separoid if and only if its dimension and its geometric dimension are equal.*

Proof. The necessity is clear. For the sufficiency, consider S as a separoid of convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. Let $\mathcal{P}$ be a choice on S and let $\varphi\colon \mathcal{P} \longrightarrow S$ be the obvious bijection. We will show that, in fact, this is an isomorphism of separoids.

In the one hand, the argument on Example 8 shows that φ is a morphism, i.e. for every $A, B \subseteq S$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

On the other hand, let $A \dagger B$ be a minimal Radon partition of S. Since S is a separoid in general position, due to Lemma 2, the cardinality of the support is $|A \cup B| \geq d + 2$. Then the preimage of this union consists of $d+2$ or more points in $\mathbb{E}^d$ and by Theorem 1 there exists a partition $D \dagger E$ of $\varphi^{-1}(A \cup B)$ in $\mathcal{P}$. Since φ is a bijective morphism, $\varphi(D) \dagger \varphi(E)$ is a Radon partition of $A \cup B$. Finally, by the previous lemma, S is a Radon separoid and $\{A, B\} = \{\varphi(D), \varphi(E)\}$. Therefore $\varphi^{-1}(A) \dagger \varphi^{-1}(B)$. Since the set of minimal Radon partitions determines all Radon partitions, it follows that for every $A, B \subseteq S$,

$$A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

Thus, φ is an isomorphism of separoids and S is a point separoid. •

This result is sharp. The hypothesis of general position cannot be dropped without adding a new ingredient. The separoid depicted in Figure 1.**g** is a 1-dimensional separoid in general position, it is a Radon separoid and it can be realized in the line but, it is not a point separoid. The small examples of non-stretchable pseudolines arrangements (i.e., which are not order types) suggest the following

Conjecture 1. *An oriented matroid is an order type (stretchable, realizable, coordinatizable, linear) if and only if its dimension (its rank minus one) is equal to its geometric dimension.*

However, it is well known that the stretchability problem is NP-hard (cf. Shor 1991) so, this problem is asking for new methods of research. E.g., a direct consequence of Theorem 5 (see also the next section) is: *A* <u>*uniform*</u> *oriented matroid M is an order type if and only if* $\mathrm{gd}(M) = \mathrm{d}(M)$.

2 The Radon complex: oriented matroids

An *oriented matroid* is a Radon separoid S whose minimal Radon partitions satisfies an extra condition, e.g. the so-called "exchange axiom" (cf. Goodman & Pollack 1988): if $A \dagger B$ is minimal, then

$$\forall i \in S \setminus (A \cup B) \quad \exists j \in A \cup B \quad A \setminus j \dagger (B \setminus j) \cup i.$$

This is a very strong assumption which yields a very rich theory (see Björner, Las Vergnas, Sturmfels, White & Ziegler 1993). It is not hard to prove that point separoids satisfy such a property. In fact, for $d+3$ points in general position (in dimension d), this property imposes a cyclic order in the points of the separoid. In this section we will see how this is a consequence (and a generalization) of Radon's theorem.

An interesting example is 5 points (in $\mathbb{E}^2$) in general position: such a cycle coincides with that of the unique conic (thought of as a subset of $\mathbb{P}^2$) spanned by the points (Bracho 2000).

2.1 The n-cube

Let $\mathcal{Q}_n$ denote the n-cube. Its vertices $V(\mathcal{Q}_n)$ will be identified with the family of subsets 2^S of the n-set S. Its faces are intervals of the natural contention partial order defined in 2^S, i.e., each face is of the form

$$[A, B] := \{X \subseteq S : A \subseteq X \subseteq B\}.$$

In fact, this definition leads to an n-ball but in the sequel, the n-cube will be thought of as an $(n-1)$-sphere so the face $[\phi, S]$ is dropped out.

Given a separoid $(S, \dagger)$, for each Radon partition $A \dagger \overline{A}$ (where $\overline{A}$ denotes the complement), take the component $A \in V(\mathcal{Q}_n)$; the complex induced by all such vertices is what we call the *Radon complex* of the separoid and we denote it by $\mathcal{R} = \mathcal{R}(S)$. Here, by *induced* we mean that a face of $\mathcal{Q}_n$ is in the complex if and only if all of its vertices are. In Figure 2 are shown the Radon complexes of all point separoids on three elements.

It is follows from the definition that

Lemma 5 *If $A \dagger B$ is a Radon partition of S then $[A, \overline{B}]$ is a face of $\mathcal{R}(S)$.*

Proof. Let $A \dagger B$ be a Radon partition of a separoid S. It is clear that for all $C \subseteq \overline{B}$ we have that $(A \cup C) \dagger B$, therefore every vertex of $[A, \overline{B}]$ is a component of the given separoid S. •

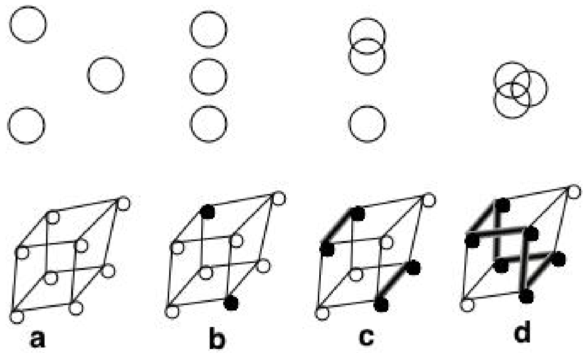

Figure 2. *Some small Radon complexes.*

The converse of Lemma 5 is not true in general. That is, there exist a separoid S such that $[A, \overline{B}]$ is a face of $\mathcal{R}(S)$ and $A \dagger B$ is not a Radon partition. Therefore the *maximal* Radon partitions does not determine the separoid. In fact, there are many separoids which yield the same Radon complex (cf. Figure 2 with Figure 1 and observe that, while there are 8 acyclic separoids of order 3, there are only 4 possible Radon complexes in the 3-cube). However, in some important cases the separoid can be reconstructed from its Radon complex. In particular oriented matroids, and therefore point separoids, are completely encoded by the Radon complex. Those separoids for which the converse of the previous lemma holds are called *full* separoids.

We will say that a separation $A \mid B$ is *maximum* if the union of its parts $A \cup B = S$ is the separoid it self. We say that a separation $A \mid B$ *conforms* to other separation $C \mid D$ if $A \subseteq C$ and $B \subseteq D$.

Lemma 6 *Let S be a separoid. If every separation of S conforms to a maximum separation then S is a full separoid.*

Proof. Let $[A, \overline{B}]$ be a face of $\mathcal{R}(S)$ and denote by $C = \overline{B} \setminus A$ the difference of those subsets. Clearly every vertex of such a face is of the form $A \cup C'$, for some $C' \subseteq C$. Then, since they are vertices of the complex, for all $C' \subseteq C$ we have that $A \cup C' \dagger \overline{A \cup C'}$, this is, the set $\{A \cup C' : C' \subseteq C\}$ is a subset of the components of S.

Now, in order to search for a contradiction, suppose that $A \mid B$. The hypothesis says that this separation conforms to a maximum one. Denote by C_a, respectively C_b, those elements of C which are on the same side of A, respectively B, so $C = C_a \cup C_b$. That is, $A \cup C_a \mid B \cup C_b$ but, as previously settled, $A \cup C_a \dagger \overline{A \cup C_a}$ which is a contradiction. •

2.2 The Sphericity Theorem

We mention one more lemma which will allow us to prove the next generalization of Radon's theorem, however its proof is technical and long so the reader is referred to Strausz 2001 for the details. In it, $\mathcal{K} \cap \ell$ denotes the intersection of a polytopal complex $\mathcal{K}$ with an affine subspace ℓ in the usual sense —thought of as a subcomplex of the first baricentric subdivision $\mathcal{K}'$— and $\mathcal{K} \sqcap \ell$ denotes the *fat intersection*; the subcomplex (of the baricentric subdivision) induced by all closed faces of $\mathcal{K}$ that "touches" ℓ (not necessarily in their interior). See Figure 3.

Lemma 7 *Let $\mathcal{K}$ be a polytopal complex, and ℓ an affine subspace that intersects it in the interior. Then there exists a strong homotopical retraction $\rho\colon |\mathcal{K} \sqcap \ell| \searrow |\mathcal{K} \cap \ell|$.* •

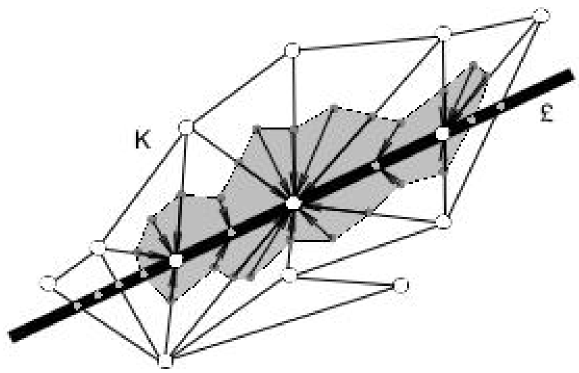

Figure 3. *The fat intersection.*

Theorem 6 *Let $\mathcal{P}$ be a d-dimensional point separoid of order n. Then, the $(n-d-2)$-sphere is an homotopical retraction of its Radon complex $\mathcal{R} = \mathcal{R}(\mathcal{P})$,*

$$\mathcal{R} \searrow \mathbb{S}^{n-d-2}.$$

More over, if the separoid is in general position, then such homotopy is in fact an homeomorphism,

$$\mathcal{R} \cong \mathbb{S}^{n-d-2}.$$

Proof. Let $\mathcal{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{R}^d)^n$ be a configuration of points, $S = S(\mathcal{P})$ its separoid and $\mathcal{R} = \mathcal{R}(S)$ its Radon complex. We will identify the configuration with the intersection of, the kernel $K = \varphi^{-1}(\mathbf{0})$ of its linear function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^d$ (where $\varphi(\mathbf{e}_i) = \mathbf{p}_i$), and the hyperplane

$$\Pi = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum x_i = 0 \right\}.$$

This $(n-d-1)$-subspace of Π will be denoted by $\ell = K \cap \Pi$. A straight forward argument shows that this assignment is well defined and, modulo affine transformations, is one-to-one.

Give to $\mathbb{R}^n$ the structure of a (Manhattan) normed space and denote by

$$|\mathcal{O}| = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum |x_i| = 2 \right\}$$

the sphere of radius 2 centered at the origin. Recall the definition of the fat intersection

$$\mathcal{O} \sqcap \ell = \mathcal{O}'[\sigma \in \mathcal{O} : \sigma \cap \ell \neq \phi]$$

and define the complex of its dual faces

$$\Re := \{\delta(\sigma) \in \mathcal{Q}_n : \sigma \in \mathcal{O} \text{ and } \sigma \cap \ell \neq \phi\},$$

where $\delta\colon \mathcal{O} \to \mathcal{Q}_n$ is the obvious duality function from the n-crosspolytope $\mathcal{O}$ onto the n-cube.

Clearly $\Re' = \mathcal{O} \sqcap \ell$. Observe also that, since ℓ is a subspace of dimension $n-d-1$, then $\mathcal{O} \cap \ell$ is a sphere of dimension $n-d-2$. Now, due to the previous lemma, $\mathcal{O} \cap \ell$ is a strong retract of $\mathcal{O} \sqcap \ell$ and therefore $\Re$ has the homotopy type of the $(n-d-2)$-sphere

$$\Re \searrow \mathfrak{S}^{n-d-2}.$$

Claim. $\Re$ is equal to $\mathcal{R}$

$\circ$ Since point separoids are full separoids —$A \dagger B$ is a Radon partition if and only if $[A, \overline{B}]$ is a face of $\mathcal{R}$ (cf. Lemma 6)— it is enough to prove that $[A, \overline{B}]$ is a face of $\Re$ if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. For this, let $\sigma \in \mathcal{O}$ be a face of the n-crosspolytope and $(z_i) \in \{-1, 0, 1\}^n$ its corresponding signed vector (i.e., $|\sigma| = \langle 2z_i\mathbf{e}_i : z_i \neq 0 \rangle$). Then σ has associated the 3-partition of $\mathcal{P}$ given by $A = \{\mathbf{p}_i \in \mathcal{P} : z_i = 1\}$, $B = \{\mathbf{p}_i \in \mathcal{P} : z_i = -1\}$ and $C = \overline{A \cup B} = \{\mathbf{p}_i \in \mathcal{P} : z_i = 0\}$ and, by the definition of δ, we have that $\delta(\sigma) = [A, A \cup C]$. Therefore, it is enough to prove that

$$\sigma \cap \ell \neq \phi \iff \langle A \rangle \cap \langle B \rangle \neq \phi.$$

For, let $\mathbf{x} \in \sigma \cap \ell$, then

$$\sum x_i \mathbf{p}_i = \mathbf{0}, \quad \sum x_i = 0 \quad \text{and} \quad \sum |x_i| = 2.$$

The first equation is due to $\mathbf{x} \in K$, the second because $\mathbf{x} \in \Pi$ (all these since $\mathbf{x} \in \ell = K \cap \Pi$) and the third one because $\mathbf{x} \in \mathcal{O}$. More over, since $\mathbf{x} \in \sigma$, we are aloud to write

$$\frac{1}{2}\mathbf{x} = \sum \lambda_i (z_i \mathbf{e}_i)$$

as a convex combination ($\sum \lambda_i = 1$ and $\lambda_i \geq 0$) of some canonic vectors or its corresponding negatives. Combining these ($x_i = 2z_i\lambda_i$) we have that

$$\sum_{\mathbf{p}_i \in A} \lambda_i \mathbf{p}_i = \sum_{\mathbf{p}_i \in B} \lambda_i \mathbf{p}_i$$

and

$$\sum_{\mathbf{p}_i \in A} \lambda_i = \sum_{\mathbf{p}_i \in B} \lambda_i = 1.$$

This last happens if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. Since all previous steps can be followed the other war around, we have conclude the proof of the claim, and therefore $\mathcal{R} \searrow \mathfrak{S}^{n-d-2}$. $\circ$

For the case of general position, observe that $\mathcal{R}$ has a face $[A, \overline{B}]$ of dimension greater than $n-d-2$ if and only if $|\overline{B} \setminus A| > n-d-2$ and this is equivalent to the existence of a partition $A \dagger B$ where $|A \cup B| < d+2$. If the separoid S is in general position this last is impossible (Lemma 2) then, since all facets have dimension $n-d-2$ we have that

$$|\mathcal{O} \sqcap \ell| = |\mathcal{O} \cap \ell|.$$

Therefore $\mathcal{R}$ is homeomorphic to the $(n-d-2)$-sphere $\mathcal{R} \cong \mathfrak{S}^{n-d-2}$ and we are done. $\bullet$

Notice that Theorem 6, for $n = d+2$ points in general position, implies that $\mathcal{R}$ is a sphere of dimension 0 (two antipodal vertices) which represents the unique Radon partition of this points. Therefore, Theorem 6 generalizes Radon's theorem —hence the name of the complex.

Theorem 6 can be extended to the more general class of oriented matroids using Edelman's theorem (1984) and Alexander duality; it is somehow the dual version of the Basic Sphericity Theorem and it was the first step to reach the characterization of the cocircuit graphs of uniform oriented matroids (see Montellano-Ballesteros & Strausz 2001 for the pertinent definitions):

Theorem 7 *A graph $\mathcal{G}$ is the (co)circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$ if and only if its order is $2\binom{n}{d+2}$ and there exists an antipodal i-metric embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that for every pair $X \neq \pm Y \in V(\mathcal{G})$, $d_{\mathcal{Q}_n^{n-d-2}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)|$, where $d_{\mathcal{Q}_n^{n-d-2}} \colon V(\mathcal{G})^2 \to \mathbb{N}$ denotes the graph distance of the 1-skeleton of the n-cube's $(n-d-2)$-skeleton's dual complex.*

3 Homomorphisms: operations with separoids

The first category defined in the class of separoids was that of functions $\varphi\colon S \longrightarrow T$ that "pull" the separation relation, i.e., for all $C, D \subseteq T$

$$C \mid D \Longrightarrow \varphi^{-1}(C) \mid \varphi^{-1}(D)$$

(cf. Arocha et al.). This is clearly equivalent to say that, for all $A, B \subseteq S$

$$A \dagger B \Longrightarrow \varphi(A) \dagger \varphi(B) \quad \text{or} \quad \varphi(A) \cap \varphi(B) \neq \phi.$$

An easy way to strength this definition is by asking to every Radon partition to map in a Radon partition (and not intersect); we will call them *strong morphisms:*

$$A \dagger B \Longrightarrow \varphi(A) \dagger \varphi(B) \quad (\text{and} \quad \varphi(A) \cap \varphi(B) = \phi).$$

Observe that, in the case of injective functions, this definition coincides with that of the first section and therefore the notion of isomorphism do not change. In this section we explore this and other categories defined on separoids.

3.1 The homomorphisms lattice

Given a category, if two objects are identified when there exist morphisms in both directions,

$$S \sim T \iff S \underset{\psi}{\overset{\varphi}{\rightleftarrows}} T\,,$$

a partially ordered class is obtained by defining $S \leq T \iff S \longrightarrow T$. Its elements are called *color classes* (cf. Hell & Nešetřil 1990). The category is called *dense* if for every $S < T$ there exists a P such that $S < P < T$. We are going to introduce a dense category on the class of separoids.

Since the constant function is a separoid morphism, the category of morphisms collapses into a single color class. In the light of this, we introduce a kind of morphisms that is a bit more restrictive; we call them *homomorphisms* because they resemble homomorphisms of relational systems (cf. Nešetřil & Tardif 2000). It can be proved (see Nešetřil & Strausz 2002) that the homomorphism category of separoids is *universal*, i.e., any partially ordered class —hence the existence of morphisms in any category— can be represented by the existence of separoids homomorphisms.

It is very easy to see (cf. Section 2) that a separoid S of order $n \in \mathbb{N}$ can be defined as an antipodal filter in the face lattice of the n-crosspolytope

$$S \subseteq \mathcal{O}_n = (\{-,0,+\}^n, \prec)$$

or by duality, as an antipodal ideal of the n-cube $S \subseteq \mathcal{Q}_n$. Recall that it is enough to know *minimal* Radon partitions to reconstruct all Radon partitions, therefore we can concentrate on the study of them. In particular, when defining an operation, it is enough to define some (minimal Radon) partitions and close the separoid to became a filter. To emphasize this, let $\mathcal{S}$ be the family of minimal Radons partitions. In other words, $\mathcal{S}$ will denote the set of generators of the antipodal filter $(S, \dagger, \prec)$, where $A \dagger B \preceq C \dagger D$ iff $A \subseteq C$ and $B \subseteq D$. Example 3 motivates the following definition.

A separoid *homomorphism* $\varphi\colon S \longrightarrow T$ is a function that sends minimal Radon partitions into minimal Radon partitions, i.e., for every $A, B \subseteq S$

$$A \dagger B \in \mathcal{S} \Longrightarrow \varphi(A) \dagger \varphi(B) \in \mathcal{T}.$$

Clearly these functions define a category on the class of all separoids. In fact it is a subcategory of separoids with *morphisms* in the sense of Section 1.2. This definition has to be contrasted with that of strong morphisms; here we are asking only to the minimal partitions not to collapse.

For the rest of the section, $S \longrightarrow T$ will denote the fact that there exists an homomorphism. Also, as mention above, we identify two separoids whenever there exist homomorphisms in both directions and call the resulting ordered class the *homomorphism order*.

The homomorphisms order is in fact a lattice. The category of separoids homomorphisms has products $\times$ and sums $+$ and they play the role of the meet (infimum) and the joint (supremum), respectively.

$$\begin{tikzcd}
& P+T & \\
P \arrow[ur] & & T \arrow[ul] \\
& P \times T \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

They satisfy the categoric properties of products and coproducts:

- $S \longrightarrow P \times T \iff S \longrightarrow P$ and $S \longrightarrow T$,
- $P + T \longrightarrow S \iff P \longrightarrow S$ and $T \longrightarrow S$,

and they have the following internal definitions.

Given two separoids P and T, their *product* is a separoid defined in the cartesian product $P \times T$, with projections π and τ respectively, such that for every $A, B \subseteq P \times T$

$$A \dagger B \in \mathcal{P} \times \mathcal{T} \iff \pi(A) \dagger \pi(B) \in \mathcal{P} \quad \text{and} \quad \tau(A) \dagger \tau(B) \in \mathcal{T}.$$

Given two separoids P and T, their *sum* is a separoid defined in the disjoin union $P \cup T$ such that for every $A, B \subseteq P \cup T$

$$A \dagger B \in \mathcal{P} + \mathcal{T} \iff A \cap P \dagger B \cap P \in \mathcal{P} \quad \text{xor} \quad A \cap T \dagger B \cap T \in \mathcal{T}.$$

It is easy to see that this category do not collapses. For this, just observe that

$$S \longrightarrow K_1 \iff S \approx \sigma^d.$$

And more over, this is a dense lattice (look for the details in Nešetřil & Strausz 2002)

Theorem 8 *Given two separoids such that $S < T$ there exists another one P such that*

$$S < S + (P \times T) < T.$$

3.2 Radon's theorem: a categorical version

If we restrict more our homomorphisms to consider only those which are also strong morphisms, we can characterize Radon's theorem in the following

Theorem 9

$$S \not\longrightarrow K_1 \quad \textit{and} \quad S \longrightarrow K_2 + \sigma^d \iff S \approx P,$$

where $P \subset \mathbb{E}^{n-1}$ is a point separoid of order $n = \mathrm{d}(P) + 2$. More over, $d = -1$ if and only if P is in general position.

Proof. A separoid S is a point separoid of order $\mathrm{d}(S) + 2$ if and only if it is determined by a unique minimal Radon partition $A \dagger B$ (cf. Theorem 6). Let $C = S \setminus (A \cup B)$ be the complement of the support and give it an arbitrary (but fixed) linear order $C = (c_0, \ldots, c_d)$. Now, let $K_2 = \{a, b\}$, where $a \dagger b$, and $\sigma^d = \{c'_0, \ldots, c'_d\}$. Clearly the function $\varphi\colon S \to K_2 + \sigma^d$, where

$$\varphi(s) = \begin{cases} a & \text{if } s \in A, \\ b & \text{if } s \in B, \\ c'_i & \text{if } s = c_i, \end{cases}$$

is a *strong* homomorphism of separoids. More over, if this is the case, S is in general position if and only if $A \cup B = S$ and therefore $C = \phi$. •

However, in this subcategory there is not any more a meaningful notion of product which made out of the projections, strong homomorphisms. To see this, consider the separoids $\Lambda_3 = \{0, 1, 2\}$ where $0 \dagger 12$, and $K_2 = \{a, b\}$ where $a \dagger b$. Let us denote by $\Lambda_3 \times K_2 = \{0a, 0b, 1a, 1b, 2a, 2b\}$ the elements of the product and by λ and κ the two projections. If $A \dagger B$ most imply that $\lambda(A) \dagger \lambda(B)$ and $\kappa(A) \dagger \kappa(B)$ then the natural candidates to A and B are $A = \{0a\}$, $B = \{1b, 2b\}$ but, since the relation is a filter, this would imply that $A \dagger B \cup \{0b\}$ but

$$\lambda(0a) \cap \lambda(0b, 1b, 2b) = \{0\} \cap \{0, 1, 2\} = \{0\} \neq \phi,$$

therefore $\Lambda_3 \times K_2 \approx \sigma^5 \sim K_1$. That is, in this subcategory most separoids "meets" in the singleton.

3.3 3-cromorphisms

We are going to introduce one more kind of morphisms (recall Example 11). First we give some names to all acyclic separoids of order 3 (modulo isomorphism) and show their monomorphisms —the names are intended to remind us their "shape"— (cf. Figure 1):

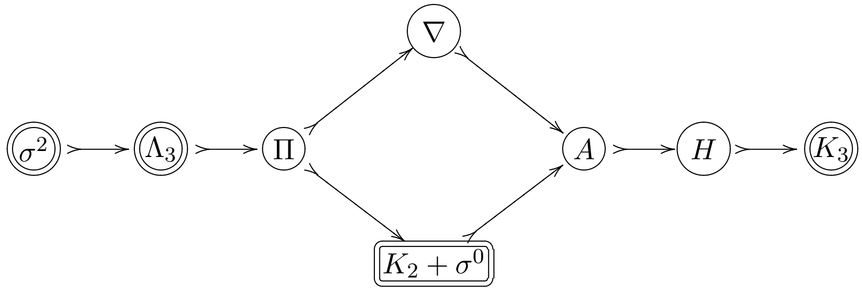

Diagram 1. *The acyclic separoids of order 3 and their monomorphisms.*

Observe that only σ^2, Λ_3, $K_2+\sigma^0$ and K_3 are point separoids.

Now, consider a separoid $(S,\dagger)$ of convex sets in $\mathbb{E}^d$. If we give a 3-coloration of its elements $\varsigma\colon S \twoheadrightarrow \{0,1,2\}$ and consider the convex hulls of each color class, then we are constructing an epimorphism onto one of these eight separoids of order 3. These morphisms satisfies the extra property that the preimage of minimal Radon partitions are Radon partitions. Such epimorphisms will be called *cromorphisms*.

As an example, let us see how this works for the point separoids of order 4 and dimension 2. There are four of them: χ_4 and Δ_4 will denote the general position point separoids of order four with unique Radon partitions of the form $12 \dagger 34$ and $1 \dagger 234$, respectively; $\Lambda_3+\sigma^0$ denotes the three colinear points with a fourth one not in line; and $K_2+\sigma^1$ is represented with the vertices of a triangle, one of them doubled. It is easy to see that we have the following combinations (where the number on each arrow counts the number of cromorphisms $|\varsigma\colon S \longrightarrow T|$):

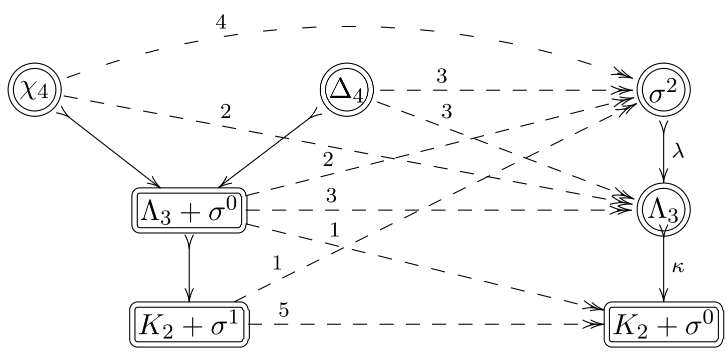

Diagram 2. *The 3-cromorphisms of 4 point in the plane.*

Observe that such cromorphisms do not commute with the monomorphisms λ and κ.

4 Hyperseparoids: Tverberg's theorem

In the rest of this paper, we focus on the famous generalization of Radon's theorem due to Tverberg (1966):

Theorem 10 *Let $P \subset \mathbb{E}^d$ be a set of $(k-1)(d+1)+1$ points. Then P can be divided into k pairwise disjoint sets $P = P_1 \cup \cdots \cup P_k$ whose convex hulls have a common point:*

$$\bigcap \langle P_i \rangle \neq \phi.$$

As we shortly will see, the magic number $(k-1)(d+1)+1$ is best possible.

The partition $P = P_1 \cup \cdots \cup P_k$ will be called a *Tverberg partition*. The reader can have a look to Eckhoff's (1993) sec. **9.3** to read more about Tverberg's theorem and its relatives. To the references there, I should add those of Bárány & Onn (1997), Matoušek (1999), Kalai (2000) and Sarkaria (2000).

Clearly, Theorem 10 reduces to Theorem 1 when $k = 2$, and for $k = 1$ it is trivial. However, even for $k = 3$, it is a hard —and deep— result. The simplest proof known by the author is based in a variant of Sarkaria's (1992) argument and uses the colorful version of Carathéodory's theorem due to Bárány (1982). It seems that, contrasting Radon's theorem which only depends on the affine structure of $\mathbb{R}^d$, Tverberg's theorem is deeply tied to the metric (and topological) properties of the Euclidean d-space.

A simple consequence of Tverberg's theorem, besides Theorem 2, is the following

Corollary 11 *If S is a separoid of order $(k-1)(\mathrm{gd}(S)+1)+1$, then there exists a cromorphism onto the complete separoid of order k: $|\varsigma\colon S \longrightarrow K_k| > 0$.*

Proof. Let us denote by $K_k = \{1, \ldots, k\}$ the elements of the complete separoid of order k and let S be a separoid of $(k-1)(d+1)+1$ convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. For any choice $\varphi\colon P \longrightarrow S$, due to Theorem 10, there exists a partition $P = P_1 \cup \cdots \cup P_k$ such that $\bigcap \langle P_i \rangle \neq \phi$. Clearly the coloration $\varsigma\colon S \to K_k$ defined as $\varsigma(s) = i \iff \varphi^{-1}(s) \in P_i$ has the desired properties. •

Observe that this result is far from imply Theorem 10 (cf. the two realizations of K_3 given in Figure 1.**h** and Figure 2.**d**). A naïve first look may suggest that it is weaker to ask for the existence of a k-partition whose convex hulls are isomorphic to K_k than to ask for a k-partition whose convex hulls have a common point —think on the vertices of a regular hexagon and perturb them a bit— and in this direction we may be tempted to reduce Tverberg's number, say to $(k-1)(d+1)$. However it is ease to see that the six points in the plane given by the vertices of a regular pentagon and it baricenter, cannot be partitioned in three sets such that the convex hulls of the parts are isomorphic to K_3.

Another direction may be to try to prove (or disprove) the following

Conjecture 2. *If S is a separoid of order $(k-1)(\mathrm{d}(S)+1)+1$, then there exists a cromorphisms onto the complete separoid of order k: $|\varsigma\colon S \longrightarrow K_k| > 0$.*

The first non-trivial example, known to be true [], is: Let $\mathcal{F}$ be a family of 5 convex sets in the plane $\mathbb{E}^2$. If every 3 elements of $\mathcal{F}$ admits a line transversal, then there exists a 3-coloration $\varsigma\colon \mathcal{F} \to \{0, 1, 2\}$ such that the convex hull of each two color classes intersect: $\langle \varsigma^{-1}(i) \rangle \cap \langle \varsigma^{-1}(j) \rangle \neq \phi$.

It seems that, while the existence of a Tverberg partition depends on the realization, the existence of a cromorphism onto K_k do not (see Figure 4).

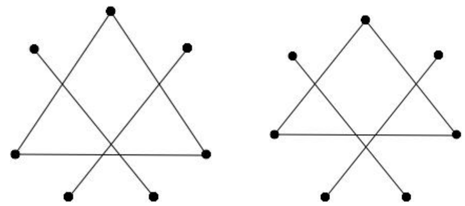

Figure 4. *Two configurations of seven points in the plane.*

The rest of this section is a first attempt to understand the combinatorial structure of Tverberg's partitions.

4.1 Stangeland's theorem

The example on Section 3.3 suggested the following Tverberg-type theorem for transversals. It basically says that, for a point separoid of order $d+2$, there is always a cromorphism onto the simploid σ^2 and, there is a cromorphism onto Λ_3 or onto $K_2+\sigma^0$. Therefore there is a coloration which admits a line transversal to each color class' convex hull.

Theorem 12 *Let $d>1$. If S is the separoid of $d+2$ points in $\mathbb{E}^d$, then*

$$|\varsigma\colon S \longrightarrow \sigma^2|\left(|\varsigma\colon S \longrightarrow \Lambda_3| + |\varsigma\colon S \longrightarrow K_2+\sigma^0|\right) > 0.$$

Proof. Given $d+2$ points $P \subset \mathbb{E}^d$, due to Radon's theorem, its separoid $S=(P,\dagger)$ is determined by a unique minimal partition $A \dagger B$ (cf. Theorem 6). To construct a cromorphism onto σ^2, take an element in each part $a \in A$, $b \in B$ and give any separation of the complement $\alpha \mid \beta$. It is easy to see that the coloration

$$\varsigma(x) = \begin{cases} 0 & x \in \{a,b\}, \\ 1 & x \in \alpha, \\ 2 & x \in \beta, \end{cases}$$

has the desired properties. Therefore, the first factor is non-zero.

If there is some element in the complement of the support $A \cup B$ (i.e., the separoid is not in general position), say $C = X \setminus (A \cup B)$, then the coloration

$$\varsigma(x) = \begin{cases} 0 & x \in A, \\ 1 & x \in B, \\ 2 & x \in C, \end{cases}$$

(cf. Theorem 10) is clearly a cromorphism onto $K_2+\sigma^0$ (with $0 \dagger 1$) and the second factor is non-zero. If not (i.e., S is in general position), A or B has more than one element, say A. Let $A_0 \cup A_1$ be a partition of A. It is easy to see that the coloration

$$\varsigma(x) = \begin{cases} 0 & x \in A_0, \\ 1 & x \in A_1, \\ 2 & x \in B, \end{cases}$$

is a cromorphism onto Λ_3 (with $2 \dagger 01$) and therefore the second factor is non-zero and we are done. •

Observe how the fact that the second factor is never zero implies, in the case $k=3$ and $\ell=1$, Stangeland's (1978) generalization of Tverberg's theorem:

Theorem 13 *Let $P \subset \mathbb{E}^d$ be a set of $(k-\ell-1)(d-\ell+1)+\ell+1$ points. Then, there exists a k-partition of the set $P = P_0 \cup \ldots \cup P_k$ and an ℓ-dimensional affine subspace L such that*

$$\langle P_i \rangle \cap L \neq \phi, \qquad \textit{for } i = 1, \ldots, k.$$

Proof ($k = 3$ and $\ell = 1$). Let P be as in the statement and S its separoid. Due to Theorem 12, there exists a coloration $\varsigma\colon S \longrightarrow T$ which makes the second factor non-zero, and the color clasess' convex hulls $\langle \varsigma^{-1}(t) \rangle$, $t \in T$, realize the corresponding separoid ($T \in \{\Lambda_3,\ K_2 + \sigma^0\}$). Finally observe that, any realization of the separoids Λ_3 and $K_2 + \sigma^0$ admits a line transversal. •

It is easy to build such kind of proofs (for $\ell > 0$) when we realize that, for example; almost any separoid of order 4 admits a plane transversal, many of them admit even a line transversal, and so on... however this method does not seems to take us too far. In particular, the case $\ell = 0$ will never be reached, even though we find how to guarantee a cromorphisms onto K_k.

That is, the structure of Tverberg partitions of a separoid is not encoded (at least, trivially) in its Radon partitions (e.g., consider the construction of Theorem 2 applied to the separoid K_3).

This fact motivates the definition of

4.2 k-separoids

A *k-separoid* is a relational system $\dagger \subseteq 2^S \times \cdots \times 2^S$ (k times) defined on a family of subsets with the following properties, for $A_i \subseteq S$, $i = 1, \ldots, k$

$$\begin{array}{cc} \circ & A_1 \dagger \cdots \dagger A_k \Longrightarrow A_{\pi(1)} \dagger \cdots \dagger A_{\pi(k)} \\ \circ\circ & A_1 \dagger \cdots \dagger A_k \Longrightarrow A_i \cap A_j = \phi, \quad 1 \leq i < j \leq k \\ \circ\circ\circ & A_1 \dagger \cdots \dagger A_k \text{ and } B \subseteq S \setminus \bigcup A_i \Longrightarrow A_1 \dagger \cdots \dagger A_k \cup B \end{array}$$

where π is any permutation of the indices. The elements of such a relational system will be called *Tverberg partitions*. Clearly separoids are 2-separoids. As before, we identify the k-separoid with the given set S. We say that the separoid is *acyclic* if $A_1 \dagger \cdots \dagger A_k \Longrightarrow \prod |A_i| > 0$.

Given k pairwise disjoint subsets of a k-separoid, $\alpha, \ldots, \gamma \subseteq S$, which are not a Tverberg partition, we say that they are a *k-separation* and denote it by $\alpha \mid \cdots \mid \gamma$.

Theorem 14 *Every acyclic k-separoid of order n can be represented with a family of convex polytopes and their Tverberg partitions in the $(n-1)$-dimensional affine space.*

Proof. The case $k = 2$ was Theorem 2. Now, in order to keep things simple, only the proof for $k = 3$ is presented. But —with the apropiate notation— the general case is totally analogous.

Let S be an acyclic 3-separoid. For each Tverberg partitions $A \dagger B \dagger C$ and each element $i \in A$, we assign a point of $\mathbb{R}^n$

$$\rho^i_{A\dagger B\dagger C} = \mathbf{e}_i + \frac{1}{3}\left[\frac{1}{|A|}\sum \mathbf{e}_a + \frac{1}{|B|}\sum \mathbf{e}_b + \frac{1}{|C|}\sum \mathbf{e}_c\right] - \frac{1}{|A|}\sum \mathbf{e}_a,$$

and realize each element $i \in S$ as the convex hull of all such points

$$i \mapsto \langle \rho^i_{A\dagger B\dagger C} : i \in A \text{ and } A \dagger B \dagger C \rangle.$$

These convex polytopes "live" in the $(n-1)$-dimensional affine subspace spanned by the basis.

The construction is made to guarantee that the Tverberg partitions are preserved, i.e., for each partition $A \dagger B \dagger C$ the vertices of the simplices $\langle \mathbf{e}_a : a \in A\rangle$, $\langle \mathbf{e}_b : b \in B\rangle$ and $\langle \mathbf{e}_c : c \in C\rangle$ moves to realize such a partition intersecting precisely in their baricenter, therefore

$$\langle \rho^a \rangle \cap \langle \rho^b \rangle \cap \langle \rho^c \rangle \neq \phi.$$

On the other hand, to prove that also the 3-separations $\alpha \mid \beta \mid \gamma$ are preserved, we use the following well-known fact: compact convex sets $\mathcal{K}_1, \ldots, \mathcal{K}_n$ in $\mathbb{R}^d$ have no point in common if and only if there are open semispaces $\ell_+^1, \ldots, \ell_+^n$ such that $\mathcal{K}_i \subset \ell_+^i$ for every i and $\bigcap \ell_+^i = \phi$. The case $n = 2$ is the basic separation theorem and the general case follows by induction (cf. Bárány & Onn 1997).

Define the affine extension $\psi = \psi_{\alpha|\beta|\gamma} \colon \mathbb{R}^n \to \mathbb{R}^2$ of the following equations, for $j = 1, \ldots, n$,

$$\psi(\mathbf{e}_j) = \begin{cases} \mathbf{u} & \text{if } j \in \alpha \\ \mathbf{v} & \text{if } j \in \beta \\ \mathbf{w} & \text{if } j \in \gamma \\ \mathbf{0} & \text{otherwise} \end{cases}, \quad \text{where } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \text{ and } \mathbf{w} = \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}.$$

Observe that the points $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subset \mathbb{S}^1$ lives in the unitary circle and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$. It follows from the definition, and with a little abuse of the notation, that

$$\psi(\rho^i) = \psi(\mathbf{e}_i) - \frac{2}{3|A|}\begin{pmatrix} |A \cap \alpha| \\ |A \cap \beta| \\ |A \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} + \frac{1}{3|B|}\begin{pmatrix} |B \cap \alpha| \\ |B \cap \beta| \\ |B \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} + \frac{1}{3|C|}\begin{pmatrix} |C \cap \alpha| \\ |C \cap \beta| \\ |C \cap \gamma| \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}.$$

Let us denote by $\psi_\alpha = \psi(\rho^i)$ when $i \in \alpha$ and analogously with β and γ.

If we have that $\psi_\alpha \cdot \mathbf{u} > 0$ and $\psi_\beta \cdot \mathbf{v} > 0$ and $\psi_\gamma \cdot \mathbf{w} > 0$, we are done (the semispaces $\psi^{-1}(\mathbf{u}_+^\perp)$, $\psi^{-1}(\mathbf{v}_+^\perp)$ and $\psi^{-1}(\mathbf{w}_+^\perp)$ will do). So let us suppose, with out loss of generality, that $\psi_\alpha \cdot \mathbf{u} = 0$. Since

$$\psi_\alpha \cdot \mathbf{u} = 1 - \tfrac{2|A\cap\alpha| - (|A\cap\beta| + |A\cap\gamma|)}{3|A|} + \tfrac{|B\cap\alpha| - \frac{1}{2}(|B\cap\beta| + |B\cap\gamma|)}{3|B|} + \tfrac{|C\cap\alpha| - \frac{1}{2}(|C\cap\beta| + |C\cap\gamma|)}{3|C|} \geq 0,$$

we have that $\psi_\alpha \cdot \mathbf{u} = 0$ if and only if $A \subseteq \alpha$ and $B \subseteq \beta \cup \gamma$ and $C \subseteq \beta \cup \gamma$. In such a case, we have also that

$$\psi_\beta \cdot \mathbf{v} = 1 + \frac{1}{3} + \frac{1}{3}\left[\left(\frac{|B \cap \beta|}{|B|} + \frac{|C \cap \beta|}{|C|}\right) - \frac{1}{2}\left(\frac{|B \cap \gamma|}{|B|} + \frac{|C \cap \gamma|}{|C|}\right)\right] \geq 1$$

and, analogously, $\psi_\gamma \cdot \mathbf{w} \geq 1$. Then we can pick any small number $0 < \epsilon < 1$, define the semispaces

$$\ell_+^\alpha = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{u} > -\epsilon\}, \quad \ell_+^\beta = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{v} > 1 - \epsilon\}, \quad \ell_+^\gamma = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{w} > 1 - \epsilon\},$$

and their preimages, $\psi^{-1}(\ell_+^\alpha)$, $\psi^{-1}(\ell_+^\beta)$ and $\psi^{-1}(\ell_+^\gamma)$, will do the work, concluding the proof. •

As an example, consider the 3-separoid of order 5 with unique Tverberg partition $1 \dagger 23 \dagger 45$ (see Figure 6, on the right). The construction in the previous theorem, realizes this point separoid with the column vectors of the matrix

$$\frac{1}{6}\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 4 & -2 & 1 & 1 \\ 1 & -2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & -2 \\ 1 & 1 & 1 & -2 & 4 \end{pmatrix},$$

and the image of these points under the map $\psi_{1|24|35}$ is given by the column vectors of the matrix

$$3\sqrt{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

Every 3-separoid has associated a 2-separoid in a natural way: each Tverberg partition $A \dagger B \dagger C$, implies the Radon partitions $A \dagger B$, $A \dagger C$ and $B \dagger C$. This separoid is alredy realized with the construction of Theorem 14. However, we miss some structure; e.g., consider the separoid of five points in the line in general position, and give the points the linear order $(1, 2, 3, 4, 5)$. This configuration has two Tverberg partitions: $14 \dagger 25 \dagger 3$ and $15 \dagger 24 \dagger 3$. If we apply the previous construction, in the final family of convex sets we will miss some Radon partitions, for example $13 \dagger 2$. To correct this 'anomaly', we can go one step further in our generalization of separoids with the following natural definition.

A *hyperseparoid* is a collection of families of subsets $\mathcal{T} \subseteq 2^{2^S}$ with the following three properties: for all $A_i \subseteq S$, $i = 1, \dots, k$

$$\begin{array}{cc} \circ & \{A_1, \dots, A_k\} \in \mathcal{T} \Longrightarrow A_i \cap A_j = \phi \\ \circ\circ & \{A_1, \dots, A_k\} \in \mathcal{T} \Longrightarrow \{A_1, \dots, A_{k-1}\} \in \mathcal{T} \\ \circ\circ\circ & \{A_1, \dots, A_k\} \in \mathcal{T} \text{ and } B \subseteq S \setminus \bigcup A_i \Longrightarrow \{A_1, \dots, A_k \cup B\} \in \mathcal{T} \end{array}$$

The elements of $\mathcal{T}$ are the Tverberg partitions. The hyperseparoid is *acyclic* if $\{\phi\} \notin \mathcal{T}$. From the second and third axioms follows that it is enough to know the *principal* partitions; those partitions $\{A_1, \dots, A_k\}$ where k is maximal and each A_i is minimal. The *morphisms* and *homomorphisms* can be defined analogously as before.

Clearly, we can use Theorem 14 several times (for each $k = 2, 3, \dots n$) to conclude that

Corollary 15 *Every acyclic hyperseparoid can be represented with convex polytopes.* •

5 Final remarks and problems

Hyperseparoids seems to be "the right concept" to study Tverberg's Theorem from a purely combinatorial point of view, but this will have to be done some where else... Here we formulate some questions which may guide such a further development.

Let us start with the most challenge (and may be difficult) one. In the spirit of Theorem 9,

1 *Find necessary and sufficient conditions for a hyperseproid to be a point separoid.*

In the light of Shor's theorem (1991), problem **1** possibly is NP-hard, however it may have a simple solution as the following argument suggest. Consider a realization of a full Radon hyperseparoid S with convex sets as "thin" as possible; if each convex set is a point, we are done. If there exist a convex set $\mathcal{K} \in S$ with dimension greater that 0, it will contain at least one segment $\langle \mathbf{a}, \mathbf{b} \rangle \subseteq \mathcal{K}$. The extreme points of such a segment, have to be participating in two different principal partitions, say $\mathbf{a} \dagger A_1 \dagger \cdots \dagger A_k$ and $\mathbf{b} \dagger B_1 \dagger \cdots \dagger B_k$, which are "far" each from the other... they are "separated". So it may be sufficient to ask for a condition of the form *if* $\mathbf{a} \dagger A_1 \dagger \cdots \dagger A_k$ *and* $\mathbf{b} \dagger B_1 \dagger \cdots \dagger B_k$

are principal then $A_i \dagger B_j \setminus A_i$ *or* $B_j \dagger A_i \setminus B_j$, in order to guarantee that S is a point separoid (see Figure 5).

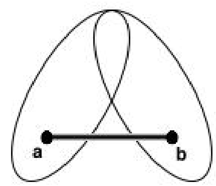

Figure 5. *A "minimal" segment whose extreme points are "separted".*

The next problem has to do with an invariant which may be called *Tverberg dimension*. Given a hyperseparoid S, define $\mathrm{d}_k(S)$ as the minimum natural number d such that every subset $X \subseteq S$ of cardinality $(k-1)(d+1)+1$ contains a k-partition $A_1 \dagger \cdots \dagger A_k$. Clearly, $\mathrm{d}(S) = \mathrm{d}_2(S)$ and $\mathrm{d}_k(S) \leq \mathrm{gd}(S) \leq |S| - 1$, but no more can be said, at least in principle (see Figure 6). So, in the spirit of Theorem 5,

2 *Find necessary and sufficient conditions to guarantee that*

$$\mathrm{d}(S) = \mathrm{d}_2(S) = \mathrm{d}_3(S) = \cdots = \mathrm{gd}(S) \leq |S| - 1.$$

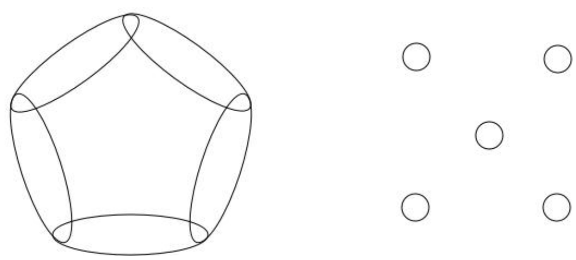

Figure 6. *Two separoids with different values of* $\mathrm{d}_2(S)$ *and* $\mathrm{d}_3(S)$*.*

Finally, let me present a problem whose character may look more technical (cf. Nešetřil 1999). For each separoid S, define the infinite vector $\Upsilon(S) \in \mathbb{N}^{\aleph}$ whose coordinates are indexed by finite separoids (modulo isomorphism) and each $\Upsilon(S)_T = |\varsigma\colon S \longrightarrow T|$ counts the number of homomorphisms (cromorphisms, strong morphisms). This definition has to be contrasted with that of Lovász (1971) where he proved that, with the arrows in the oposite direction, such a vector characterizes each object of a relational system.

3 *Is it true that* $S \approx T$ *(or* $S \sim T$*) if and only if* $\Upsilon(S) = \Upsilon(T)$*?*

If we restric to finite families of separoids, the answer may be negative as the following (and last) diagram shows. In it, χ_5 and Δ_5 denotes the general position point separoids with Radon partitions $12 \dagger 345$ and $1 \dagger 2345$, respectively.

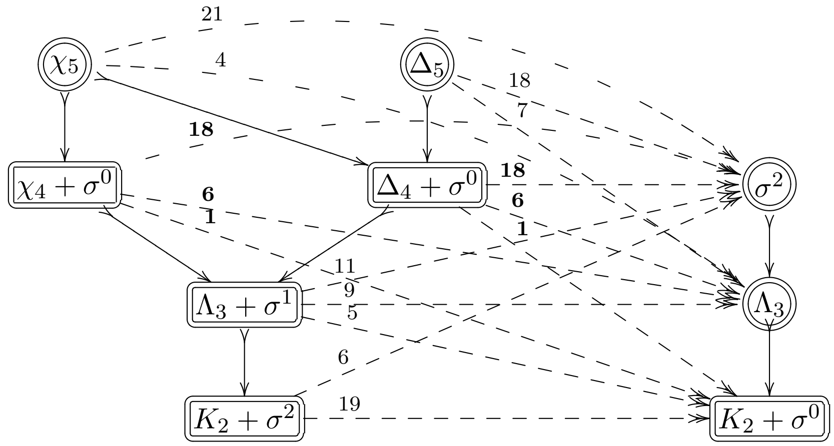

Diagram 3. *The 3-cromorphisms of 5 point in the space.*

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