

Covering the Convex Quadrilaterals of Point Sets

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Abstract. For a point set P on the plane, a four element subset $S \subset P$ is called a *4-hole* of P if the convex hull of S is a quadrilateral and contains no point of P in its interior. Let R be a point set on the plane. We say that a point set B *covers* all the 4-holes of R if any 4-hole of R contains an element of B in its interior. We show that if $|R| \geq 2|B| + 5$ then B cannot cover all the 4-holes of R . A similar result is shown for a point set R in convex position. We also show a point set R for which any point set B that covers all the 4-holes of R has approximately $2|R|$ points.

Key words. k -Hole, Bicolored point set, Covering

1. Introduction

Throughout this paper, P denotes a point set on the plane. P is said to be in *general position* if no three of its elements lie on a line. We denote by $\text{Conv}(P)$ the convex hull of P , and we say that P is in *convex position* if P is in general position and all of its elements lie on the boundary of $\text{Conv}(P)$.

For an integer $k \geq 3$ and any point set P on the plane, a k -subset S of P is called a *k-hole* of P if S is in convex position and no element of P lies in the interior of $\text{Conv}(S)$. A *k-hole* is often identified with its convex hull. In 1931, Esther Klein proved that any point set P in general position with at least 5 elements contains a 4-subset in convex position [4,6]. It is easy to see that it also contains a 4-hole.

We say that P is a *bicolored* point set if P is the union of disjoint point sets R and B . Call the elements of R and B the red and blue points of P respectively. A *monochromatic 4-hole* of P is a 4-hole of P such that all its elements are either in R or in B . In [2], O. Devillers et al. showed many results on k -holes of m -colored point sets, $k \geq 3$, $m \geq 2$. Concerning monochromatic 4-holes of a bicolored point set, they conjecture:

Conjecture A. *Let P be a bicolored point set in general position consisting of a sufficiently large number of points. Then P contains a monochromatic 4-hole.*

For a 4-hole S of a red point set R and a blue point $b \in B$, we say that b *covers* S if $\text{Conv}(S)$ contains b in its *interior*. Furthermore, we say that B covers all the 4-holes

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of R if each 4-hole of R is covered with a point of B . In this paper, we will study the following question: Given a red point set R , how many points must a blue point set B have so that it covers all the 4-holes of R ? Our objective here is to continue the study of a similar problem concerning coverings of 3-holes of point sets. The problem of coverings of 3-holes has been studied by M. Katchalski et al.[5] and independently by J. Czyzowicz et al.[1]. They proved:

Theorem A. *For any red point set R in general position, $2|R| - K - 2$ blue points are necessary and sufficient to cover all 3-holes of R , where K denotes the number of points of R on the boundary of $\text{Conv}(R)$.*

Observe that any triangulation of R (in the language of this paper, a set of 3-holes of R with disjoint interiors such that their union is $\text{Conv}(R)$) contains exactly $2|R| - K - 2$ triangles. Thus it follows that if a point set B of $2|R| - K - 2$ blue points covers all the triangles of R , then each 3-hole of R contains exactly one element of B .

In Section 2 we prove the following results:

Theorem 1. *Let $P = R \cup B$ be a bicolored point set such that R is in general position. Then if $|R| \geq 2|B| + 5$, P contains a red 4-hole.*

(The inequality of Theorem 1 is sharp when $|B| = 0$ and $|B| = 1$.) If R is in convex position, we obtain a better bound.

Theorem 2. *Let $P = R \cup B$ be a bicolored point set such that R is in convex position. Then if $|R| \geq \frac{3}{2}|B| + 4$, P contains a red 4-hole.*

For a red point set R , let $\beta(R)$ denote the minimum number of blue points that cover all 4-holes of R . From Theorems 1 and 2, it follows that $\beta(R) > \frac{|R|-5}{2}$ for any red point set R in general position, and $\beta(R) > \frac{2|R|-8}{3}$ for any red point set R in convex position. In Section 4 we prove:

Theorem 3. *Let n be a positive integer. Then*

$$\max_{\substack{|R|=n \\ R \text{ is in convex position}}} \beta(R) = n + o(n).$$

For a positive integer n , let

$$\beta_n = \max_{|R|=n} \beta(R).$$

It follows from Theorem A that $\beta_n \leq 2n - 5$ (it is not difficult to show $\beta_n \leq 2n - 6$ for the problem of covering red 4-holes). In Section 4, we also show that there exist red point sets R for which approximately $2|R|$ blue points are needed to cover all 4-holes of R :

Theorem 4. $\beta_n = 2n + o(n)$.

2. Proof of Theorems 1 and 2

For a point set $\{p_1, p_2, \dots, p_m\}$, we denote by $p_1p_2 \dots p_m$ the polygon whose vertices in the clockwise are p_1, p_2, \dots, p_m .

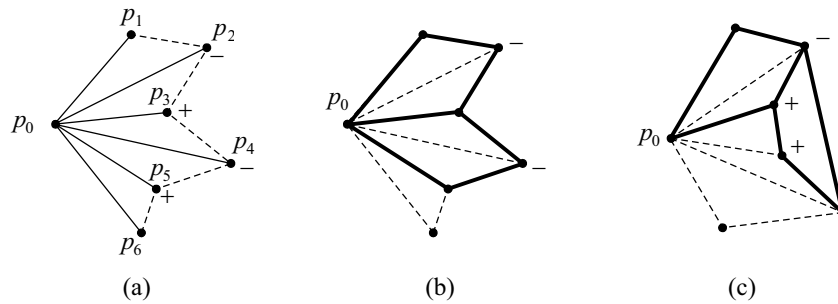


Fig. 1. Labeling of the points and two convex quadrilaterals with disjoint interiors.

2.1. Proof of Theorem 1

First note that B covers every red 4-hole if and only if so does a blue point set obtained by any slight perturbation of points in B . Thus it suffices to prove Theorem 1 for the case where $P = R \cup B$ is in general position.

We proceed by induction on $|B|$. The result is true for the case where $|B| = 0$ (the Esther Klein's Theorem mentioned above). Next we show the result for the case where $|B| = 1$. For this purpose, it suffices to prove the following proposition:

Proposition 1. *Any point set R with exactly seven elements in general position contains the vertices of two convex quadrilaterals with disjoint interiors.*

Proof. Choose the leftmost vertex on the convex hull of R , assuming without loss of generality that this point is unique, and let it be labeled p_0 . Label the elements of $R - \{p_0\}$ by p_1, \dots, p_6 in descending order according to the slope of the segments joining p_i to p_0 , $i = 1, \dots, 6$; see Fig. 1(a). For each $i = 2, \dots, 5$, assign the signature $+$ or $-$ to p_i according to whether the inner angle at p_i of the quadrilateral $p_0p_{i-1}p_i p_{i+1}$ is greater than or less than 180° .

If the sequence of signatures assigned to p_2, \dots, p_5 contains two non-consecutive minus signs, then our result follows (Fig. 1(b)). Our result also follows if the sequence contains a minus sign and consecutive plus signs, see Fig. 1(c). The remaining cases to be analyzed, are for the sequences $+- -+$ or $++++$. For the latter case, $p_1p_2p_3p_4$ and $p_1p_4p_5p_6$ are convex quadrilaterals with disjoint interiors.

Assume then that our sequence is $+- -+$. Let l denote the straight line connecting p_2 and p_5 . If at least one of p_1 or p_6 is in the same side of l as p_0 , then $p_2p_3p_4p_5$ and at least one of $p_0p_1p_2p_5$ or $p_0p_2p_5p_6$ are convex quadrilaterals with disjoint interiors. Thus assume that both of p_1 and p_6 are in the opposite side of l to that containing p_0 . Let m denote the straight line connecting p_3 and p_4 , and D the half-plane bounded by m and containing p_2 and p_5 . If $p_1 \in D$ (resp. $p_6 \in D$), then $p_1p_2p_4p_3$ and $p_0p_2p_4p_5$ (resp. $p_3p_5p_6p_4$ and $p_0p_2p_3p_5$) are convex quadrilaterals with disjoint interiors. Assume next that $p_1 \notin D$ and $p_6 \notin D$. In this case, $p_1p_3p_4p_6$ and $p_2p_3p_4p_5$ are convex quadrilaterals with disjoint interiors. \square

Returning to the proof of Theorem 1, we consider the case where there are at least two blue points. Let p and p' be consecutive vertices of the boundary of $\text{Conv}(B)$, and let D denote a half-plane bounded by the straight line pp' , and containing no element of $B - \{p, p'\}$. If D contains at least five red points, then the result follows from the Esther Klein's Theorem. Thus we may assume that D contains at most four red points. Then if

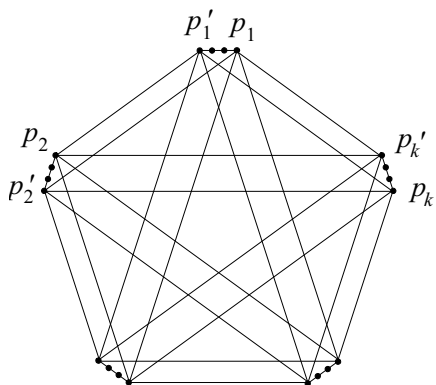


Fig. 2. p_i and p'_i .

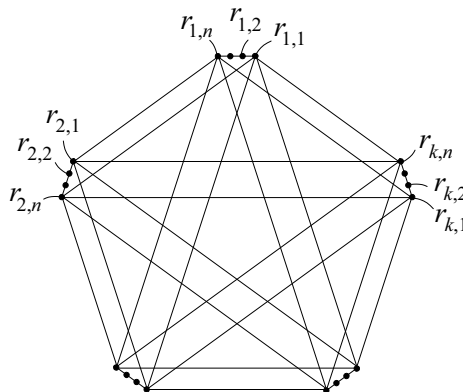


Fig. 3. Red point set $R = R(k, \varepsilon, n)$.

we write $D' = \mathbb{R}^2 - D$, $|R \cap D'| \geq |R| - 4 \geq 2(|B| - 2) + 5 = 2|B \cap D'| + 5$, and the desired conclusion follows from the induction hypothesis. \square

2.2. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. In place of Proposition 1, we use the fact that any point set R with $|R| = 6$ in convex position contains two convex quadrilaterals with disjoint interiors. \square

3. Point set $R(k, \varepsilon, n)$ and Theorem 5

3.1. Point set $R(k, \varepsilon, n)$

Let $k \geq 3$ be an odd integer and n an integer, and consider a regular k -gon $P_0 = p_1 p_2 \dots p_k$ inscribed in a unit circle. Rotate each point p_i by a sufficiently small angle ε around the center of P_0 to obtain a point p'_i (Fig. 2). We may assume that

$$\frac{p_1 p'_k}{p_1 p'_1} > kn \quad (1)$$

in particular (we use this inequality in the proof of Lemma 1). Furthermore, since no three diagonals of P_0 meet at a point (see [3]), we may assume that no three quadrilaterals $p_{i_1} p'_{i_1} p_{j_1} p'_{j_1}$, $p_{i_2} p'_{i_2} p_{j_2} p'_{j_2}$ and $p_{i_3} p'_{i_3} p_{j_3} p'_{j_3}$ ($i_1, i_2, i_3, j_1, j_2, j_3$ are all different) have a common point.

Take n red points $r_{i,1} = p_i, r_{i,2}, \dots, r_{i,n-1}, r_{i,n} = p'_i$ at regular intervals on segment $p_i p'_i$ (Fig. 3). Set $R_i = \{r_{i,1}, r_{i,2}, \dots, r_{i,n}\}$ and define the set $R = R(k, \varepsilon, n)$ consisting of kn red points by $R = R(k, \varepsilon, n) = \cup_{i=1}^k R_i$.

3.2. Theorem 5

To prove Theorems 3 and 4, we show the following result:

Theorem 5. *Let k and n be positive integers. Then*

$$\frac{\beta(R(k, \varepsilon, n))}{|R(k, \varepsilon, n)|} \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \frac{k}{n} \rightarrow \infty.$$

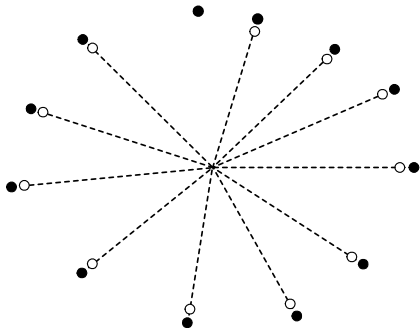


Fig. 4. ●: red point, ○: blue point.

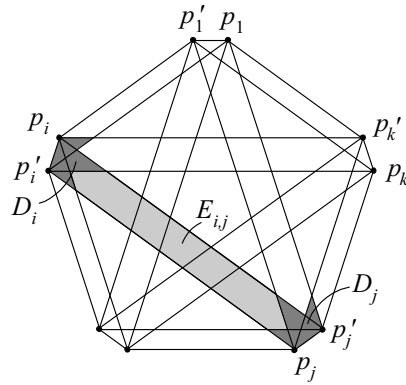


Fig. 5. D_i and $E_{i,j}$.

Observe that for a red point set R in convex position, if we place $|R| - 1$ blue points inside the convex hull of R (shown in Fig. 4 as small empty circles) all the 4-holes of R are covered. From a similar observation, we see that

$$\beta(R(k, \varepsilon, n)) \leq kn - 1. \quad (2)$$

3.3. Proof of Theorem 5

We will prove that

$$\frac{\beta(R(k, \varepsilon, n))}{kn} \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \frac{k}{n} \rightarrow \infty.$$

Define D_i and $E_{i,j}(= E_{j,i})$ by

$$D_i = \bigcup_{\substack{1 \leq l < m \leq k \\ l \neq i, m \neq i}} (p_i p_i' p_l p_l' \cap p_i p_i' p_m p_m')$$

$$E_{i,j} = p_i p_i' p_j p_j' - (D_i \cup D_j), \quad \text{where } i \neq j; \text{ see Fig. 5.}$$

Lemma 1. *The intersection of quadrilaterals $r_{i,m} r_{i,m+1} r_{j,l} r_{j,l+1}$ and $r_{i,m'} r_{i,m'+1} r_{j,l'} r_{j,l'+1}$ is contained in $E_{i,j}$ for any $i, j, m, m', l, l', 1 \leq i < j \leq k, 2 \leq m+1 < m' \leq n-1$ and $2 \leq l+1 < l' \leq n-1$.*

Proof. It suffices to consider the case where $p_i p_j' < p_i' p_j$ (Fig. 6). Take any point x in the intersection of quadrilaterals $r_{i,m} r_{i,m+1} r_{j,l} r_{j,l+1}$ and $r_{i,m'} r_{i,m'+1} r_{j,l'} r_{j,l'+1}$. For this x , we can take points p, q, p' and q' such that they are on the segments $r_{i,m} r_{i,m+1}$, $r_{j,l} r_{j,l+1}$, $r_{i,m'} r_{i,m'+1}$ and $r_{j,l'} r_{j,l'+1}$, respectively, and the intersection point of the segments pq and $p'q'$ is x . Take the point r such that $\vec{q'r} = \vec{p_i p_j'}$, and let y be the intersection point of pr and $p'q'$. Then we have $p'x > p'y$ and $\frac{p'y}{q'y} = \frac{p'r}{r q'} \geq \frac{1}{n-1}$, and hence

$$p'x > \frac{p'y}{n} > \frac{p_i p_j'}{n} \geq \frac{p_1 p_k'}{n}.$$

On the other hand, since the convex hull of D_i is a regular k -gon which has $p_i p_i'$ as one of its sides (Fig. 7), the diameter of D_i is less than $k p_i p_i'$, which is the perimeter of the regular k -gon. Since $k p_i p_i' < \frac{p_1 p_k'}{n}$ by (1), $x \notin D_i$, as desired. \square

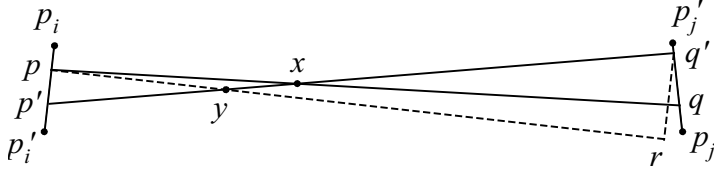


Fig. 6. A point x in the intersection and the point y .

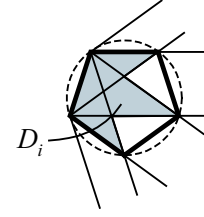


Fig. 7. D_i and its convex hull.

Let B be a set of blue points of minimum cardinality which cover all the 4-holes of R . For each $1 \leq i \leq k$, let $E_i = \cup_{j \neq i} E_{i,j}$. Then two cases arise:

Case 1. For each $1 \leq i \leq k$, either $|D_i \cap B| > n - \sqrt{n}$ or $|E_i \cap B| > 4(n - \sqrt{n})$.

Case 2. There exists i with $1 \leq i \leq k$ such that $|D_i \cap B| \leq n - \sqrt{n}$ and $|E_i \cap B| \leq 4(n - \sqrt{n})$.

Case 1. Let $I = \{1, 2, \dots, k\}$,

$$I_1 = \{i \mid |D_i \cap B| > n - \sqrt{n}, 1 \leq i \leq k\} \quad \text{and} \\ I_2 = I - I_1.$$

Then

$$|E_i \cap B| > 4(n - \sqrt{n}) \quad \text{for each } i \in I_2.$$

Since any point $b \in \cup_{1 \leq i \leq k} (E_i \cap B)$ is contained in at most four E_i 's, we have $|\cup_{i \in I_2} (E_i \cap B)| \geq \frac{1}{4} \sum_{i \in I_2} |E_i \cap B|$. Thus

$$\begin{aligned} |B| &\geq \left| \bigcup_{i \in I_1} (D_i \cap B) \right| + \left| \bigcup_{i \in I_2} (E_i \cap B) \right| \\ &\geq \sum_{i \in I_1} |D_i \cap B| + \frac{1}{4} \sum_{i \in I_2} |E_i \cap B| \\ &> |I_1|(n - \sqrt{n}) + \frac{1}{4}(k - |I_1|) \cdot 4(n - \sqrt{n}) \\ &= k(n - \sqrt{n}) = k(n + o(n)). \end{aligned}$$

From this, together with (2), we obtain the desired conclusion.

Case 2. Take i with $1 \leq i \leq k$ such that $|D_i \cap B| \leq n - \sqrt{n}$ and $|E_i \cap B| \leq 4(n - \sqrt{n})$. From this point on, we fix i . Let

$$J = \{j \mid E_{i,j} \cap B = \emptyset, j \neq i\}.$$

Since $|\{j' \mid E_{i,j'} \cap B \neq \emptyset, j' \neq i\}| \leq |E_i \cap B| \leq 4(n - \sqrt{n})$,

$$|J| \geq (k - 1) - 4(n - \sqrt{n}) = k + o(k) \tag{3}$$

for k, n with $\frac{k}{n} \rightarrow \infty$. Take any $j \in J$. Since $n - 1$ blue points arranged suitably close to the midpoints of segments $r_{j,m}r_{j,m+1}$, $1 \leq m \leq n - 1$, are sufficient to cover all 4-holes of R containing these segments, it follows from the minimality of $|B|$ that

$$|D_j \cap B| \leq n - 1. \tag{4}$$



Fig. 8. m_2 must be greater than or equal to m'_1 .

Lemma 2. $|D_j \cap B| = n + o(n)$.

To prove this lemma, we introduce some notation. Let e_l be the segment connecting $r_{i,l}$ and $r_{i,l+1}$, and f_m be the segment connecting $r_{j,m}$ and $r_{j,m+1}$, where $1 \leq l \leq n-1$ and $1 \leq m \leq n-1$. We denote by $[e_l, f_m]$ the quadrilateral containing e_l and f_m as its sides.

Let $b \in D_i \cap B$. Then it can be easily observed that the set \mathcal{Q}_b of quadrilaterals $[e_l, f_m]$ which are covered with b is expressed in the following form:

$$\mathcal{Q}_b = \{[e_{l_1}, f_m] \mid m_1 \leq m \leq m'_1\} \cup \{[e_{l_1+1}, f_m] \mid m_2 \leq m \leq m'_2\} \cup \dots \cup \{[e_{l_1+h}, f_m] \mid m_{h+1} \leq m \leq m'_{h+1}\},$$

where l_1 and h are integers with $1 \leq l_1 \leq l_1 + h \leq n-1$, and the m_t and the m'_t , $1 \leq t \leq h+1$, are integers with $1 \leq m_t \leq m'_t \leq n-1$.

Lemma 3. $h \in \{0, 1\}$; and in the case where $h = 1$, $m_2 \geq m'_1$.

Proof. We have $h \in \{0, 1\}$ from Lemma 1. Next assume $h = 1$. Then since $b \in \text{Int}([e_{l_1}, f_{m'_1}]) \cap \text{Int}([e_{l_1+1}, f_{m_2}])$, where $\text{Int}(X)$ denotes the interior of X , we must have $\text{Int}([e_{l_1}, f_{m'_1}]) \cap \text{Int}([e_{l_1+1}, f_{m_2}]) \neq \emptyset$, and hence $m_2 \geq m'_1$ (Fig. 8). \square

Consider an $(n-1) \times (n-1)$ table whose (l, m) -component corresponds to the quadrilateral $[e_l, f_m]$. By Lemma 3, each \mathcal{Q}_b , $b \in D_i \cap B$, is expressed as a set of components as shown in Fig. 9 (the case where $m_2 = m'_1$). We identify such a set of components with the set \mathcal{Q}_b . Set

$$\mathcal{Q} = \bigcup_{b \in D_i \cap B} \mathcal{Q}_b \quad (\text{Fig. 10}).$$

For each $b \in D_i \cap B$ such that \mathcal{Q}_b is expressed in the form of $\mathcal{Q}_b = \{[e_{l_1}, f_m] \mid m_1 \leq m \leq m'_1\} \cup \{[e_{l_1+1}, f_m] \mid m'_1 \leq m \leq m'_2\}$, let $U_b = [e_{l_1+1}, f_{m'_1}]$ (Fig. 9), and let \mathcal{U} denote the set of all such U_b 's. We have

$$|\mathcal{U}| \leq |D_i \cap B| \leq n - \sqrt{n}. \quad (5)$$

For each m with $1 \leq m \leq n-1$, call the column consisting of $n-1$ components $[e_1, f_m], [e_2, f_m], \dots, [e_{n-1}, f_m]$ the *column* f_m . Let \mathcal{F} be the set of columns f_1, f_2, \dots, f_{n-1} , and $\mathcal{F}^* (\subseteq \mathcal{F})$ the set of columns f_m each of which contains at most $\sqrt[4]{n}$ components corresponding to elements of \mathcal{U} . Since we consider the case where n and k are sufficiently large, we may assume $n \geq 7$ in particular.

Lemma 4. Let $f_m \in \mathcal{F}^*$. Then there is a component $[e_l, f_m]$ which does not belong to any \mathcal{Q}_b , i.e., there is a quadrilateral $[e_l, f_m]$ which is covered with some $b' \in D_j \cap B$.

Proof. We identify the column f_m with the set of components contained in it, i.e., we let $f_m = \{[e_l, f_m] \mid 1 \leq l \leq n-1\}$. Since

$$|\mathcal{Q} \cap f_m| \leq |D_i \cap B| + |\mathcal{U} \cap f_m| \leq (n - \sqrt{n}) + \sqrt[4]{n},$$

there are at least $\sqrt{n} - \sqrt[4]{n} - 1 (> 0 \text{ for } n \geq 7)$ components $[e_l, f_m] \in f_m$ which do not belong to any \mathcal{Q}_b . \square

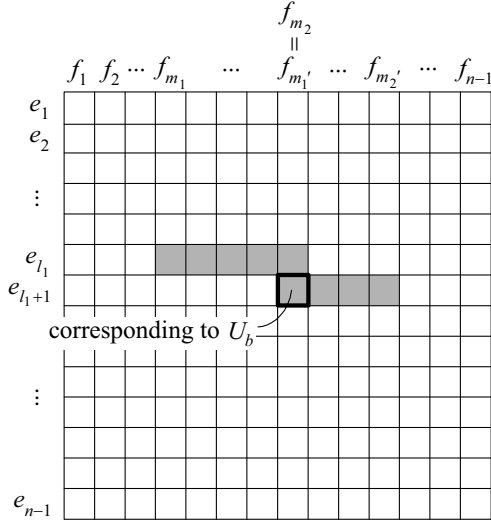


Fig. 9. Components corresponding to \mathcal{Q}_b .

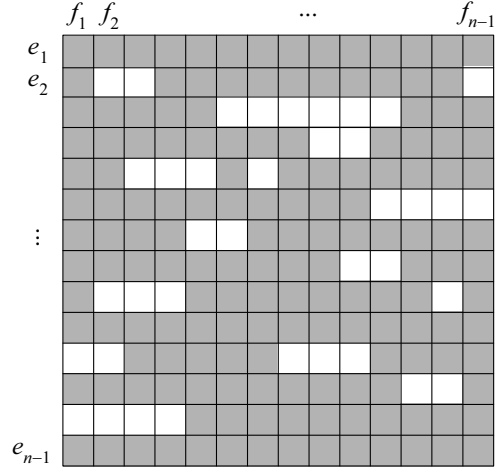


Fig. 10. Components corresponding to \mathcal{Q} .

Lemma 5. $|\mathcal{F}^*| \geq n - \sqrt[4]{n^3} + \sqrt[4]{n} - 1$.

Proof. By way of contradiction, suppose that $|\mathcal{F}^*| < n - \sqrt[4]{n^3} + \sqrt[4]{n} - 1$. Then we must have

$$\begin{aligned} |\mathcal{U}| &> \sqrt[4]{n} \times |\mathcal{F} - \mathcal{F}^*| \\ &> \sqrt[4]{n} \times (\sqrt[4]{n^3} - \sqrt[4]{n}) \\ &= n - \sqrt[4]{n}. \end{aligned}$$

This contradicts (5). \square

Now consider points of $D_j \cap B$. For $b' \in D_j \cap B$, we use the same notation used above: denote by $\mathcal{Q}_{b'}$ the set of quadrilaterals $[e_l, f_m]$ which are covered with b' . In the same way as in the proof of Lemma 3, the following lemma follows:

Lemma 6. $\mathcal{Q}_{b'}$ is in the form of $\{[e_l, f_m] \mid l_1 \leq l \leq l'_1\}$ or $\{[e_l, f_m] \mid l_1 \leq l \leq l'_1\} \cup \{[e_l, f_{m+1}] \mid l_2 \leq l \leq l'_2\}$, where $l_2 \geq l'_1$ (Fig. 11).

Let $\mathcal{S} (\subseteq \mathcal{F}^*)$ be the set of columns $f_m \in \mathcal{F}^*$ which satisfy at least one of the following conditions (i), (ii) or (iii):

- (i) $m = 1$;
- (ii) f_{m-1} is not contained in \mathcal{F}^* ;
- (iii) no two quadrilaterals $[e_l, f_m]$ and $[e_{l'}, f_{m-1}]$ are covered with a single point $b' \in D_j \cap B$.

Let $s = |\mathcal{S}|$ and write $\mathcal{S} = \{f_{m_1}, f_{m_2}, \dots, f_{m_s}\}$. Then \mathcal{F}^* is expressed in the following form:

$$\begin{aligned} \mathcal{F}^* = \{f_{m_1}, f_{m_1+1}, \dots, f_{m'_1}\} \cup \{f_{m_2}, f_{m_2+1}, \dots, f_{m'_2}\} \cup \\ \dots \cup \{f_{m_s}, f_{m_s+1}, \dots, f_{m'_s}\}. \end{aligned} \quad (6)$$

Let

$$\mathcal{F}_t^* = \{f_{m_t}, f_{m_t+1}, \dots, f_{m'_t}\}, \quad 1 \leq t \leq s$$

(so $\mathcal{F}^* = \mathcal{F}_1^* \cup \mathcal{F}_2^* \cup \dots \cup \mathcal{F}_s^*$), and let B_t denote the set of points $b' \in D_j \cap B$ each of which covers a quadrilateral of $\{[e_l, f_m] \mid 1 \leq l \leq n-1, f_m \in \mathcal{F}_t^*\}$. From Lemmas 4, 6 and the definition of \mathcal{F}_t^* , the following lemma follows:

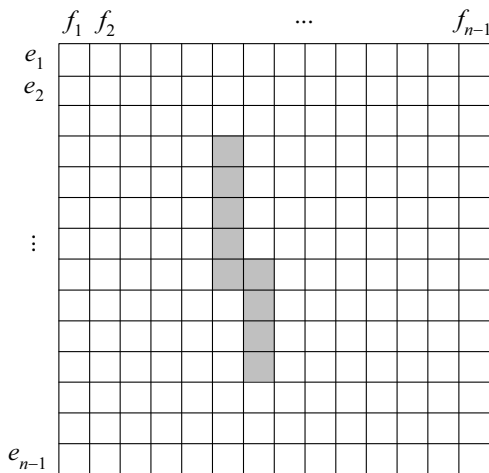


Fig. 11. Components corresponding to $Q_{b'}$.

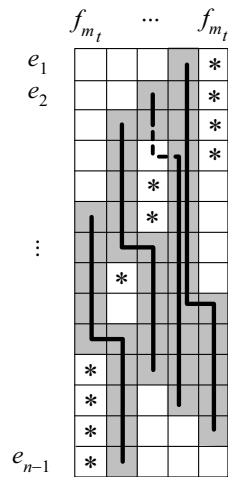


Fig. 12. Components corresponding to $Q_{b'}$'s.

Lemma 7. $|B_t| \geq |\mathcal{F}_t^*| - 1$ for $1 \leq t \leq s$.

Furthermore, the following holds.

Lemma 8. Let $1 \leq t \leq s$, and suppose $|\mathcal{F}_t^*| \leq \sqrt[4]{n} - 1$. Then $|B_t| \geq |\mathcal{F}_t^*|$.

Proof. By way of contradiction, suppose $|B_t| \leq |\mathcal{F}_t^*| - 1$ (so $|B_t| = |\mathcal{F}_t^*| - 1$ by Lemma 7). Then it follows from Lemmas 4, 6 and the definition of \mathcal{F}_t^* again that for each m with $m_t \leq m \leq m'_t - 1$, there exists exactly one point $b' \in B_t$ such that

$$Q_{b'} = \{[e_l, f_m] \mid l_1 \leq l \leq l'_1\} \cup \{[e_l, f_{m+1}] \mid l_2 \leq l \leq l'_2\}, \quad (7)$$

where l_1, l'_1, l_2, l'_2 are some positive integers with $l_1 \leq l'_1, l_2 \leq l'_2$ and $l'_1 \leq l_2$; and furthermore, there exists no point $b'' \in D_j \cap B$ such that $Q_{b''}$ is expressed in the form $\{[e_l, f_m] \mid l_1 \leq l \leq l'_1\}$.

For $b' \in B_t$ such that $l'_1 = l_2$ holds in the expression (7), let $V_{b'} = [e_{l_2}, f_{m+1}]$, and let \mathcal{V} denote the set of all such $V_{b'}$'s. Furthermore, for each l with $1 \leq l \leq n - 1$, call the (sub)row consisting of $[e_l, f_{m_t}], [e_l, f_{m_t+1}], \dots, [e_l, f_{m'_t}]$, the row e_l (a row of the table shown in Fig. 12). Let \mathcal{R} denote the set of rows e_l containing no element of \mathcal{V} . Since $|\mathcal{V}| \leq |B_t| \leq |\mathcal{F}_t^*| - 1$ by assumption, $|\mathcal{R}| \geq (n - 1) - |\mathcal{V}| \geq n - |\mathcal{F}_t^*|$. On the other hand, among $|\mathcal{F}_t^*|$ components of each row $e_l \in \mathcal{R}$, at most $|B_t|$ ($\leq |\mathcal{F}_t^*| - 1$) components correspond to quadrilaterals which are covered with points of B_t . Hence, from each $e_l \in \mathcal{R}$, we can take one component $[e_l, f_m]$ which is not covered with any point of B_t (e.g. each component marked with $*$ in Fig. 12). Let \mathcal{W} be the set of these components. Then

$$|\mathcal{W}| = |\mathcal{R}| \geq n - |\mathcal{F}_t^*|. \quad (8)$$

Since distinct components of $\mathcal{W} - \mathcal{U}$ must be covered with distinct points of $D_i \cap B$, we must have

$$|\mathcal{W} - \mathcal{U}| \leq |D_i \cap B| \leq n - \sqrt{n} \quad (9)$$

from the assumption of Case 2. On the other hand, since the columns $f_{m_t}, \dots, f_{m'_t}$ belong to \mathcal{F}^* , it follows from the definition of \mathcal{F}^* and (8) that

$$|\mathcal{W} - \mathcal{U}| \geq (n - |\mathcal{F}_t^*|) - \sqrt[4]{n} |\mathcal{F}_t^*|$$

$$\begin{aligned}
&= n - (\sqrt[4]{n} + 1)|\mathcal{F}_t^*| \\
&\geq n - (\sqrt[4]{n} + 1)(\sqrt[4]{n} - 1) \quad (\text{by the assumption of Lemma 8}) \\
&= n - \sqrt{n} + 1,
\end{aligned}$$

which contradicts (9). □

Now let $T = \{1, 2, \dots, s\}$, $T_1 = \{t \in T \mid |\mathcal{F}_t^*| > \sqrt[4]{n} - 1\}$ and $T_2 = T - T_1$. Since

$$|T_1| < \frac{|\mathcal{F}^*|}{\sqrt[4]{n} - 1} \leq \frac{|\mathcal{F}|}{\sqrt[4]{n} - 1} \leq \frac{n - 1}{\sqrt[4]{n} - 1} = O(n^{\frac{3}{4}}), \quad (10)$$

and since $B_t \cap B_{t'} = \emptyset$ for $1 \leq t < t' \leq s$, it follows from Lemmas 7 and 8 that

$$\begin{aligned}
|D_j \cap B| &\geq \sum_{t \in T_1} |B_t| + \sum_{t \in T_2} |B_t| \\
&\geq \sum_{t \in T_1} (|\mathcal{F}_t^*| - 1) + \sum_{t \in T_2} |\mathcal{F}_t^*| \\
&= \sum_{t \in T} |\mathcal{F}_t^*| - |T_1| \\
&= |\mathcal{F}^*| - |T_1| \\
&= n + o(n) \quad (\text{by Lemma 5 and (10)}).
\end{aligned}$$

From this together with (4), Lemma 2 follows. This completes the proof of Lemma 2.

Now the conclusion of Theorem 5 follows in Case 2 as well from (2), (3) and Lemma 2.

4. Proofs of Theorems 3 and 4

Concerning Theorem 5, note that the same conclusion holds even if we construct the red point set by taking n red points at regular intervals, on each *arc* $p_i p'_i$, $1 \leq i \leq k$, of the unit circle in which the regular k -gon P_0 is inscribed (recall the construction of $R(k, \varepsilon, n)$ stated in Section 3.1). The point set we obtain in this way is in convex position, and hence Theorem 3 holds.

We can construct another red point set with the desired property by placing n red points at regular intervals, around consecutive k vertices of a regular k' -gon, $k' > k$, as shown in Fig. 13 (all the points lie on a circle). Let $R_{k'}(k, \varepsilon, n)$ denote a point set obtained in this way. We now prove Theorem 4. First choose n , and k sufficiently large with respect to n , and K sufficiently large with respect to kn , and construct $R(K, \varepsilon, kn)$ for ε sufficiently small. To obtain the final point set, we replace each point set $\{r_{i,1}, r_{i,2}, \dots, r_{i,kn}\}$ by a copy of $R_{k'}(k, \varepsilon', n)$ as shown in Fig. 14, where k' is a sufficiently large number with respect to K , so that Lemma 1 can be applied. Denote by R^* the red point set we obtain in this way. Then we have $|R^*| = Kkn$ and if we let $m = kn$ and $M = Kkn (= Km)$,

$$\beta(R^*) = K(m + o(m)) + (M + o(M)) = 2M + o(M)$$

as $n \rightarrow \infty$, $\frac{k}{n} \rightarrow \infty$ and $\frac{K}{kn} \rightarrow \infty$ (though not all pairs of adjacent points of $R_{k'}(k, \varepsilon', n)$ are at regular intervals), and hence Theorem 4 holds.

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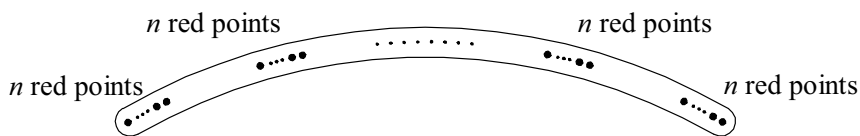


Fig. 13. Red point set R' with $|R'| = kn$.

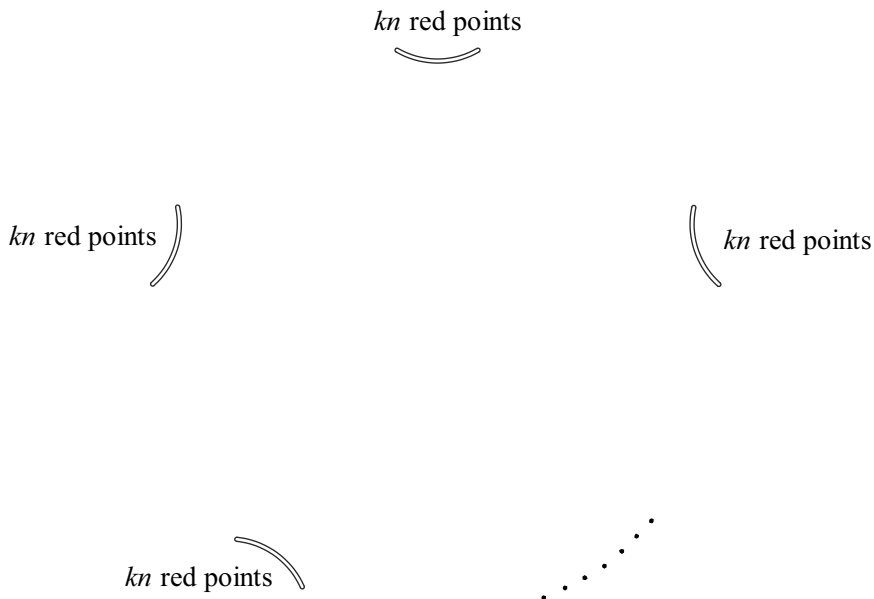


Fig. 14. Red point set R^* with $|R^*| = Kkn$.

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